Functiones et Approximatio
XXXIV (2005), 27–146

FOURIER COEFFICIENTS OF MODULAR FORMS AND EIGENVALUES OF A HECKE OPERATOR

NIGEL WATT

Abstract: We prove results analogous to certain theorems of Deshouillers and Iwaniec [Invent. Math. **70** (1982), 219-288]. Our proofs parallel theirs in the use made of the summation formulae of Bruggeman and Kuznetsov: where they require a lower bound on eigenvalues $\lambda_j = 1/4 + \kappa_j^2$ of the hyperbolic Laplacian operator (using that of Selberg) we need instead upper bounds on the moduli of the eigenvalues of a Hecke operator, obtaining these from recent work of Kim and Sarnak [J. Amer. Math. Soc. **16** (2003), 139-183]. Specifically, we give new bounds for sums $\sum_{Q/2<q\leqslant Q}\sum_{|\kappa_j|\leqslant K}|\sum_{N/2<n\leqslant N} b_n\rho_j(Dn)|^2$, where (b_n) is a complex sequence, and j indexes the elements, $u_j(z)$, of a suitable orthonormal basis of the space spanned by the Maass cusp forms for the Hecke congruence subgroup $\Gamma_0(q)$, while $\rho_j(n)$ is the n-th Fourier coefficient at the cusp ∞ for $u_j(z)$, and D is a large positive integer. Our bounds are strongest in cases where every prime factor of D is a small power of D.

One application (briefly discussed in the paper) is a new mean-square bound for the modulus of a certain multiple sum involving Dirichlet characters modulo D. It is hoped this will be useful in the study of Carmichael numbers.
Keywords: Maass cusp form, Fourier coefficient, Hecke operator, eigenvalue, mean value, Kloosterman sum, Dirichlet character.

1. Introduction

In this paper we establish two results that are in a certain sense (to be made clear below) analogous to results obtained by Deshouillers and Iwaniec in [7]. That seminal work of Deshouillers and Iwaniec has served as our guide in the construction of our proofs. We also take the opportunity to apply a bound for Hecke eigenvalues that was proved recently by Kim and Sarnak in [16], Appendix 2. The initial motivation for this work has come from its potential applications in the context of our earlier work with Glyn Harman and Kam Wong in [10].

The objects of primary interest to us will be Fourier coefficients of modular forms and Eisenstein series associated with a subgroup Γ of the full modular group $SL_2(\mathbb{Z})$. Our work, and this Introduction, relates only to cases in which Γ

2001 Mathematics Subject Classification: 11F30, 11A25, 11F11, 11F25, 11F70, 11F72, 11L05, 11L40, 11M06, 33C10

is a Hecke congruence subgroup,

$$\Gamma = \Gamma_0(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \; : \; c \equiv 0 \pmod{q} \right\},$$

of some given 'level' $q \in \mathbb{N}$. With apologies to experts, we use the first half of this section to present some basic information about modular forms and Eisentein series, just sufficient to put our results (which follow) into some perspective, while laying out some other facts for later reference. Our notation is largely identical to that introduced in [7], Sections 1.1 and 1.2, and the reader should look there, or in works such as [4], [12], [14] and [15], for an account of those elements of the theory that we omit.

The elements of Γ (or indeed those of $SL_2(\mathbb{R})$) act on the upper half plane, $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$, and on the extended real line $\mathbb{R} \cup \{\infty\}$ through the linear fractional transformations,

$$\gamma z = \frac{az+b}{cz+d} \qquad \left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \; z \in \mathbb{C} \cup \{\infty\} \right),$$

where the usual conventions regarding '∞' are observed. As $\Gamma \subseteq SL_2(\mathbb{Z})$, the cusps of Γ are just the elements of $\mathbb{Q} \cup \{\infty\}$. An equivalence relation on the set of cusps is defined by writing $\mathfrak{a} \sim \mathfrak{b}$ ('$\mathfrak{a}$ is equivalent to $\mathfrak{b}$'), or $\mathfrak{a} \overset{\Gamma}{\sim} \mathfrak{b}$ ('$\mathfrak{a}$ is Γ-equivalent to $\mathfrak{b}$'), if and only if the orbits $\Gamma\mathfrak{a}$, $\Gamma\mathfrak{b}$ are equal. For example, one has $\infty \sim 1/q$, since there exists $\gamma \in \Gamma$ with $\gamma z = z/(qz+1)$ and (consequently) $\gamma\infty = 1/q$.

For each cusp $\mathfrak{a}$ we may choose a 'scaling matrix' $\sigma_\mathfrak{a} \in SL_2(\mathbb{R})$ with

$$\sigma_\mathfrak{a}\infty = \mathfrak{a} \quad \text{and} \quad \sigma_\mathfrak{a}^{-1}\Gamma_\mathfrak{a}\sigma_\mathfrak{a} = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{Z} \right\}, \tag{1.1}$$

where $\Gamma_\mathfrak{a}$ is the stabiliser subgroup. In the particular case $\mathfrak{a} = \infty$ we may take here

$$\sigma_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.2}$$

Given that $k/2 \in \mathbb{N}$, a function $f : \mathbb{H} \longrightarrow \mathbb{C}$ is a holomorphic cusp form of weight k with respect to Γ if and only if it satisfies $f(z) = ((\mathrm{d}/\mathrm{d}z)(\gamma z))^{-k/2} f(\gamma z)$, for $\gamma \in \Gamma$ and $z \in \mathbb{H}$, and, for each cusp $\mathfrak{a}$, possesses a Fourier expansion,

$$\left(\frac{\mathrm{d}}{\mathrm{d}z} \sigma_\mathfrak{a} z \right)^{-k/2} f(\sigma_\mathfrak{a} z) = \sum_{m=1}^{\infty} \psi(\mathfrak{a}, m) \mathrm{e}(mz), \tag{1.3}$$

absolutely convergent for $z \in \mathbb{H}$.

A function $u : \mathbb{H} \longrightarrow \mathbb{C}$ is a non-holomorphic modular form of weight zero with respect to Γ if and only if $u(z)$ is an eigenfunction of the hyperbolic Laplacian operator,

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \qquad (x = \operatorname{Re}(z), \; y = \operatorname{Im}(z)),$$

and is such that $u(z) = u(\gamma z)$, for $\gamma \in \Gamma$ and $z \in \mathbb{H}$. Such a function $u(z)$ is called a Maass cusp form if and only if, for each cusp $\mathfrak{a}$, it has a Fourier expansion of the form

$$u(\sigma_{\mathfrak{a}} z) = y^{1/2} \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \rho_{\mathfrak{a}}(m) K_{i\kappa}(2\pi |m| y)\, \mathrm{e}(mx) \tag{1.4}$$

(absolutely convergent for $z = x + iy$ with $x \in \mathbb{R}$ and $y > 0$), where

$$\tfrac{1}{4} + \kappa^2 = \lambda = (\Delta u)/u \tag{1.5}$$

and $K_\nu(z)$ is the Bessel function conventionally so denoted (see our Lemma 2.3 for an integral representation). Note that since $K_\nu(z)$ is an even function of ν, the Fourier series in (1.4) does not depend upon the choice of κ satisfying (1.5).

For any cusp $\mathfrak{c}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the Eisenstein series $E(z) = E_{\mathfrak{c}}(z,s)$ given by [7], Equation (1.13), is a function $E : \mathbb{H} \longrightarrow \mathbb{C}$ satisfying $\Delta E = s(1-s)E$ and $E(z) = E(\gamma z)$, for $\gamma \in \Gamma$ and $z \in \mathbb{H}$. At each cusp $\mathfrak{a}$, there is a Fourier series expansion,

$$\begin{aligned} E_{\mathfrak{c}}(\sigma_{\mathfrak{a}} z, s) = \delta_{\mathfrak{c}\mathfrak{a}} y^s + \frac{\sqrt{\pi}\,\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)} \varphi_{\mathfrak{c}\mathfrak{a}}(0,s) y^{1-s} + \\ + y^{1/2} \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{2\pi^s |m|^{s-1/2}}{\Gamma(s)} \varphi_{\mathfrak{c}\mathfrak{a}}(m,s) K_{s-1/2}(2\pi|m|y)\, \mathrm{e}(mx) \end{aligned} \tag{1.6}$$

(see 'Notations (II)', at the end of this section, for a definition of $\delta_{\mathfrak{c}\mathfrak{a}}$ here). All the Fourier coefficients $\varphi_{\mathfrak{c}\mathfrak{a}}(m,s)$ here are holomorphic for $s \in \mathbb{C} - \{1\}$ with $\mathrm{Re}(s) \geqslant 1/2$. For such s, and $z = x + iy \in \mathbb{H}$ as in (1.4), the Fourier expansions (1.6) are absolutely convergent (see [15], Chapter 6, which gives an account of Selberg's approach to the meromorphic continuation of the Eisenstein series). In fact (1.6) for the cusp $\mathfrak{c}$ and just one cusp $\mathfrak{a}$ yields, for $z \in \mathbb{H}$, a meromorphic continuation of $E_{\mathfrak{c}}(z,s)$ to all of $\mathbb{C}$.

Given $\mathfrak{c}$ and $s = \frac{1}{2} + ir$, with $r \in \mathbb{R}$, the function $E(z) = E_{\mathfrak{c}}(z,s)$ is a non-holomorphic form of weight zero with respect to Γ, but, due to the presence of terms independent of x in its Fourier expansion at cusp $\mathfrak{c}$, it is not a Maass cusp form (see (1.6), (1.4) and the comment following [15], Proposition 6.12). These particular non-holomorphic forms correspond, in respect of the operator Δ, to eigenvalues $\lambda = s(1-s) = \frac{1}{4} + r^2$, which, as r runs over $\mathbb{R}$, range over the 'continuous spectrum' of values satisfying

$$\lambda \in [1/4, \infty). \tag{1.7}$$

For $k/2 \in \mathbb{N}$, the Petersson inner product $< f, g >_k$ (defined in [7], Section 1.1) makes the space of holomorphic cusp forms of weight k with respect to Γ into a finite dimensional Hilbert space, $\mathfrak{M}_k(\Gamma)$. Each space $\mathfrak{M}_k(\Gamma)$ is equipped with a sequence $(T_n^{(k)})$ of Hecke operators, where, for $f \in \mathfrak{M}_k(\Gamma)$, $n \in \mathbb{N}$ and $z \in \mathbb{H}$,

$$\left(T_n^{(k)} f\right)(z) = \frac{1}{n} \sum_{\substack{ad=n \\ (a,q)=1}} a^k \sum_{b \bmod d} f\left(\frac{az+b}{d}\right) \tag{1.8}$$

(see [14], Chapter 6). Deligne [6] has, in a setting more general than our own, shown the eigenvalues of these operators on $\mathfrak{M}_k(\Gamma)$ to be of the magnitude predicted by the Ramanujan-Petersson conjecture.

Theorem 1.1. [Deligne] *Let $k/2 \in \mathbb{N}$ and $n \in \mathbb{N}$ with $(n,q) = 1$. If λ is an eigenvalue of $T_n^{(k)}$, then*

$$|\lambda| \leqslant \tau(n) n^{(k-1)/2},$$

where $\tau(n)$ is the divisor function.

The Maass cusp forms $u(z)$ span an infinite dimensional Hilbert space $L^2_{\text{cusp}}(\Gamma\backslash\mathbb{H})$ with Petersson inner-product $< f, g >_0$. This space too has its Hecke operators T_n, where, as in [7], Section 1.2,

$$(T_n f)(z) = \frac{1}{\sqrt{n}} \sum_{\substack{ad=n \\ (a,q)=1}} \sum_{b \bmod d} f\left(\frac{az+b}{d}\right), \tag{1.9}$$

for $n \in \mathbb{N}$, $f \in L^2_{\text{cusp}}(\Gamma\backslash\mathbb{H})$ and $z \in \mathbb{H}$. Note that $n^{-1/2}T_n$ would be the operator $T_n^{(0)}$ given by (1.8). It is conjectured that, for $n \in \mathbb{N}$ with $(n,q) = 1$, all the eigenvalues τ of T_n satisfy $|\tau| \leqslant \tau(n)$ (the constant functions being excluded by virtue of their orthogonality to all Maass cusp forms). This non-holomorphic Ramanujan-Petersson conjecture has been reformulated in representation theoretic terms by Satake [21] so as to embrace also Selberg's conjecture that, in respect of the space $L^2_{\text{cusp}}(\Gamma\backslash\mathbb{H})$, all eigenvalues λ of Δ satisfy the same inequality (1.7) that holds quite trivially in the case of Eisenstein series (both conjectures allowing Γ to be any congruence subgroup of the full modular group). Following significant progress towards one or other of these conjectures by Selberg [22], by Serre (see [20]), and by the authors of [5] and [18], there has quite recently been striking further progress on both fronts by Kim and Shahidi [17] and Kim [16]. With further assistance from methods developed in [7] and [18], Kim and Sarnak [16], Appendix 2, have achieved the strongest results to date. These include (as a special case) the following theorem.

