PROBLEM SET 02 - Basic counting rules

1 Basic rules

Exercise 1.1. How many strings of three decimal digits
   a) do not contain the same digit three times?
   b) begin with an odd digit?
   c) have exactly two digits that are 4s?

Exercise 1.2. Six different airlines fly from New York to Denver and seven fly from Denver to San Francisco. Moreover
   three different airlines fly from New York to San Francisco directly. How many different configurations of airlines can
   you choose on which to book a trip from New York to San Francisco and back? What if you do not want to repeat any
   connection, when you go back?

Exercise 1.3. Each user on a computer system has a password, which is six to eight characters long, where each character
   is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

Exercise 1.4. How many ways are there to seat four of a group of ten people around a circular table where two seatings
   are considered the same when everyone has the same immediate left and immediate right neighbor?

Exercise 1.5. In how many ways can a photographer at a wedding arrange 6 people in a row from a group of 10 people,
   where the bride and the groom are among these 10 people, if
   a) the bride must be in the picture?
   b) both the bride and groom must be in the picture?
   c) exactly one of the bride and the groom is in the picture?
   d) at least one of the bride and the groom is in the picture?

Exercise 1.6. How many bit strings of length seven either begin with two 0s or begin with three 1s?

Exercise 1.7. A multiple-choice test contains 10 questions. There are four possible answers for each question.
   a) In how many ways can a student answer the questions on the test if the student answers every question?
   b) In how many ways can a student answer the questions on the test if the student can leave answers blank?

Exercise 1.8. How many ways there are to put fruits in a basket, so that the basket is not empty, if we have
   n indistinguishable apples and m indistinguishable oranges.

Exercise 1.9. There are 32 microcomputers in a computer center. Each microcomputer has 24 ports. How many different
   ports to a microcomputer in the center are there?

Exercise 1.10. How many subsets of a set with 100 elements have more than one element?

Exercise 1.11. Suppose that a password for a computer system must have at least 8, but no more than 12, characters,
   where each character in the password is a lowercase English letter, an uppercase English letter, a digit, or one of the six
   special characters *, >, <, !, +, and =.
   a) How many different passwords are available for this computer system?
   b) How many of these passwords contain at least one occurrence of at least one of the six special characters?
   c) How many of these passwords contain a special character at the beginning or at the end?
   d) Using your answer to part (a), determine how long it takes a hacker to try every possible password, assuming that
      it takes one nanosecond for a hacker to check each possible password. (HINT: ns = 10^{-9} sec)

Exercise 1.12. Every student in a discrete mathematics class is either a computer science or a mathematics major or is
   a joint major in these two subjects. How many students are in the class if there are 38 computer science majors (including
   joint majors), 23 mathematics majors (including joint majors), and 7 joint majors?

Exercise 1.13. How many positive integers not exceeding 100 are divisible by 4 or by 6?
2 Bijections

Exercise 2.1. In each case find a bijection (a one-to-one correspondence) between sets A and B:

a. A – the set of all bit strings with \( k \) "0" and \( n \) "1";
   B – the set of all shortest paths from A to B on the lattice

b. A – the set of all bit strings of length \( n \) (\( n \)-digit binary sequences);
   B – the family of all subsets of the set \{1, \ldots, n\}.

c. A – the set of all arrangement of \( n \) distinguishable balls in \( k \) numbered boxes;
   B – the set of sequences of length \( n \) with elements from the set \{1, \ldots, k\};

d. A – the set of arrangements of \( n \) distinguishable balls in \( k \) (\( n \leq k \)) numbered boxes, such that each box contains at most one ball;
   B – the set of sequences of length \( n \) with elements from the set \{1, \ldots, k\} (\( n \leq k \)), in which no two elements are equal;

e. A – the set of arrangements of \( n \) identical balls in \( k \) (\( n \leq k \)) numbered boxes, such that each box contains at most one ball;
   B – the family of \( k \)-element subsets of the set \{1, \ldots, n\}

f. A – the set of \( n \)-digit binary sequences with exactly \( k \) ones (\( k \leq n \));
   B – the family of \( k \)-element subsets of the set \{1, \ldots, n\} (\( k \leq n \));

g. A – the set of all solutions of the equation
   \[ x_1 + x_2 + \ldots + x_{2k} = 0, \quad x_i \in \{-1, 1\}; \]
   B – the set of all shortest paths between opposite corners of the lattice with side of length \( k \);

h. A – the set of all solutions of the equation
   \[ x_1 + x_2 + \ldots + x_{2k} = 4, \quad x_i \in \{-1, 1\}; \quad k \geq 2 \]
   B – the family of \( k + 2 \)-element subsets of the set \{1, 2, \ldots, 2k\}, \( k \geq 2 \);

i. A – the set of all integer solutions of the equation
   \[ x_1 + x_2 + \ldots + x_k = n, \quad x_i \geq 2 \]
   B – the set of all integer solutions of the equation
   \[ y_1 + y_2 + \ldots + y_k = n - k, \quad y_i \geq 1 \]

j. A – the set of all integer solutions of the equation
   \[ x_1 + x_2 + \ldots + x_k = n, \quad x_i \geq 1 \]
   B – the set of \( n + k - 1 \)-digit binary sequences with \( n \) zeros and \( k - 1 \) ones, starting and ended by zero with no two consecutive ones.

k. A – the set of all integer solutions of the equation
   \[ x_1 + x_2 + \ldots + x_k = n, \quad x_i \geq 1 \]
   B – the family of \( k - 1 \)-element subsets of the \( n - 1 \)-element set.

l. A – the set of all 6-letter words, which may be created using letters from the word TAMTAM
   B – the family of ordered divisions of the set \{1, 2, 3, 4, 5, 6\} into two-element subsets.