Theorem 1.2. [Kim-Sarnak] *Let $n \in \mathbb{N}$ with $(n,q) = 1$. Suppose that τ is an eigenvalue of T_n, and λ an eigenvalue of Δ, where, in each case, the corresponding eigenfunction belongs in $L^2_{\text{cusp}}(\Gamma\backslash\mathbb{H})$. Then*

$$|\tau| \leqslant \tau(n) n^{\vartheta} \tag{1.10}$$

and

$$\lambda \geqslant \frac{1}{4} - \vartheta^2, \tag{1.11}$$

where

$$\vartheta = \frac{7}{64}. \tag{1.12}$$

By [15], Theorem 4.7, for example, $L^2_{\text{cusp}}(\Gamma\backslash\mathbb{H})$ has an orthonormal basis $\{u_j : j \in \mathbb{N}\}$ with elements u_j that are all Maass cusp forms with respect to Γ. By (1.11) and the Weyl law (see [15], Corollary 11.2), it may be assumed that, subject to some renumbering, one has here:

$$\tfrac{1}{4} - \vartheta^2 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \ldots \quad \text{and} \quad \lambda_n \longrightarrow \infty \ \text{ as } \ n \longrightarrow \infty, \tag{1.13}$$

where $\lambda_j = (\Delta u_j)/u_j$. We will take $\rho_{j\mathfrak{a}}(m)$ to be the coefficient $\rho_{\mathfrak{a}}(m)$ in the expansion of $u(z) = u_j(z)$ given by (1.4).

We follow [7] in parameterising the eigenvalues of Δ in terms of the 'κ' of (1.5), so that $\lambda_j = \frac{1}{4} + \kappa_j^2$. By (1.13) it follows that, for all sufficiently large $j \in \mathbb{N}$, one has

$$\lambda_j \geqslant 1/4 \qquad \text{and} \qquad \kappa_j \in \mathbb{R}. \tag{1.14}$$

Adding the prefix 'Δ' to the terminology used in [7], we would classify any λ_j not satisfying (1.14) as 'Δ-exceptional'. If there are any Δ-exceptional eigenvalues for Γ, then they are finite in number, and (by (1.13) again) must satisfy:

$$\lambda_j \in [1/4 - \vartheta^2, 1/4) \qquad \text{and} \qquad i\kappa_j \in (0, \vartheta] \tag{1.15}$$

(where we allow ourselves to substitute $-\kappa_j$ for κ_j if necessary).

As observed in [7], Section 1.2, the facts that the Hecke operators T_n with $(n,q) = 1$ are bounded, self-adjoint, and commute with each other and with Δ, enable us to assume, additionally, that each element u_j of the above basis is an eigenfunction of all those operators T_n:

$$T_n u_j = \tau_j(n) u_j \qquad (j, n \in \mathbb{N} \text{ and } (n,q) = 1), \tag{1.16}$$

where the 'Hecke' eigenvalues, $\tau_j(n)$ are real.

Note that another possible basis of $L^2_{\text{cusp}}(\Gamma\backslash\mathbb{H})$ is the set $\{v_j : j \in \mathbb{N}\}$, where $v_j(z) = u_j(-\bar{z})$, for $j \in \mathbb{N}$ and $z \in \mathbb{H}$. This alternative basis inherits all of the properties that we are assuming the basis of u_j's to have. If we were to substitute $v_j(z)$ for $u_j(z)$ (for all j), then the effect (see (1.4) and (1.2)) would be to change $\rho_{j\infty}(m)$ into $\rho_{j\infty}(-m)$, for all j and m. Therefore bounds for sums involving coefficients $\rho_{j\infty}(m)$ with positive m (only) will imply corresponding bounds for sums involving $\rho_{j\infty}(n)$ with negative n (only). This explains why we will not explicitly state any results regarding the latter type of sum.

For $k/2 \in \mathbb{N}$, the Hecke operators $T_n^{(k)}$ with $(n,q) = 1$ are (like the corresponding operators T_n) bounded, self-adjoint and pairwise commuting. It follows that we may suppose each space $\mathfrak{M}_k(\Gamma)$ with $k/2 \in \mathbb{N}$ to have an assigned orthonormal basis, $\{f_{jk} : 1 \leqslant j \leqslant \theta_k(q)\}$, where every basis element f_{jk} is an eigenfunction of all Hecke operators $T_n^{(k)}$ with $(n,q) = 1$:

$$T_n^{(k)} f_{jk} = \lambda_{jk}(n) f_{jk} \qquad (k/2, j, n \in \mathbb{N},\ j \leqslant \theta_k(q) \text{ and } (n,q) = 1). \tag{1.17}$$

As a matter of notational convenience (in stating subsequent results) we define here

$$\lambda_{jk}^*(n) = \lambda_{jk}(n)/n^{(k-1)/2} \tag{1.18}$$

and also

$$\psi_{jk}^*(\mathfrak{a}, m) = \psi_{jk}(\mathfrak{a}, m)/m^{(k-1)/2}, \tag{1.19}$$

where $\psi_{jk}(\mathfrak{a}, m)$ is the mth Fourier coefficient $\psi(\mathfrak{a}, m)$ in the expansion of $f(z) = f_{jk}(z)$ given by (1.3). Note that by Kim and Sarnak's bound ((1.10) and (1.12) of Theorem 1.2), and by (1.18) and Theorem 1.1 of Deligne we have now, for $j \in \mathbb{N}$,

$$|\tau_j(n)| \leqslant \tau(n)n^{\vartheta} \qquad (n \in \mathbb{N} \text{ with } (n,q) = 1) \tag{1.20}$$

and, for $k/2 \in \mathbb{N}$ and $1 \leqslant j \leqslant \theta_k(q)$,

$$\left|\lambda_{jk}^*(n)\right| \leqslant \tau(n) \qquad (n \in \mathbb{N} \text{ with } (n,q) = 1). \tag{1.21}$$

Comparison of (1.16) and (1.17) with the operations of T_n and $T_n^{(k)}$ (indicated by (1.9) and (1.8)) upon the respective Fourier expansions, at cusp $\mathfrak{a} = \infty$, of $u(z) = u_j(z)$ and $f(z) = f_{jk}(z)$ (shown in (1.4) and (1.3)) leads one to the following well-known identities (in which $j, m, n \in \mathbb{N}$ is assumed):

$$\rho_{j\infty}(m)\tau_j(n) = \sum_{g|(m,n)} \rho_{j\infty}\left(\frac{m}{g}\frac{n}{g}\right) \qquad ((n,q) = 1), \tag{1.22}$$

and, for $k/2 \in \mathbb{N}$ and $j = 1, \ldots, \theta_k(q)$,

$$\psi_{jk}^*(\infty, m)\lambda_{jk}^*(n) = \sum_{g|(m,n)} \psi_{jk}^*\left(\infty, \frac{m}{g}\frac{n}{g}\right) \qquad ((n,q) = 1)$$

(see (1.18) and (1.19) for our notation here). It follows by Möbius inversion that, for $j, m \in \mathbb{N}$, and $n \in \mathbb{N}$ with $(n,q) = 1$,

$$\rho_{j\infty}(mn) = \sum_{g|(m,n)} \mu(g)\tau_j\left(\frac{n}{g}\right)\rho_{j\infty}\left(\frac{m}{g}\right), \tag{1.23}$$

and, for $k/2 \in \mathbb{N}$, $j = 1, \ldots, \theta_k(q)$, $m \in \mathbb{N}$, and $n \in \mathbb{N}$ with $(n,q) = 1$,

$$\psi_{jk}^*(\infty, mn) = \sum_{g|(m,n)} \mu(g)\lambda_{jk}^*\left(\frac{n}{g}\right)\psi_{jk}^*\left(\infty, \frac{m}{g}\right). \tag{1.24}$$

As for the Eisenstein series $E_{\mathfrak{c}}(z,s)$, it is shown in [7], pages 227 and 246, that their Fourier coefficients, in (1.6), are given by:

$$\varphi_{\mathfrak{c}\mathfrak{a}}(m,s) = \sum_{\gamma}^{\Gamma} \gamma^{-2s} S_{\mathfrak{c}\mathfrak{a}}(0, m; \gamma) \quad (\mathrm{Re}(s) > 1), \tag{1.25}$$

where $S_{\mathfrak{c}\mathfrak{a}}(0,m;\gamma)$ is a special instance of the generalised Kloosterman sums that feature in [7] (see (2.3) and (2.4) for a definition). It is well known, and unsurprising, given the analogy with the Ramanujan sum $S(0,m;c) = \sum_{a \bmod c}^{*} e(am/c)$, that quite simple formulae for $S_{\mathfrak{c}\mathfrak{a}}(0,m;\gamma)$ exist. These can lead, by (1.25), to the expression of $\varphi_{\mathfrak{c}\mathfrak{a}}(m,s)$ in terms of reciprocals of Dirichlet L-functions (for characters mod q), and so to the meromorphic continuation of $E_{\mathfrak{c}}(z,s)$ discussed earlier (see [11] and [19] for examples). We do not take this path. Instead we shall, in Section 3, work to exploit the formula for $S_{\mathfrak{c}\infty}(0,m;\gamma)$ so as to obtain (in Lemma 3.4) a useful 'near analog' of (1.23) and (1.24) for the coefficients $\varphi_{\mathfrak{c}\infty}(m,s)$.

In [7] Deshouillers and Iwaniec obtained 'large sieve inequalities' for the following three expressions, in which $\mathbf{b} = (b_n)$ denotes an arbitrary complex sequence,

$$\mathcal{S}^{(0)}_{\mathfrak{a},q,K}(\mathbf{b},N) = \sum_{\substack{2\leqslant k\leqslant K\\ k \text{ even}}} \frac{(k-1)!}{(4\pi)^{k-1}} \sum_{j=1}^{\theta_k(q)} \left| \sum_{N/2<n\leqslant N} b_n \psi^*_{jk}(\mathfrak{a},n)\right|^2, \tag{1.26}$$

$$\mathcal{S}^{(1)}_{\mathfrak{a},q,K}(\mathbf{b},N) = \sum_{|\kappa_j|\leqslant K}^{(q)} \frac{1}{\cosh(\pi\kappa_j)} \left| \sum_{N/2<n\leqslant N} b_n \rho_{j\mathfrak{a}}(n)\right|^2, \tag{1.27}$$

$$\mathcal{S}^{(2)}_{\mathfrak{a},q,K}(\mathbf{b},N) = \sum_{\mathfrak{c}}^{\Gamma} \int_{-K}^{K} \left| \sum_{N/2<n\leqslant N} b_n n^{ir} \varphi_{\mathfrak{c}\mathfrak{a}}\left(n,\tfrac{1}{2}+ir\right)\right|^2 dr \tag{1.28}$$

(note that, by (1.14), (1.15) and (1.12), the hyperbolic cosine in (1.27) is real and bounded below by the positive quantity $\cos(\vartheta\pi)$). The following theorem is a restatement of [7], Theorem 2,

Theorem 1.3. [Deshouillers-Iwaniec] *Let $\varepsilon, N > 0$, $q \in \mathbb{N}$ and $K \geqslant 1$. Let $\mathbf{b} = (b_n)$ be a complex sequence. Then, when $\mathfrak{a}$ is a cusp of $\Gamma = \Gamma_0(q)$, each of the three sums $\mathcal{S}^{(i)}_{\mathfrak{a},q,K}(\mathbf{b},N)$ $(i = 0,1,2)$ is majorised by a term*

$$O_\varepsilon\left(\left(K^2 + \mu(\mathfrak{a})N^{1+\varepsilon}\right)\|\mathbf{b}_N\|_2^2\right),$$

where

$$\|\mathbf{b}_N\|_p = \left(\sum_{N/2<n\leqslant N} |b_n|^p\right)^{1/p} \qquad (p>0) \tag{1.29}$$

and where, for $\mathfrak{a} \sim u/w$ with $w|q$ and $(u,w) = 1$,

$$\mu(\mathfrak{a}) = \left(w, \frac{q}{w}\right) q^{-1}. \tag{1.30}$$

Note that [7], Lemma 2.3 (Lemma 2.1 of this paper), shows the $\mu(\mathfrak{a})$ of (1.30) to be a well-defined function from $\mathbb{Q}\cup\{\infty\}$ into the set $\{1/|v| : v|q\}$. Note also that Deshouillers and Iwaniec have subsequently been able to replace the factor $N^{1+\varepsilon}$ in Theorem 1.3 by just $N\log(2N)$ (this is reported in [15], Section 8.4). Nevertheless we will find Theorem 1.3 sufficient for our use, since we cannot prevent other unwanted factors, similar in size to N^{ε}, from entering into our calculations later on.

In this paper we replace the sequence $\mathbf{b} = (b_n)$, of Theorem 1.3, by a sequence $\mathbf{b}^{(D)} = (b_n^{(D)})$, where $D \in \mathbb{N}$ and

$$b_n^{(D)} = \begin{cases} b_{n/D}, & \text{if } n \equiv 0 \pmod{D}, \\ 0, & \text{otherwise.} \end{cases} \tag{1.31}$$

By Theorem 1.3, one has, for $i = 0, 1, 2$,

$$\mathcal{S}^{(i)}_{\mathfrak{a},q,K}(\mathbf{b}^{(D)}, DN) \ll_{\varepsilon} \left(K^2 + \mu(\mathfrak{a})(DN)^{1+\varepsilon}\right) \|\mathbf{b}_N\|_2^2. \tag{1.32}$$

Here it should be observed that,

$$\mathcal{S}^{(0)}_{\mathfrak{a},q,K}(\mathbf{b}^{(D)}, DN) = \sum_{\substack{2\leqslant k\leqslant K \\ k \text{ even}}} \frac{(k-1)!}{(4\pi)^{k-1}} \sum_{j=1}^{\theta_k(q)} \left| \sum_{N/2<n\leqslant N} b_n \psi^*_{jk}(\mathfrak{a}, Dn) \right|^2, \tag{1.33}$$

$$\mathcal{S}^{(1)}_{\mathfrak{a},q,K}(\mathbf{b}^{(D)}, DN) = \sum_{|\kappa_j|\leqslant K}^{(q)} \frac{1}{\cosh(\pi\kappa_j)} \left| \sum_{N/2<n\leqslant N} b_n \rho_{j\mathfrak{a}}(Dn) \right|^2, \tag{1.34}$$

$$\mathcal{S}^{(2)}_{\mathfrak{a},q,K}(\mathbf{b}^{(D)}, DN) = \sum_{\mathfrak{c}}^{\Gamma} \int_{-K}^{K} \left| \sum_{N/2<n\leqslant N} b_n n^{ir} \varphi_{\mathfrak{c}\mathfrak{a}}\left(Dn, \tfrac{1}{2} + ir\right) \right|^2 dr. \tag{1.35}$$

Note that the sums over n in (1.33)-(1.35) each have only $O(N)$ terms. Moreover, in cases where $(D, q) = 1$, so that Lemma 3.4 applies and (1.23) and (1.24) apply with $n = D$, the Ramanujan-Petersson conjecture suggests that the moduli of these $O(N)$ terms tend not to exceed the corresponding moduli in the case $D = 1$ by more than a factor $O_\varepsilon(D^\varepsilon)$, where $\varepsilon > 0$ is arbitrary. It is therefore disappointing to have the factor $(DN)^{1+\varepsilon}$ in the upper bound (1.32). A natural conjecture is that the offending factor could be replaced with just $D^\varepsilon N^{1+\varepsilon}$. Underpinning some of the work in [10] is the following theorem, which may be regarded as an approach to the case $\mathfrak{a} = \infty$ of the conjecture just mentioned.

Theorem 1.4. [Harman-Watt-Wong] *Let $\varepsilon, N > 0$, $K \geqslant 1$, $q, D \in \mathbb{N}$ with $(q, D) = 1$, and take $\mathbf{b} = (b_n)$ to be a sequence of complex numbers. Then*

$$\mathcal{S}^{(1)}_{\infty,q,K}(\mathbf{b}^{(D)}, DN) \ll_{\varepsilon} D^{2\vartheta}\tau^4(D)\left(K^2 + q^{-1}N^{1+\varepsilon}\right) \|\mathbf{b}_N\|_2^2 \tag{1.36}$$

and, for $i \in \{0, 2\}$,

$$\mathcal{S}^{(i)}_{\infty,q,K}(\mathbf{b}^{(D)}, DN) \ll_{\varepsilon} \tau^4(D)\left(K^2 + q^{-1}N^{1+\varepsilon}\right) \|\mathbf{b}_N\|_2^2. \tag{1.37}$$

Section 3 gives details of the proof of this result, which was only briefly sketched in [10].

For comparison of Theorem 1.4 with (1.32), note that as $\infty \sim 1/q$ we obtain $\mu(\infty) = (q,1)/q = 1/q$ from (1.30).

Theorem 1.4 is clearly less generally applicable than Theorem 1.3 of Deshouillers and Iwaniec, for it only gives information relating to Fourier expansions about the cusp $\mathfrak{a} = \infty$. Nevertheless, if $\mathfrak{c}$ is a cusp such that the generalised Kloosterman sums $S_{\mathfrak{c}\mathfrak{c}}(m,n;\gamma)$ and $S_{\infty\infty}(m,n;\gamma)$ are identically equal (and therefore defined for the same set of positive γ), then, as was observed by Iwaniec in [13], the 'Kloosterman sum' side of the identity given by the Bruggeman-Kuznetsov summation formula (Theorem 2.4 of Section 2) is the same for $\mathfrak{a} = \mathfrak{b} = \mathfrak{c}$ as it is for $\mathfrak{a} = \mathfrak{b} = \infty$, so that (through the implied invariance of the sum $\mathfrak{Z}_1 + \mathfrak{Z}_2$ in Theorem 2.4) it follows by Theorem 1.4 itself that substition of '$\mathfrak{c}$' in place of the subscript '∞' in (1.36) would not invalidate that result. This situation occurs if (for example) $\mathfrak{c} = 1/s$ and $q = rs$ with $(r,s) = 1$ (see [24], page 195, and, for an application, [24], page 204).

In view of the phenomenon just discussed, and since we consider results concerning the Fourier coefficients of holomorphic forms and Eisenstein series to be of secondary interest (here), we have felt it reasonable in this paper to limit ourselves to working with the Fourier coefficients from expansions about the cusp ∞. This also means that we avoid some distractions from the main ideas.

We are interested in bounding the average value of $\mathbb{S}^{(1)}_{\infty,q,K}(\mathbf{b}^{(D)}, DN)$ as the 'level', q, runs over integer values in an interval $(Q/2, Q]$. Deshouillers and Iwaniec introduced the idea of averaging over the level in [7], Section 8.2. Progress beyond what Theorem 1.4 implies, would follow (by (1.23) for $n = D$) if one had better bounds for sums

$$S^*_{Q,K}(\mathbf{b}, N; D) = \sum_{\substack{Q/2<q\leqslant Q \\ (q,D)=1}} \sigma^*_{q,K}(\mathbf{b}, N; D), \tag{1.38}$$

where

$$\sigma^*_{q,K}(\mathbf{b}, N; D) = \sum_{|\kappa_j|\leqslant K}^{(q)^*} \frac{1}{\cosh(\pi\kappa_j)} (\tau_j(D))^2 \left| \sum_{N/2<n\leqslant N} b_n \rho_{j\infty}(n) \right|^2 \tag{1.39}$$

with the asterisk indicating that summation is further restricted to $j \in \mathbb{N}$ such that

$$1 < (\tau_j(D)/\tau(D))^2 \leqslant D^{2\vartheta}. \tag{1.40}$$

Such progress is our goal in this paper.

Note that, since $\tau_j(D)$ is always real, the bound (1.20) implies that the rightmost inequality of (1.40) holds whenever $(q,D) = 1$. By analogy with (1.15), any $\tau_j(D)$ satisfying (1.40) may be regarded as a 'T_D-exceptional' eigenvalue, in that it would provide a counterexample to the Ramanujan-Petersson conjecture in respect of the Hecke operator T_D. Indeed, the entire sum $S^*_{Q,K}(\mathbf{b}, N; D)$, given by (1.39) and (1.38), may be seen as analogous to the sum

$$S(Q,Y,N;0) = \sum_{Q<q\leqslant 16Q} \sum_{\substack{j\geqslant 1\\ \lambda_j<1/4}}^{(q)} \left(Y^{i\kappa_j}\right)^2 \left| \sum_{N<n\leqslant 2N} a_n \rho_{j\infty}(n) \right|^2 ,$$

considered by Deshouillers and Iwaniec in [7], Section 8. Note that in view of (1.15) the factor $1/\cosh(\pi\kappa_j)$, present in (1.39), would make little difference if inserted into the sum $S(Q,Y,N;0)$. In respect of this sum, Deshouillers and Iwaniec observed (see [7], Theorem 8) that (1.15) implies

$$1 < \left(Y^{i\kappa_j}\right)^2 \leqslant Y^{2\vartheta} \qquad (j \in \mathbb{N},\ \lambda_j < 1/4), \tag{1.41}$$

given that $Y > 1$ (the non-trivial case in [7], Section 8.2). As (1.40) and (1.41) have such an apparent similarity in respect of their implications regarding the sums that they relate to, the analogy between $S^*_{Q,K}(\mathbf{b}, N; D)$ in (1.38)-(1.39) and the sums $S(Q,Y,N;0)$ of [7] appears strong.

Our main idea in this paper has been to extend the above analogy so that it encompasses certain proofs and results in [7] (especially the work of [7], Section 8). As this suggests, the real concern in our work is with the possibility of T_D-exceptional eigenvalues (the Δ-exceptional eigenvalues will play only an incidental part in what we do). At the same time we have tried to take full advantage of the leverage granted by Kim and Sarnak's results (of which (1.10) is the most relevant): the best analogous result available when [7] was written being Selberg's bound $\lambda_j \geqslant 3/16$ (corresponding to the inequality $0 < i\kappa_j \leqslant 1/4$ in place of (1.15)).

Although the sums defined in (1.39) and (1.38) are helpful in explaining our main ideas, we have found them rather awkward to use in some of our arguments (though doubtless those difficulties could be overcome). We have preferred to do all our work with the slightly different sums:

$$\begin{aligned}\sigma_{q,K}(\mathbf{b}, N; D, y) = {} & \frac{1}{\pi} \sum_{\mathfrak{c}}^{\Gamma_0(q)} \int_{-K}^{K} \left| \sum_{N/2<n\leqslant N} b_n n^{i(r+y)} \varphi_{\mathfrak{c}\infty}\left(Dn, \tfrac{1}{2}+ir\right) \right|^2 \mathrm{d}r + \\ & + \sum_{|\kappa_j|\leqslant K}^{(q)} \frac{1}{\cosh(\pi\kappa_j)} \left| \sum_{N/2<n\leqslant N} b_n n^{iy} \rho_{j\infty}(Dn) \right|^2 ,\end{aligned} \tag{1.42}$$

and

$$S_{Q,K}(\mathbf{b}, N; D, y) = \sum_{Q/2<q\leqslant Q} \sigma_{q,K}(\mathbf{b}, N; D, y), \tag{1.43}$$

where the real parameter y is a essentially a technical convenience (one could restrict y throughout to being $O_\varepsilon((QDN)^\varepsilon)$ for arbitrary $\varepsilon > 0$). By (1.22) and

(1.23), there exists a close connection between the sums in (1.39) and (1.38) and the sums in (1.42), (1.43). The latter sums are easier to work with, and are also more directly related to the sum bounded in (1.36) of Theorem 1.4 (the result all our work aims to improve on). Specifically, by (1.42), (1.31) and (1.27) and (1.28), we have:

$$\begin{aligned}\sigma_{q,K}(\mathbf{b},N;D,y) &= \sigma_{q,K}\left(\mathbf{b}(y),N;D,0\right) = \\ &= \sigma_{q,K}\left(\mathbf{b}^{\langle D\rangle}(y),ND;1,0\right) = \\ &= \mathbb{S}^{(1)}_{\infty,q,K}(\mathbf{b}^{\langle D\rangle}(y),DN) + \frac{1}{\pi}\,\mathbb{S}^{(2)}_{\infty,q,K}(\mathbf{b}^{\langle D\rangle}(y),DN),\end{aligned} \tag{1.44}$$

where, given $y \in \mathbb{R}$ and $\mathbf{v} = (v_n)$, the sequence $\mathbf{v}(y) = (v_n(y))$ is given by:

$$v_n(y) = v_n n^{iy} \qquad (n \in \mathbb{N}). \tag{1.45}$$

Note the ambiguity as to whether it is $v_n^{\langle k\rangle} n^{iy}$ or $v_n^{\langle k\rangle}(n/k)^{iy}$ that should be the value of $v_n^{\langle k\rangle}(y)$: so long as one chooses consistently (for all n) the difference is immaterial here, since the two alternatives differ by a factor k^{iy}, and since all we ever require in our work is that sums of the form

$$\sum_m \sum_n \omega_{mn} \overline{v_m^{\langle k\rangle}(y)} v_n^{\langle k\rangle}(y)$$

be well-defined. Therefore we may sometimes use $\mathbf{v}^{\langle k\rangle}(y) = (v_n^{\langle k\rangle} n^{iy})$, while at other times using $\mathbf{v}^{\langle k\rangle}(y) = \mathbf{u}^{\langle k\rangle}$, where $\mathbf{u} = \mathbf{v}(y)$. A similar excuse can be made for the ambiguity in the compound notation $\mathbf{v}^{\{k\}}(y)$, where, for $g_0 \in \mathbb{N}$ and $\mathbf{b} = (b_n)$, the sequence $\mathbf{b}^{\{g_0\}}$ is given by (1.52) below.

Through the Bruggeman-Kuznetsov summation formula [7], Lemma 4.7 (Theorem 2.4 in Section 2), the sums $\sigma_{q,K}(\mathbf{b},N;D,y)$ in (1.42) are bounded in terms of sums of Kloosterman sums:

$$\begin{aligned}\alpha_{q,H}(\mathbf{b},N;D,y) &= \\ &= \sum\sum_{N/2<m,n\leqslant N} \bar{b}_m b_n \left(\frac{m}{n}\right)^{-iy} \sum_{\ell=1}^{\infty} \frac{1}{q\ell}\Phi_H\left(\frac{4\pi D\sqrt{mn}}{q\ell}\right) S(Dm,Dn;q\ell)\end{aligned} \tag{1.46}$$

and

$$A_{Q,H}(\mathbf{b},N;D,y) = \sum_q \omega(q/Q)\alpha_{q,H}(\mathbf{b},N;D,y), \tag{1.47}$$

where, for $H > 0$,

$$\Phi_H(x) = H^3 x \int_0^{\infty} \sin(x\cosh(\xi))\xi\tanh(\xi)e^{-(H\xi)^2}\,\mathrm{d}\xi, \tag{1.48}$$

while the 'classical' Kloosterman sum $S(a,b;c)$ is as in (2.8), and $\omega(x)$ is some infinitely differentiable real function satisfying:

$$\omega(x) \geqslant \begin{cases} 1 & \text{if } 1/2 \leqslant x \leqslant 1, \\ 0, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \omega(x) = 0 \quad (x \notin (1/4,2)). \tag{1.49}$$

The new parameter H here is strongly linked to the 'spectral' parameter K in (1.42)-(1.43): in effect, we work throughout with H satisfying $1 \leqslant H \ll D^{\varepsilon} K$, for some arbitrarily small positive absolute constant ε (see Lemma 4.9 and its applications in the proofs of Lemmas 5.3 and 8.1).

In Section 4 we treat these sums of Kloosterman sums almost by a formal manipulation (analysis playing a supporting part). Through Lemmas 4.6-4.8 we escape the constraints resulting from the condition $(n,q) = 1$ in (1.20), (1.23), (1.24) (and from the condition $(D,q) = 1$ in Lemma 3.4). Our first significant result (the proposition below) is then a quite straightforward consequence of (1.20), (1.23) and Lemma 3.4.

Proposition 1.1. *Let $\varepsilon > 0$ and $\vartheta = 7/64$. Then for $K \geqslant 1$, $Q, N > 0$, $y \in \mathbb{R}$, any complex sequence $\mathbf{b} = (b_n)$, $D \in \mathbb{N}$ and any $D_1 \in \mathbb{N}$ with $D_1 | D$, one has*

$$\frac{1}{K^2} S_{Q,K}(\mathbf{b}, N; D, y) \ll_{\varepsilon} \frac{1}{G^2} S_{Q_1,G}\left(\mathbf{b}^{\{g_0\}}, \frac{N}{g_0}; \frac{D_1}{g_1}, y\right) \left(\frac{D}{D_1}\right)^{2\vartheta+\varepsilon},$$

for some $G \geqslant 1$ and some Q_1, g_0, g_1 and sequence $\mathbf{b}^{\{g_0\}}$ satisfying:

$$\frac{Q}{2D/D_1} < Q_1 \leqslant Q, \tag{1.50}$$

$$g_0, g_1 \in \mathbb{N}, \quad (g_0, D_1) = 1, \quad g_0 \Big| \frac{D}{D_1}, \quad g_1 \Big| \left(\frac{D}{D_1}, D_1\right) \tag{1.51}$$

and

$$b_n^{\{g_0\}} = b_{g_0 n} \qquad (n \in \mathbb{N}). \tag{1.52}$$

Proposition 1.1 is proved in Section 5. It should be regarded as the analog for the simpler of the two processes by which Deshouillers and Iwaniec transform their sum $S(Q,Y,N;0)$ (or related sums $S(Q,Y,N;it)$) in [7], Section 8: corresponding, in fact, to their application of Selberg's result that $0 < i\kappa_j \leqslant 1/4$ for Δ-exceptional eigenvalues. The next proposition is our analog of Deshouillers and Iwaniec's other transforming process [7], Lemma 8.1, which (like our proposition) allows the swapping of one set of 'levels' (q) for another such set (hence the 'swapping of levels' referred to in a couple of our section headings).

Proposition 1.2. *Take $C \geqslant 64\pi$ to be a sufficiently large absolute constant. Let $\varepsilon > 0$ and $j \in \mathbb{N}$ with $j \geqslant 2$. Then, for $K \geqslant 1$, $Q, N > 0$, $y \in \mathbb{R}$, any complex* *sequence* $\mathbf{b} = (b_n)$, *and* $D \in \mathbb{N}$, *one has*

$$\frac{1}{K^2} S_{Q,K}(\mathbf{b}, N; D, y) \ll_{\varepsilon,j}$$

$$\ll_{\varepsilon,j} (DN)^{\varepsilon} \left(\left(Q + \frac{DN}{Q} + N \right) \|\mathbf{b}_N\|_2^2 + \int_{-\infty}^{\infty} \frac{1}{G^2} S_{L,G}(\mathbf{b}, N; D, y+t) \frac{\mathrm{d}t}{(1+|t|)^j} \right),$$

for some $G \geqslant 1$ *and some* L *satisfying*

$$0 < L \leqslant CQ^{-1}DN. \tag{1.53}$$

For our proof of Proposition 1.2 we utilize (in Section 6) a 'smooth partitioning' of the sums $\alpha_{q,H}(\mathbf{b}, N; D, y)$ in (1.46) and (1.47), so that we are reduced to considering related sums that are dependent upon a new parameter $X \in \{2^j : j \in \mathbb{Z}\}$:

$$\alpha_{q,H,X}(\mathbf{b}, N; D, y) = \tag{1.54}$$

$$= \sum_{N/2 < m,n \leqslant N} \sum \bar{b}_m b_n \left(\frac{m}{n}\right)^{-iy} \sum_{\ell=1}^{\infty} \frac{1}{q\ell} \Phi_{H,X}\left(\frac{4\pi D\sqrt{mn}}{q\ell}\right) S(Dm, Dn; q\ell)$$

and

$$A_{Q,H,X}(\mathbf{b}, N; D, y) = \sum_{q} \omega(q/Q) \alpha_{q,H,X}(\mathbf{b}, N; D, y), \tag{1.55}$$

where

$$\Phi_{H,X}(x) = \Omega_0(x/X)\Phi_H(x). \tag{1.56}$$

with $\Omega_0 : \mathbb{R} \longrightarrow [0,1]$ being an infinitely differentiable function such that

$$\Omega_0(x) = 0 \qquad (x \notin (1/2, 2)). \tag{1.57}$$

The cases where $X \leqslant \Delta$ (a sufficiently small positive absolute constant) are dealt with in Section 6, essentially by using the Kuznetsov summation formula (Theorem 2.2 in Section 2) to effect a localised reversal of the summation of Bruggeman and Kuznetsov that gave rise to the sums $\alpha_{q,H}(\mathbf{b}, N; D, y)$ in the first place.

In cases where $X > \Delta$ we borrow from Deshouillers and Iwaniec (see [7], page 272) their idea of swapping the rôles of the variables q and ℓ in (1.54) and (1.55), so that the Kuznetsov summation formula is applied for $\Gamma = \Gamma_0(\ell)$, rather than for $\Gamma = \Gamma_0(q)$. Technical preparation (concerned with the Bessel transforms defined in (2.15) and (2.16) of Theorem 2.2) is carried out in Section 7. We complete the swapping process (and so prove Proposition 1.2) in Section 8.

Using Selberg's bound for exceptional eigenvalues, together with their own transforming process [7], Lemma 8.1, Deshouillers and Iwaniec proved the following result, which is [7], Theorem 6.

Theorem 1.5. [Deshouillers-Iwaniec] *Let* $\varepsilon > 0$ *and* $Y, Q, N \geqslant 1$. *Then, for any complex sequence* $\mathbf{b} = (b_n)$,

$$\sum_{q \leqslant Q} \sum_{\substack{j \geqslant 1 \\ \lambda_j < 1/4}}^{(q)} \left(Y^{i\kappa_j}\right)^4 \left| \sum_{N/2 < n \leqslant N} b_n \rho_{j\infty}(n) \right|^2 \ll_\varepsilon (QN)^\varepsilon (Q + NY) \|\mathbf{b}_N\|_2^2 .$$

With the help of Propositions 1.1 and 1.2, and the 'initial result' (9.1) (a direct corollary of Lemma 5.1) we are able to prove, in Section 9, the following analog of Theorem 1.5.

Theorem 1.6. *Let* $\varepsilon > 0$, $\vartheta = 7/64$, $\varrho = 2\vartheta$ *and* $\zeta = 1 - 4\vartheta$. *Then there exist sufficiently large constants,* $M_0(\varepsilon) \geqslant 1$, $C_0(\varepsilon) \geqslant 1$, *depending only upon* ε, *such that the following is true. If*

$$M \geqslant M_0(\varepsilon), \tag{1.58}$$

then, for $Q > 0$, $K \geqslant 1$, $D \in \mathbb{N}$,

$$N \in (0, M], \tag{1.59}$$

$$P \geqslant \max_{\substack{d \mid D \\ \tau(d) \leqslant 2}} d, \tag{1.60}$$

$y \in \mathbb{R}$, *and any complex sequence* $\mathbf{b} = (b_n)$, *one has*

$$\frac{1}{K^2} S_{Q,K}(\mathbf{b}, N; D, y) \leqslant \tag{1.61}$$
$$\leqslant C_0(\varepsilon)(QDN)^\varepsilon \left(Q + D^\varrho M + (PDN)^\varrho \left(\min\left(Q, \sqrt{DN}\right)\right)^\zeta\right) \|\mathbf{b}_N\|_2^2 .$$

As we now seek to explain, the analogy with Theorem 1.5 may be seen in the result itself (not only in its proof). In view of the close connection already noted between $S_{Q,K}(\mathbf{b}, N; D, 0)$ and the sum $S^*_{Q,K}(\mathbf{b}, N; D)$ of (1.38)-(1.39), and of the analogy between (1.40) and (1.41), we hope the reader is persuaded that an appropriate analogy of Theorem 1.5 in our context might be the bound:

$$S_{Q,K}(\mathbf{b}, N; D, 0) \ll_\varepsilon (QN)^\varepsilon \left(Q + N\sqrt{D}\right) \|\mathbf{b}_N\|_2^2 K^2, \tag{1.62}$$

where $D = Y^2$ (we assume that $Y^2 \in \mathbb{N}$, and are also not really concerned with the dependence on K). In cases where $N \geqslant M_0(\varepsilon)$ and no prime factor of D exceeds D^ε, the application of Theorem 1.6 with $M = N$, $P = D^\varepsilon$ yields a bound:

$$S_{Q,K}(\mathbf{b}, N; D, 0) \ll_\varepsilon D^{2\varepsilon}(QN)^\varepsilon \left(Q + ND^\varrho + \sqrt{ND}\right) \|\mathbf{b}_N\|_2^2 K^2.$$

Discounting the 'slowly growing' factor $D^{2\varepsilon}$, Theorem 1.6 appears, in this instance, to be significantly stronger than (1.62) (the suggested analog of Theorem 1.5). In a sense, however, it is simply the 'updated' version of that analog, since the term ND^{ϱ} in our last bound would have been $ND^{1/2}$ had we been working with the bound $|\tau| \leqslant \tau(n)n^{1/4}$ (analogous to Selberg's bound $\lambda \geqslant 3/16$) in place of Kim and Sarnak's bound (1.10).

In [7], Theorem 7, it was shown that [7], Theorem 6, could be improved for some sequences $\mathbf{b} = \mathbf{\Psi}$ of the form

$$\Psi_n = \begin{cases} 1, & \text{if } N_1 < n \leqslant N, \\ 0, & \text{otherwise,} \end{cases} \qquad (n \in \mathbb{N}), \tag{1.63}$$

where $N_1 \geqslant 0$. The following result is equivalent to [7], Theorem 7.

Theorem 1.7. [Deshouillers-Iwaniec] *Let $\varepsilon > 0$ and $Y, Q, N \geqslant 1$. Then, for $N_1 \in (N/2, N]$,*

$$\sum_{q \leqslant Q} \sum_{\substack{j \geqslant 1 \\ \lambda_j < 1/4}}^{(q)} \left(Y^{i\kappa_j}\right)^4 \left| \sum_{N_1 < n \leqslant N} \rho_{j\infty}(n) \right|^2 \ll_\varepsilon (QN)^\varepsilon \left(Q + N + \sqrt{N}Y\right) N.$$

The proof of this (in [7], Section 8.3) required an 'initial result' derived from the bound:

$$\sum_{t \leqslant T} \left| \sum_{m \leqslant M} \sum_{n \leqslant N} S(m, n; t) \right| \ll_\varepsilon (TMN)^\varepsilon T(T + MN)$$

(for $\varepsilon, T, M, N > 0$), which is [7], Theorem 14. This bound does not provide the kind of 'initial result' that might help us to improve on Theorem 1.6, so in Section 10 we work to establish a suitable substitute (Lemma 10.12). In Section 11 we obtain the desired 'initial result', (11.7), essentially as a corollary of Lemma 10.12. We are then able to employ Propositions 1.1 and 1.2 so as to deduce the following analog of [7], Theorem 7.

Theorem 1.8. *Let ε, ϑ, ϱ and ζ be as in Theorem 1.6. Then there exist sufficiently large positive constants, $M_1(\varepsilon)$, $C_1(\varepsilon)$, depending only upon ε, such that the following is true. If*

$$M \geqslant M_1(\varepsilon), \tag{1.64}$$

then, for $y \in \mathbb{R}$, $K \geqslant 1$, $Q > 0$, $D \in \mathbb{N}$, N satisfying (1.59), P satisfying (1.60), and any sequence $\mathbf{\Psi} = (\Psi_n)$ given by (1.63) with

$$N_1 \in [N/2, N), \tag{1.65}$$

one has

$$\frac{1}{(1+y^2)\,K^2}\, S_{Q,K}(\Psi, N; D, y) \leqslant \tag{1.66}$$
$$\leqslant C_1(\varepsilon)(QDN)^{\varepsilon}\Big(Q + M + (PDM)^{\varrho}\left(\min\big(Q, \sqrt{DN}\,\big) + M\right)^{\varsigma}\Big)N.$$

To confirm that this result does represent an analog of Theorem 1.7, we begin by recalling our remarks before and after (1.62). These suggest that an appropriate analog of Theorem 1.7 in our context might be the bound:

$$S_{Q,K}(\Psi, N; D, 0) \ll_{\varepsilon} (QN)^{\varepsilon}\left(Q + N + \sqrt{ND}\right)NK^2, \tag{1.67}$$

where $D = Y^2$ (as was the case in (1.62)) and $\Psi = (\Psi_n)$ is any sequence given by (1.63) with $N_1 \in (N/2, N]$. Supposing that $N \geqslant M_1(\varepsilon)$, and that no prime factor of D exceeds D^{ε}, our Theorem 1.8 would imply (in place of (1.67)) the bound:

$$S_{Q,K}(\Psi, N; D, 0) \ll_{\varepsilon} D^{2\varepsilon}(QN)^{\varepsilon}\left(Q + N + (DN)^{1/2} + D^{\varrho}N^{1-\varrho}\right)NK^2.$$

Since $2\varrho = 4\vartheta = 7/16 \in (0,1)$, we have here

$$N + (DN)^{1/2} \geqslant N^{1-2\varrho}\left(\sqrt{DN}\right)^{2\varrho} = D^{\varrho}N^{1-\varrho},$$

so that, in this instance, Theorem 1.8 is indeed essentially equivalent to (1.67), the proposed analog of Theorem 1.7 (we discount the 'slowly growing' factor $D^{2\varepsilon}$). It seems that Kim and Sarnak's bound (1.10) does not lead to a bound superior to (1.67), but it can shape the result given by Theorem 1.8 in cases where $Q^2 = o(DN)$, or where D has some 'large' prime factors (so that P in (1.60) must be greater than D^c for some positive absolute constant c).

Note the homogeneity with respect to K of our bounds for $S_{Q,K}(\mathbf{b}, N; D, y)$ and $S_{Q,K}(\Psi, N; D, y)$ in Theorems 1.6 and 1.8. Comparison of those bounds with the bounds of Theorem 1.4, which are not homogeneous in K, leads us to expect that it should be possible to replace (1.61) and (1.66) with similar (but not so homogeneous) bounds that would be sharper for large K. As this homogeneity in K arises from our use of Propositions 1.1 and 1.2, which oversimplify conclusions reached in their direct antecedents, Lemmas 5.3 and 8.5, it might therefore be possible to achieve the suggested improvement by instead working directly with the latter pair of lemmas.

As mentioned at the beginning of this Introduction, we have undertaken this work motivated by issues raised in our paper [10] with Harman and Wong. That paper was concerned with upper bounds for the mean-value

$$\frac{1}{\phi(D)}\sum_{\chi \bmod D}\int_{-T}^{T}\left|L\left(\tfrac{1}{2}+it,\chi\right)\right|^4\left|\sum_{n\leqslant N} a_n\chi(n)n^{-it}\right|^2 dt, \tag{1.68}$$

where $L(s,\chi)$ is Dirichlet's L-function for the Dirichlet character χ, while $\mathbf{a} = (a_n)$ is an arbitrary complex sequence. A possible application of our work arises in connection with the proof of the proposition in [10], Section 3, where it is necessary to bound an average of certain sums,

$$b_{r,s} = \sum_{\substack{j\geqslant 1 \\ \lambda_j<1/4}}^{(rs)} \left(Y^{i\kappa_j}\right)^2 \left| \sum_{h,k} \alpha(h)\beta(k)\overline{\rho_{j\infty}(Dhk)} \sum_{\ell} \omega(\ell)\rho_{j\,1/s}(\ell) \right|$$

(with or without the complex-conjugation shown). There would be an extra layer of complexity in such an application: one would have to be concerned with T_D-exceptional eigenvalues, $\tau_j(D)$, for which the corresponding λ_j was Δ-exceptional.

Our results have a simpler, more direct, application to the problem of obtaining a good upper bound for a 'pure character' variant of the mean-value in (1.68):

$$I_D(f,M;\mathbf{a},N) = \frac{1}{\phi(D)} \sum_{\substack{\chi \bmod D \\ \chi\neq\chi_0}} \left| \sum_{m\leqslant M} f(m/M)\chi(m) \right|^4 \left| \sum_{n\leqslant N} a_n\chi(n) \right|^2,$$

where $f : \mathbb{R} \longrightarrow \mathbb{C}$ is assumed to be infinitely differentiable and supported in $[1/2, 1]$. The cases of particular interest are those where $M = O(D^{1/2})$, but $D = o(M^2N)$. Using only Theorem 1.4 one can, in such cases, obtain essentially the best possible upper bound for $I_D(f,M;\mathbf{a},N)$ if $N = O(D^{(1-2\vartheta)/4})$. This gives another approach to one special case of a result first proved in Section 5.2 of [3] (that special case being an upper bound for a mean-value similar to (1.68), but without the averaging over t). In cases where the greatest prime factor of D is a sufficiently small power of D Theorems 1.6 and 1.8 enable one to obtain an essentially best-possible upper bound on $I_D(f,M;\mathbf{a},N)$ for even larger N: detailed results are to appear in [26]. A further improvement would follow if one could establish a suitable analog of [24], Theorem 2.

Theorems 1.6 and 1.8 are unlikely to be useful in applications where D might have a relatively large prime factor. Nevertheless we believe that, through their application to the mean-value $I_D(f,M;\mathbf{a},N)$ (above), these theorems will lead to an improved lower bound for the number of Carmichael numbers less than a given positive number x (see [1] and [9]).

Acknowledgement. The author is indebted to Prof. Glyn Harman, both for his encouragement to work on the questions addressed in this paper, and for pointing out the relevance of bounds for $I_D(f,M;\mathbf{a},N)$ to the Carmichael number problem.

Notations (I): special definitions and conventions. Throughout this paper ϑ, ϱ and ζ are as in (1.12) of Theorem 1.2 (and in Theorems 1.6 and 1.8), so that $\vartheta = 7/64$, $\varrho = 2\vartheta = 7/32$ and $\zeta = 1 - 4\vartheta = 9/16$.

Outside of the Introduction (where it may refer to the hyperbolic Laplacian operator) Δ denotes a 'sufficiently small' positive absolute constant. More precisely, we assume that $\Delta \in (0, 1/2]$, and is small enough for the penultimate step in the proof of Lemma 6.5 to go through. We take C to be any absolute constant satisfying $C \geqslant 32\pi/\Delta$. We consider Δ and C as given from the start, and at no point will either change its value.

In statements or proofs of lemmas, propositions and theorems, ε always denotes an arbitrarily small positive constant (only permitted to change its value between lemmas, propositions, or theorems, or at certain other points where we indicate and discuss the change).

To standardise $\omega(x)$ and $\Omega_0(x)$ in (1.47), (1.49), (1.55) and (1.56), (1.57), we suppose given, once and for all, an infinitely differentiable function $\Omega(x)$ such that

$$\Omega(x) = \begin{cases} 1 & \text{if } x \leqslant 1, \\ 0 & \text{if } x \geqslant 2, \end{cases} \tag{1.69}$$

and

$$\Omega'(x) \leqslant 0 \qquad (x \in \mathbb{R}). \tag{1.70}$$

For $x \in \mathbb{R}$, the values of $\omega(x)$ and $\Omega_0(x)$ are given by:

$$\omega(x) = \Omega(x) - \Omega(4x) \qquad \text{and} \qquad \Omega_0(x) = \Omega(x) - \Omega(2x) \tag{1.71}$$

(these choices ensuring that (1.49) and (1.57) both hold). The function $\Omega(x)$ proves useful in its own right in Section 7, where (also) we define several other functions ($\Omega_1(x)$, $\alpha(u)$ and $\beta(u)$) in terms of it.

We have some special notations, $\|\mathbf{b}_N\|_p$, $\mathbf{b}^{\langle D \rangle}$, $\mathbf{b}(y)$ and $\mathbf{b}^{\{g\}}$, relating to sequences $\mathbf{b} = (b_n)$: see (1.29), (1.31), (1.45) and (1.52), respectively, for the relevant definitions (and note the discussion of the compound notation $\mathbf{b}^{\langle D \rangle}(y)$ under (1.45)). The sequences denoted by $\mathbf{\Psi}$ or (Ψ_n) have a special form, being given by (1.63) for some choice of $N > 0$ and $N_1 \in [N/2, N]$ (although in Section 10 and in Lemma 11.1 we work with the definition (10.18), which differs superficially from that given in (1.63)).

For the definition of $\mathcal{S}^{(i)}_{\mathfrak{a},q,K}(\mathbf{b}, N)$ (for $i = 0, 1, 2$), see (1.26), (1.27) and (1.28); for $\sigma_{q,K}(\mathbf{b}, N; D, y)$ and $S_{Q,K}(\mathbf{b}, N; D, y)$ see (1.42) and (1.43); for $\alpha_{q,H}(\mathbf{b}, N; D, y)$, $A_{Q,H}(\mathbf{b}, N; D, y)$ and $\Phi_H(x)$, see (1.46)-(1.48) (and note that Lemma 4.4 reconciles (1.48) with the alternative definition of $\Phi_H(x)$ in (4.3)); and for $\alpha_{q,H,X}(\mathbf{b}, N; D, y)$, $A_{Q,H,X}(\mathbf{b}, N; D, y)$ and $\Phi_{H,X}(x)$, see (1.54)-(1.56). The function $R_H(r)$, given by (4.2) of Lemma 4.2, is a transform of the function $\mathcal{H}(r, t)$ defined in (2.17) of Theorem 2.4. The terms $\mathcal{A}_\varepsilon(q)$ and $\check{\sigma}_q(H_1, H)$, defined in (6.14) of Lemma 6.4 and (4.27) of Lemma 4.8 (respectively), are also dependent upon the parameters $\mathbf{b}$, N, D, and (in the case of the latter) y. Sums important in Section 10 are $B_{Q,X}(\mathbf{b}, N; D, \theta)$ and $B^{(\delta)}_{Q,X}(\mathbf{\Psi}, N; D, \theta)$ and $\mathcal{V}(D_\delta; A/c_1)$ defined in (10.3), (10.4) and (10.19)-(10.21) and (10.23)-(10.26), respectively.

Notations (II): definitions for 'Kloostermania'. Most of our notation here is borrowed from [7], Sections 1.1-1.3, although we introduce some small innovations in our notation for the Fourier coefficients of holomorphic forms and Eisenstein series.

The multiplicative group of 2-by-2 matrices with real entries and determinant 1 is denoted by $SL_2(\mathbb{R})$. By Γ we mean a Hecke congruence subgroup $\Gamma_0(q)$, where $q \in \mathbb{N}$. Not far into our Introduction we defined $\Gamma_0(q)$ as a subgroup of $SL_2(\mathbb{Z})$, the multiplicative group of 2-by-2 matrices with integer entries and determinant 1. There we also defined: the upper half-plane $\mathbb{H}$, the action of Γ and $SL_2(\mathbb{R})$ on $\mathbb{H}$ (and on $\mathbb{R} \cup \{\infty\}$), the cusps $\mathfrak{a}$ (for Γ), and the equivalence relation $\overset{\Gamma}{\sim}$ (Γ-equivalence) on the set of cusps. The function $\mu(\mathfrak{a})$, defined on the set of cusps for Γ, is given by (1.30) (Lemma 2.1 also being relevant).

We have two distinct 'Dirac delta' notations, δ_{mn} and $\delta_{\mathfrak{a}\mathfrak{b}}$ (distinguishable from one another by their subscripts, since the symbols $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, ∞, or non-integer rationals, always signify cusps): both these notations are defined in Theorem 2.3.

Following a brief discussion of the relevant cusp forms (and of the hyperbolic Laplacian Δ), the definitions of the spaces $\mathfrak{M}_k(\Gamma)$ and $L^2_{\text{cusp}}(\Gamma\backslash\mathbb{H})$ are made in the paragraphs before and after Theorem 1.1 (see [7], Section 1.1, regarding the Petersson inner-product $< f, g >_k$). The orthonormal bases $\{u_j : j \in \mathbb{N}\}$ and $\{f_{jk} : 1 \leqslant j \leqslant \theta_k(q)\}$ are introduced and discussed in the paragraphs between (1.12) and (1.17): note that λ_j and κ_j relate to $u_j(z)$ (and to each other) as do λ, κ and $u(z)$ in (1.5); that the u_j's are ordered so that (1.13) holds; and that κ_j is chosen to satisfy (1.14) or (1.15) (whichever is appropriate), with the sign of κ_j being left unspecified in the former case.

The eigenvalues $\tau_j(n)$ and $\lambda_{jk}(n)$ are given by (1.16) and (1.17), in terms of the relevant Hecke operators in (1.9) and (1.8) (respectively). We define $\psi_{jk}(\mathfrak{a}, m)$ and $\rho_{j\mathfrak{a}}(m)$ under (1.18) and (1.19) (where our own special notations, $\lambda^*_{jk}(n)$ and $\psi^*_{jk}(\mathfrak{a}, m)$, are introduced). See [7], Equation (1.13), for a definition of the Eisenstein series $E_{\mathfrak{c}}(z, s)$. The coefficients $\varphi_{\mathfrak{c}\mathfrak{a}}(m, s)$, from the Fourier expansion of $E_{\mathfrak{c}}(z, s)$ in (1.6), are explicitly defined in (1.25): note that sums involving a generalised Kloosterman sum $S_{\mathfrak{a}\mathfrak{b}}(m, n; \gamma)$ (such as (1.25), or the first sum in Theorem 2.2) are sums over exactly those $\gamma > 0$ for which that Kloosterman sum is defined (see (2.3) and (2.4) for the definition of $S_{\mathfrak{a}\mathfrak{b}}(m, n; \gamma)$, and (1.1), (1.2) and Lemma 2.2 regarding the relevant scaling matrices $\sigma_{\mathfrak{a}}$, $\sigma_{\mathfrak{b}}$). The classical Kloosterman sum $S(m, n; c)$ is given by (2.8).

A superscript '$\theta_k(q)$', '(q)', 'Γ', '$\Gamma_0(q)$' above a summation sign (as in Theorem 2.2, for example) indicates that the terms being summed are defined with reference to the group $\Gamma = \Gamma_0(q)$. In (2.15), (2.16) of Theorem 2.2 we define the transforms $\tilde{\phi}(\ell)$ and $\hat{\phi}(r)$. See Lemma 2.3 regarding the standard Bessel functions $J_\nu(z)$, $K_\nu(z)$; see (2.18) for the function $\mathfrak{D}_\nu(x)$.

Notations (III): standard definitions. By $a|b$ we mean that a divides b. We use (a, b) to denote the greatest common divisor of the integers a, b. The sum $\sum_{d|n} 1$ is the divisor function, $\tau(n)$. The Möbius function, $\mu(n)$, should not be confused with either the '$\mu(\mathfrak{a})$' of (1.30), or the plain 'μ' sometimes used to denote a variable.

The greatest integer not exceeding x is $[x]$, and the distance from x to the nearest integer is $\|x\| = \min(x - [x],\ [x] + 1 - x)$. By $\log(x)$ we mean the natural logarithm, $\log_e(x)$, which is the inverse of the exponential function $\exp(x) = e^x$. For $x \in \mathbb{R}$ and $i = \sqrt{-1}$ we write $\exp(2\pi i x)$ as $\mathrm{e}(x)$. By $\Gamma(z)$ we denote the standard Gamma function (so that $\Gamma(n) = (n-1)!$). For $z \in \mathbb{C}$ we use $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ to denote the real and imaginary parts of z, so that $z = \mathrm{Re}(z) + i\mathrm{Im}(z)$.

In sums where a runs over the residue classes mod b, we may indicate restriction of a to classes for which $(a, b) = 1$ by placing an asterisk just after the relevant summation sign. In expressions such as '$\overline{a} \pmod b$', '$\mathrm{e}(c\overline{a}/b)$', or $\|\beta + c\overline{a}/b\|$, it is implicit that $(a, b) = 1$, and we take '$\overline{a}$' to denote a solution, x, for the congruence $ax \equiv 1 \pmod b$. In other contexts $\overline{y}$ may simply denote the complex-conjugate of y.

The $o(x)$ and $O(x)$ notation is standard. We use $a \ll b$ (or $b \gg a$) and $a \asymp b$ to mean $a = O(b)$ and $a \ll b \ll a$, respectively. The constants implicit in such notations depend (at most) upon parameters explicitly declared to be absolute constants, or upon parameters appended to the notation as subscripts.

For any function $\phi : \mathbb{R} \longrightarrow \mathbb{C}$ we have (when they are defined) the standard norms: $\|\phi\|_\infty = \sup_{x \in \mathbb{R}} |\phi(x)|$ and $\|\phi\|_1 = \int_{-\infty}^{\infty} |\phi(x)| \mathrm{d}x$.

The symbols '$\Rightarrow$' and '$\Leftrightarrow$' mean 'only if' and 'if and only if' (respectively).

Where we have reproduced a noteworthy result due to other authors, we try to indicate this, either explicitly, or by including their name(s), or some keywords (e.g. 'Bessel functions'), in the heading of the relevant lemma, proposition, or theorem.

2. Summation formulae of Bruggeman and Kuznetsov

Theorem 2.2 in this section gives a restatement of one case of [7], Theorem 1, the Kuznetsov summation formula for $\Gamma = \Gamma_0(q)$ (Deshouillers and Iwaniec being the first to establish this result for cases other than $q = 1$). Two precursors to that theorem, [7], Lemma 4.7, and [7], Equation (4.4), are presented here as Theorems 2.3 and 2.4. Note that we ascribe Theorem 2.4 to Bruggeman and Kuznetsov, although our source is [7] and the result is not quite what Bruggeman and Kuznetsov actually achieved.

As a preface to Theorems 2.2, 2.3 and 2.4 we will begin with some relevant results and definitions relating to Kloosterman sums and Bessel functions.

At the end of the section we have a lemma giving bounds (from [7], Lemma 7.1) for the Bessel transforms ((2.15) and (2.16)) of Theorem 2.2. There is also a very simple lemma on the Gamma function (needed for the proof of Lemma 6.2).

Lemma 2.1. *Every cusp of Γ is equivalent to one of the form u/w, where*

$$u, w \in \mathbb{N}, \qquad w | q \qquad \textit{and} \qquad (u, w) = 1. \tag{2.1}$$

Moreover, for cusps u/w, u_1/w_1 of this form, one has $u/w \overset{\Gamma}{\sim} u_1/w_1$ if and only if

$$w_1 = w \qquad \textit{and} \qquad u_1 \equiv u \pmod{(w, q/w)}. \tag{2.2}$$

Proof. This is merely a restatement of [7], Lemma 2.3. ■

Suppose that each cusp $\mathfrak{a}$ for Γ has been assigned a scaling matrix $\sigma_{\mathfrak{a}} \in \in SL_2(\mathbb{R})$ such that (1.1) holds. Let $\mathfrak{a}$ and $\mathfrak{b}$ be cusps of Γ. Then, for $m, n \in \mathbb{Z}$, and $\gamma > 0$ such that there exists a matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \sigma_{\mathfrak{a}}^{-1} \Gamma \sigma_{\mathfrak{b}}, \tag{2.3}$$

we define

$$S_{\mathfrak{a}\mathfrak{b}}(m, n; \gamma) = \sum_{\delta (\mathrm{mod}\ \gamma\mathbb{Z})}^{*} \mathrm{e}\left(m\frac{\alpha}{\gamma} + n\frac{\delta}{\gamma} \right), \tag{2.4}$$

where the summation is over δ's (modulo $\gamma\mathbb{Z}$) for which it is possible to find α, β satisfying (2.3) (with the α of (2.4) being any one of the instances of α in (2.3)). The correctness of this definition is verified in [7], Lemma 2.2. Our next lemma summarises remarks from [7], Section 2.1, concerning the dependence of the Kloosterman sum upon the choice of scaling matrices, and the behaviour of the sum under a permutation of cusps that leaves fixed the equivalence classes (modulo Γ).

Lemma 2.2. *Let $\mathfrak{a}, \mathfrak{a}'$ and $\mathfrak{b}, \mathfrak{b}'$ be Γ-equivalent pairs of cusps, with a given choice of $\sigma_{\mathfrak{a}}, \sigma_{\mathfrak{a}'}, \sigma_{\mathfrak{b}}, \sigma_{\mathfrak{b}'}$, and take any $\tau_1, \tau_2 \in \Gamma$ such that $\tau_1 \mathfrak{a}' = \mathfrak{a}$, $\tau_2 \mathfrak{b}' = \mathfrak{b}$. If $\sigma_{\mathfrak{a}}^{-1} \tau_1 \sigma_{\mathfrak{a}'} = \rho_1$ and $\sigma_{\mathfrak{b}}^{-1} \tau_2 \sigma_{\mathfrak{b}'} = \rho_2$, then*

$$\rho_1 = \begin{pmatrix} 1 & \beta_1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \rho_2 = \begin{pmatrix} 1 & \beta_2 \\ 0 & 1 \end{pmatrix}, \tag{2.5}$$

where β_1, β_2 are some real numbers, and

$$S_{\mathfrak{a}'\mathfrak{b}'}(m, n; \gamma) = \mathrm{e}\left(-m\beta_1 + n\beta_2 \right) S_{\mathfrak{a}\mathfrak{b}}(m, n; \gamma), \tag{2.6}$$

for $m, n \in \mathbb{Z}$ and $\gamma > 0$ (both sides of the last equation being defined for the same set of γ's).

Proof. By (1.1) both ρ_1 and ρ_2 must fix ∞. It is moreover the case that $\rho_1^{-1} \Gamma_\infty \rho_1 = \sigma_{\mathfrak{a}'}^{-1} \tau_1^{-1} \Gamma_{\mathfrak{a}} \tau_1 \sigma_{\mathfrak{a}'} = \sigma_{\mathfrak{a}'}^{-1} \Gamma_{\mathfrak{a}'} \sigma_{\mathfrak{a}'} = \Gamma_\infty$ (see (1.1)-(1.2)), and (similarly) that $\rho_2^{-1} \Gamma_\infty \rho_2 = \Gamma_\infty$. Therefore the ρ_i are upper triangular matrices from $SL_2(\mathbb{R})$ satisfying $\Gamma_\infty \rho_i = \rho_i \Gamma_\infty$ $(i = 1, 2)$, which is only possible if they have the form shown in (2.5). Since $\sigma_{\mathfrak{a}'}^{-1} \Gamma \sigma_{\mathfrak{b}'} = \rho_1^{-1} \sigma_{\mathfrak{a}}^{-1} \tau_1 \Gamma \tau_2^{-1} \sigma_{\mathfrak{b}} \rho_2 = \rho_1^{-1} (\sigma_{\mathfrak{a}}^{-1} \Gamma \sigma_{\mathfrak{b}}) \rho_2$, the result (2.6) follows from (2.5) and the definition of the Kloosterman sum given in (2.3)-(2.4). ■

Given (1.2), it is immediate from the definition in (2.3)-(2.4) that

$$S_{\infty\infty}(m,n;\gamma) = \begin{cases} S(m,n;\gamma), & \text{if } \gamma/q \in \mathbb{N}, \\ \text{undefined}, & \text{otherwise}, \end{cases} \tag{2.7}$$

where

$$S(m,n;c) = \sum_{d \bmod c}{}^{*} \mathrm{e}\left(m\frac{\overline{d}}{c} + n\frac{d}{c}\right) \tag{2.8}$$

(the classical Kloosterman sum). In respect of this last sum we have at our disposal the following important result.

Theorem 2.1. [Weil's bound] *For $m, n \in \mathbb{Z}$ and $c \in \mathbb{N}$,*

$$|S(m,n;c)| \leqslant (m,n,c)^{1/2} c^{1/2} \tau(c).$$

Proof. This is a corollary of A. Weil's bound for $S(m,n;p)$ with prime p. See [14], Section 4.3, for some of the details. ■

Lemma 2.3. [Bessel functions] *Let $x > 0$. If $\nu \in \mathbb{C}$, then*

$$J_\nu(x) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!\,\Gamma(\ell+1+\nu)} \left(\frac{x}{2}\right)^{2\ell+\nu}, \tag{2.9}$$

and if, moreover, $|\mathrm{Re}(\nu)| < 1$, then

$$J_\nu(x) = \frac{2}{\pi} \int_0^\infty \sin\left(x\cosh(\xi) - \tfrac{\pi}{2}\nu\right) \cosh(\nu\xi)\mathrm{d}\xi \tag{2.10}$$

and

$$\frac{J_\nu(x) - J_{-\nu}(x)}{\sinh\left(\frac{\pi}{2}i\nu\right)} = \frac{4i}{\pi} \int_0^\infty \cos(x\cosh(\xi)) \cos(i\nu\xi)\mathrm{d}\xi. \tag{2.11}$$

If $k \in \mathbb{N}$ is even, then

$$J_{k-1}(x) = -\frac{i^k}{\pi} \int_{-\pi/2}^{\pi/2} e^{-(k-1)i\eta} \sin\left(x\cos(\eta)\right) \mathrm{d}\eta. \tag{2.12}$$

If $t \in \mathbb{R}$ and $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$, then

$$K_{2it}(z) = \int_0^\infty e^{-z\cosh(\xi)} \cos(2t\xi)\mathrm{d}\xi, \tag{2.13}$$

and if, moreover, $t \neq 0$, *then*

$$K_{2it}(z) = \frac{z}{2t} \int_0^\infty e^{-z\cosh(\xi)} \sinh(\xi)\sin(2t\xi)\mathrm{d}\xi. \tag{2.14}$$

Proof. For (2.9) see [27], page 359. Result (2.10) is the Mehler-Sonine formula of [8], page 82, and directly implies (2.11), since

$$\sin\left(\theta - \tfrac{\pi}{2}\nu\right) - \sin\left(\theta + \tfrac{\pi}{2}\nu\right) = 2\cos(\theta)\sin\left(-\tfrac{\pi}{2}\nu\right) = 2i\cos(\theta)\sinh\left(\tfrac{\pi}{2}i\nu\right).$$

Formula (2.12) follows from Bessel's integral,

$$J_{k-1}(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-(k-1)i\theta + ix\sin(\theta)}\mathrm{d}\theta$$

(see [27], page 362), on writing $\sin(\theta) = \cos(\theta - \frac{\pi}{2})$, then substituting $\theta - \frac{\pi}{2} = \eta$, if $0 \leqslant \theta \leqslant \pi$, and $\theta - \frac{\pi}{2} = \eta - \pi$, if $-\pi \leqslant \theta < 0$, and then, finally, using $e^{iz} + e^{(k-1)i\pi - iz} = 2i\sin(z)$ (with $z = x\cos(\eta) = -x\cos(\eta - \pi)$).

Formula (2.13) is from [8], page 82, and implies (2.14) through integration by parts. ■

Theorem 2.2. [Kuznetsov-Deshouillers-Iwaniec] *Let* ϕ *be a three times continuously differentiable function, with compact support in* $(0,\infty)$. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be cusps of* Γ. *Then, for* $m, n \in \mathbb{N}$,

$$\sum_{\gamma}^{\Gamma} \frac{1}{\gamma} S_{\mathfrak{a}\mathfrak{b}}(m,n;\gamma)\phi\left(\frac{4\pi\sqrt{mn}}{\gamma}\right) = \mathcal{K}_0 + \mathcal{K}_1 + \mathcal{K}_2,$$

where:

$$\mathcal{K}_0 = \frac{1}{2\pi}\sum_{k \text{ even}}\sum_{j=1}^{\theta_k(q)} \frac{i^k(k-1)!}{(4\pi)^{k-1}} \overline{\psi^*_{jk}(\mathfrak{a},m)}\psi^*_{jk}(\mathfrak{b},n)\tilde{\phi}(k-1),$$

$$\mathcal{K}_1 = \sum_{j\geqslant 1}^{(q)} \frac{1}{\cosh(\pi\kappa_j)}\overline{\rho_{j\mathfrak{a}}(m)}\rho_{j\mathfrak{b}}(n)\hat{\phi}(\kappa_j),$$

$$\mathcal{K}_2 = \frac{1}{\pi}\sum_{\mathfrak{c}}^{\Gamma}\int_{-\infty}^{\infty}\left(\frac{m}{n}\right)^{-ir}\overline{\varphi_{\mathfrak{c}\mathfrak{a}}(m,\tfrac{1}{2}+ir)}\varphi_{\mathfrak{c}\mathfrak{b}}(n,\tfrac{1}{2}+ir)\hat{\phi}(r)\mathrm{d}r,$$

with

$$\tilde{\phi}(\ell) = \int_0^\infty J_\ell(y)\phi(y)\frac{\mathrm{d}y}{y} \tag{2.15}$$

and

$$\hat{\phi}(r) = \frac{\pi}{\sinh(\pi r)} \int_0^\infty \frac{J_{2ir}(x) - J_{-2ir}(x)}{2i} \phi(x) \frac{dx}{x}. \tag{2.16}$$

Theorem 2.3. [Petersson] *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be cusps of* Γ, $m, n \in \mathbb{N}$ *and* $k/2 \in \mathbb{N}$. *Then*

$$\mathcal{J}_0 = \delta_{\mathfrak{a}\mathfrak{b}} \delta_{mn} \mathrm{e}\left(\beta_{\mathfrak{a}\mathfrak{b}} n\right) + 2\pi i^k \sum_\gamma^\Gamma \frac{1}{\gamma} S_{\mathfrak{a}\mathfrak{b}}(m, n; \gamma) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{\gamma}\right),$$

where

$$\mathcal{J}_0 = \frac{(k-2)!}{(4\pi)^{k-1}} \sum_{j=1}^{\theta_k(q)} \overline{\psi^*_{jk}(\mathfrak{a}, m)} \psi^*_{jk}(\mathfrak{b}, n),$$

$$\delta_{\mathfrak{a}\mathfrak{b}} = \begin{cases} 1, & \text{if } \mathfrak{a} \overset{\Gamma}{\sim} \mathfrak{b}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\delta_{mn} = \begin{cases} 1, & \text{if } m = n, \\ 0, & \text{otherwise}, \end{cases}$$

and $\beta_{\mathfrak{a}\mathfrak{b}}$ *is a real number satisfying*

$$\beta_{\mathfrak{a}\mathfrak{b}} = 0 \quad \text{if } \mathfrak{a} = \mathfrak{b} \text{ and } \sigma_\mathfrak{a} = \sigma_\mathfrak{b}.$$

Theorem 2.4. [Bruggeman-Kuznetsov] *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be cusps of* Γ, $m, n \in \mathbb{N}$ *and* $t \in \mathbb{R}$. *Then*

$$\mathcal{J}_1 + \mathcal{J}_2 = \frac{t \delta_{\mathfrak{a}\mathfrak{b}} \delta_{mn}}{\pi \sinh(\pi t)} \mathrm{e}\left(\beta_{\mathfrak{a}\mathfrak{b}} n\right) + \sum_\gamma^\Gamma \frac{4\pi\sqrt{mn}}{\gamma^2} S_{\mathfrak{a}\mathfrak{b}}(m, n; \gamma) \mathcal{D}_{2it}\left(\frac{4\pi\sqrt{mn}}{\gamma}\right),$$

where $\delta_{\mathfrak{a}\mathfrak{b}}$, δ_{mn} *and* $\beta_{\mathfrak{a}\mathfrak{b}}$ *are as in Theorem 2.3,*

$$\mathcal{J}_1 = \sum_{j \geqslant 1}^{(q)} \frac{\mathcal{H}(\kappa_j, t)}{\cosh(\pi \kappa_j)} \overline{\rho_{j\mathfrak{a}}(m)} \rho_{j\mathfrak{b}}(n),$$

$$\mathcal{J}_2 = \frac{1}{\pi} \sum_\mathfrak{c}^\Gamma \int_{-\infty}^\infty \left(\frac{m}{n}\right)^{-ir} \overline{\varphi_{\mathfrak{c}\mathfrak{a}}(m, \tfrac{1}{2} + ir)} \varphi_{\mathfrak{c}\mathfrak{b}}(n, \tfrac{1}{2} + ir) \mathcal{H}(r, t) \mathrm{d}r,$$

$$\mathcal{H}(r, t) = \frac{\pi \cosh(\pi r)}{\cosh(\pi(r - t)) \cosh(\pi(r + t))} \tag{2.17}$$

and

$$\mathcal{D}_{2it}(x) = -\frac{2it}{\sinh(\pi t)} \int_{-i}^{i} K_{2it}(x\nu) \frac{\mathrm{d}\nu}{\nu} \tag{2.18}$$

(with integration along the contour $\{\exp(i\theta) \,:\, -\pi/2 \leqslant \theta \leqslant \pi/2\}$*).*

Lemma 2.4. [Bessel transforms] *Let* $\delta, X, Y, F > 0$. *Suppose that* $\phi(x)$ *is a complex-valued function of a real variable, possessing a continuous second derivative, and vanishing outside of the interval* $[X, 8X]$. *Suppose also that*

$$\|\phi\|_\infty \leqslant F, \qquad \|\phi'\|_1 \leqslant F \qquad \text{and} \qquad \|\phi''\|_1 \leqslant FYX^{-1}. \tag{2.19}$$

Then

$$\hat{\phi}(r), \tilde{\phi}(n) \ll \frac{F\,(1 + |\log(X)|)}{1 + X} \qquad (r \in \mathbb{R},\ n \in \mathbb{N}), \tag{2.20}$$

and, for $r \in \mathbb{R}$,

$$\left(1 + X^{3/2}\right) YF|r|^{-5/2} \gg \begin{cases} \hat{\phi}(r) & \text{if } |r| \geqslant 1, \\ \tilde{\phi}(r) & \text{if } r \in \mathbb{N}. \end{cases} \tag{2.21}$$

Moreover, if $r \in \mathbb{R}$ *with*

$$0 < |r| \leqslant \frac{1}{2} - \delta, \tag{2.22}$$

then

$$\hat{\phi}(ir) \ll \begin{cases} \left(\delta^{-1} + \min\left(|r|^{-1}, |\log(X)|\right)\right) FX^{-2|r|}, & \text{if } X \in (0,1], \\ \delta^{-1} FX^{-1}, & \text{if } X > 1. \end{cases} \tag{2.23}$$

Proof. The bounds in (2.20) are, in the case $F = 1$, bounds given by [7], Lemma 7.1: the cases where $F \neq 1$ following by linearity of the transforms.

The bounds of (2.21) would follow from the slightly stronger results in [7], (7.4). However, as has been noted on page 7 of [3], there is reason to doubt [7], (7.4) in cases where $|r| = o(X)$. According to the authors of [3] these doubtful cases of [7], (7.4) are never actually used in [7], so their loss is not significant there, but does leave us needing an alternative justification for (2.21).

For the bound on $\hat{\phi}(r)$ in (2.21) we argue as Deshouillers and Iwaniec did for the bound on $\hat{f}(r)$ in [7], (7.4) (see [7], page 266), but with the line of integration in the Mellin-Barnes integral moved to $\sigma = -3/2$ rather than $\sigma = -1$ (no extra pole being encountered). Following the two integrations by parts (with respect to x), one obtains the desired bound on $\hat{\phi}(r)$ by appealing to the bound $\|\phi''\| \leqslant FYX^{-1}$, and bounds of the form:

$$|\Gamma(x + iy)| \asymp_{\alpha,\beta} (1 + |y|)^{x-1/2}\, e^{-(\pi/2)|y|}$$

(valid for $\alpha \leqslant x \leqslant \beta$ and $y \in \mathbb{R}$ if $[\alpha, \beta] \cap \mathbb{Z} \subseteq \mathbb{N}$).

With regard to the bound on $\tilde{\phi}(r)$ in (2.21), we begin by remarking that the argument used in [7], page 267, to justify a bound $\tilde{f}(r) \ll r^{-3}XY$, actually only supports the weaker conclusion that $\tilde{f}(r) \ll r^{-2}XY$, for $r = 2, 3, 4, \ldots$. Nevertheless, this does permit us to conclude that $\tilde{\phi}(r) \ll FYXr^{-2}$ for $r = 2, 3, 4, \ldots$. Moreover, for $r = 3, 4, 5, \ldots$, one can modify the argument slightly, by starting it with the Mellin-Barnes integral along the contour from $-2 - i\infty$ to $-2 + i\infty$ (see [8], page 21). The modified argument then shows that $\tilde{\phi}(r) \ll FYX^2r^{-3}$ for $r = 3, 4, 5, \ldots$. By the last two bounds it follows that $\tilde{\phi}(r) \ll FY\min(Xr^{-2}, X^2r^{-3}) \leqslant FYX^{3/2}r^{-5/2}$, for $r = 3, 4, 5, \ldots$. If $r \in \{1, 2\}$, then one has $|J_r(x)| \leqslant 1$ for $x \in \mathbb{R}$ (see Bessel's integral in [27], subsection 17.23) and $\|\phi\|_\infty \leqslant \|\phi'\|_1 \ll X\|\phi'\|_\infty \leqslant X\|\phi''\|_1$, so it follows from (2.15) and (2.19) that $|\tilde{\phi}(r)| \leqslant \|\phi\|_\infty \log(8) \ll X\|\phi''\|_1 \leqslant FY \ll FYr^{-5/2}$. We conclude that $\tilde{\phi}(r)$ does satisfy (2.21) for all $r \in \mathbb{N}$.

The result (2.23) is only slightly different from the corresponding result of [7], Lemma 7.1. We prove it here by first noting (as is explained on [7], page 265) that one can use (2.11), (2.19) and (implicitly) (2.22) to show:

$$\hat{\phi}(ir) \ll F \int_0^\infty e^{2|r|\xi} \min\left(1, X^{-1}e^{-\xi}\right) \mathrm{d}\xi.$$

For $X > 1$ this bound simplifies to:

$$\hat{\phi}(ir) \ll FX^{-1} \int_0^\infty e^{-(1-2|r|)\xi} \mathrm{d}\xi = \frac{F}{(1-2|r|)X},$$

so that (2.23) follows by (2.22). For $0 < X \leqslant 1$, the bound becomes (after evaluation of integrals):

$$\hat{\phi}(ir) \ll F\left(\frac{1}{2|r|}\left(\left(\frac{1}{X}\right)^{2|r|} - 1\right) + \frac{1}{(1-2|r|)}\left(\frac{1}{X}\right)^{2|r|}\right) \ll$$
$$\ll FX^{-2|r|}\left(|r|^{-1} + \delta^{-1}\right)$$

(by (2.22) again). There is another option here, since

$$\frac{1}{2|r|}\left(\left(\frac{1}{X}\right)^{2|r|} - 1\right) = \int_0^{\log(1/X)} e^{2|r|\xi} \mathrm{d}\xi \leqslant \log\left(\frac{1}{X}\right)\left(\frac{1}{X}\right)^{2|r|},$$

showing that the last bound for $\hat{\phi}(ir)$ will hold with $\log(1/X)$ substituted for the bracketed term $|r|^{-1}$ in that bound (exactly as claimed in (2.23)). ■

Lemma 2.5. [Gamma function] *For $n \in \mathbb{N}$ and $r \in \mathbb{R}$ with $r \neq 0$,*

$$|\Gamma(n + 1 + 2ir)| = \left(\frac{\pi}{2r\sinh(2\pi r)}\right)^{1/2} \prod_{m=0}^{n} |m + 2ir| =$$
$$= \sqrt{\frac{2\pi r}{\sinh(2\pi r)}} \prod_{m=1}^{n} |m + 2ir|.$$

Proof. By the functional equation $\Gamma(z+1) = z\Gamma(z)$, it suffices to consider $|\Gamma(2ir)|$. Now, for r as given,

$$|\Gamma(2ir)|^2 = \Gamma(2ir)\Gamma(-2ir) = \\ = \frac{\Gamma(2ir)\Gamma(1-2ir)}{-2ir} = \frac{\pi}{\sin(\pi(2ir))(-2ir)} = \frac{\pi}{2r\sinh(2\pi r)},$$

which is all that we need to complete the proof. ∎

3. Multiplicativity: proving Theorem 1.4

Using the multiplicativity expressed in (1.23), (1.24) one may effectively extract, as a certain factor, the dependence on D of the terms in the sums $\mathbb{S}^{(0)}_{\mathfrak{a},q,K}(\mathbf{b}^{(D)}, DN)$ and $\mathbb{S}^{(1)}_{\mathfrak{a},q,K}(\mathbf{b}^{(D)}, DN)$ shown in (1.33), (1.34). That factor can then be estimated through (1.21) and (1.20), yielding the cases $i = 0, 1$ of Theorem 1.4. Fuller details of these steps, and a treatment of the case $i = 2$, are given at the end of the section, where we prove Theorem 1.4.

We begin with a discussion of Fourier coefficients, $\varphi_{\mathfrak{c}\infty}(n,s)$, for Eisenstein series $E_{\mathfrak{c}}(z,s)$ (see (1.6), (1.25)), leading up to Lemma 3.4, which plays a rôle analogous to that of the multiplicativity relations, (1.23), (1.24).

Given (1.25) and Lemma 2.2 it at first appears that $\varphi_{\mathfrak{c}\infty}(n,s)$ might depend on both the cusp $\mathfrak{c}$ and the choice of scaling matrix $\sigma_{\mathfrak{c}}$ (see (1.1) and (1.2)). Our next lemma addresses the extent of this dependence.

Lemma 3.1. *Let $n \in \mathbb{N}$ and $s \in \mathbb{C}$ with* $\mathrm{Re}(s) \geqslant 1/2$. *Then, given our fixed choice of σ_∞ in (1.2), the Fourier coefficient $\varphi_{\mathfrak{c}\infty}(n,s)$ is a function of the equivalence class of $\mathfrak{c}$ modulo Γ.*

Proof. As the relevant cases of (2.6) will have $m = 0$ and (by (1.2)) $\beta_2 = 0$, Lemma 2.2 shows that the terms $S_{\mathfrak{c}\infty}(0,n;\gamma)$ in (1.25) would be unaltered by the mere substitution of a Γ-equivalent cusp for $\mathfrak{c}$. This proves the lemma for $\mathrm{Re}(s) > 1$. The remaining cases, where $1/2 \leqslant \mathrm{Re}(s) \leqslant 1$, follow by the meromorphic continuation discussed under (1.6). ∎

In light of Lemmas 2.1 and 3.1, we may assume that the cusp $\mathfrak{c}$ is of the form u/w (satisfying (2.1)) and follow [7], Section 3.3, in taking

$$\sigma_{\mathfrak{c}} = \sigma_{u/w} = \begin{pmatrix} u\sqrt{q/(w^2,q)} & 0 \\ w\sqrt{q/(w^2,q)} & \left(u\sqrt{q/(w^2,q)}\right)^{-1} \end{pmatrix}. \tag{3.1}$$

Lemma 3.2. *Let u/w be as in (2.1), with σ_∞ and $\sigma_{u/w}$ given by (1.2) and (3.1). Then the set,*

$$\left\{ -\frac{\delta}{\gamma} \ :\ \gamma > 0 \ \text{ and } \ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \sigma_{u/w}^{-1}\Gamma\sigma_\infty \ \text{ for some } \alpha, \beta \right\},$$

is identical to the set of rational cusps $B/A \overset{\Gamma}{\sim} u/w$:

$$\left\{\frac{B}{A} : A \in \mathbb{N}, B \in \mathbb{Z}, (A,B) = 1, (A,q) = w, AB \equiv uw \pmod{(w^2,q)}\right\},$$

with a one-to-one correspondence given by the relation

$$-\frac{\delta}{B} = \frac{\gamma}{A} = \sqrt{\frac{q}{(w^2,q)}}.$$

Proof. This follows by [7], Lemma 3.6, and the argument immediately preceding it in [7], Section 3.3. ∎

By Lemma 3.2, (2.3)-(2.4), (1.2) and (3.1), it follows that $S_{\mathfrak{c}\infty}(0,n;\gamma)$ (where $\mathfrak{c} = u/w$) is defined only if $\gamma = A\sqrt{q/(w^2,q)}$ with $A \in \mathbb{N}$ and $(A,q) = w$, and, in such a case, it is given by

$$S_{\mathfrak{c}\infty}(0,n;\gamma) = \sum_{\substack{B \bmod A \\ (B,A)=1 \\ AB \equiv uw \pmod{(w^2,q)}}} \mathrm{e}\left(-n\frac{B}{A}\right) \qquad (n \in \mathbb{Z}). \tag{3.2}$$

By (2.1), we may rewrite the conditions necessary and sufficient for the definition of $S_{\mathfrak{c}\infty}(m,n;\gamma)$ as:

$$\gamma = \ell\sqrt{qw/(w,q/w)} \quad \text{with} \quad \ell \in \mathbb{N} \quad \text{and} \quad (\ell,\, q/w) = 1. \tag{3.3}$$

Lemma 3.3. *Let* $\mathfrak{c} = u/w$ *be as in* (2.1), *with* σ_∞, $\sigma_\mathfrak{c}$ *given by* (1.2), (3.1). *Then, for* $n \in \mathbb{Z}$ *and* γ *satisfying* (3.3), *one has*

$$S_{\mathfrak{c}\infty}(0,n;\gamma) = \sum_{\substack{rt = \ell w/(w,q/w) \\ t \mid n}} \mu(r) t \mathrm{e}\left(-\frac{\overline{\ell r}}{(q/w,w)}(n/t)u\right),$$

where it is an implicit condition of the summation that $(r,(q/w,w)) = 1$.

Proof. On noting that (3.3) implies $A = \ell w$ in (3.2) (so that $(A,q) = w(\ell,q/w) = w$), we rewrite the sum over B there, using

$$\begin{aligned} AB \equiv uw \pmod{(w^2,q)} &\Leftrightarrow \ell w B \equiv uw \pmod{(w^2,q)} \\ &\Leftrightarrow \ell B \equiv u \pmod{(w,q/w)} \\ &\Leftrightarrow \ell^2 \frac{w}{(w,q/w)} B \equiv \ell \frac{w}{(w,q/w)} u \pmod{\ell w}. \end{aligned}$$

The sum in (3.2) then appears in the form

$$c_{\ell w}\left(\ell^2 \frac{w}{(w,q/w)},\, u\ell\frac{w}{(w,q/w)};\, -n\right), \tag{3.4}$$

where

$$c_\lambda(b,h;k) = \sum^*_{\substack{a \bmod \lambda \\ ab\equiv h \pmod{\lambda}}} \mathrm{e}\left(\frac{a}{\lambda}k\right). \tag{3.5}$$

This sum occurs in [24], Lemmas 2.3 and 2.4, and is there found equal to:

$$\sum_{\substack{rt=v \\ t|k}} \mu(r)t\mathrm{e}\left(\frac{\overline{b/t}}{\lambda/v}(k/t)(h/v)\right) = \sum_{\substack{rt=v \\ t|k}} \mu(r)t\mathrm{e}\left(\frac{\overline{br/v}}{\lambda/v}(k/t)(h/v)\right), \tag{3.6}$$

subject to a condition

$$(b,\lambda) = (h,\lambda) = v \tag{3.7}$$

(which, when met, defines v). By (3.4)-(3.5), we have

$$(b,\lambda) = \left(\ell^2\frac{w}{(w,q/w)}, \ell w\right) = \frac{\ell w}{(w,q/w)}\left(\ell, \left(w, \frac{q}{w}\right)\right) = \frac{\ell w}{(w,q/w)}$$

(see (3.3)), and

$$(h,\lambda) = \left(u\ell\frac{w}{(w,q/w)}, \ell w\right) = \left(u, \left(w, \frac{q}{w}\right)\right)\frac{\ell w}{(w,q/w)},$$

so that, by (2.1), the condition (3.7) is met with $v = \ell w/(w,q/w)$. The lemma therefore follows by (3.6) with λ, b, h and k as indicated by (3.4)-(3.5). ■

By (3.3), Lemma 3.3 and (1.25) one has, for $\mathfrak{c} = u/w$ as in (2.1), $n \in \mathbb{Z}$ and $\mathrm{Re}(s) > 1$,

$$\varphi_{\mathfrak{c}\infty}(n,s) = \frac{(w,q/w)^s}{q^s w^s} \sum_{\substack{\ell=1 \\ (\ell,q/w)=1}}^{\infty} \ell^{-2s} \sum_{\substack{rt=\ell w/(w,q/w) \\ t|n}} \mu(r)t\mathrm{e}\left(\frac{-\overline{\ell r}u}{(w,q/w)}(n/t)\right). \tag{3.8}$$

Lemma 3.4. *Let $\mathfrak{c} = u/w$ be as in (2.1). Let $n \in \mathbb{N}$ and let $D \in \mathbb{N}$ satisfy*

$$(D,q) = 1. \tag{3.9}$$

Then, for $\mathrm{Re}(s) \geqslant 1/2$,

$$\varphi_{\mathfrak{c}\infty}(Dn,s) = \sum_{g|(D,n)} \sum_{f|(D/g)} \mu(g)(fg)^{1-2s}\varphi_{\mathfrak{c}(f,g)\infty}(n/g,s), \tag{3.10}$$

where

$$\mathfrak{c}(f,g) = \frac{(D/fg)\overline{f}u}{w} = \frac{u_{f,g}}{w} \quad \textit{(say)}. \tag{3.11}$$

Proof. Both sides of (3.10) are functions of s analytic on the strip $1/2 \leqslant \mathrm{Re}(s) < 1$ (for $\mathrm{Re}(s) \geqslant 1/2$ in fact). It therefore will certainly suffice to establish (3.10) for all s lying in the open half plane where $\mathrm{Re}(s) > 1/2$. By (3.8),

$$\varphi_{c\infty}(Dn, s) = \frac{(w, q/w)^s}{q^s w^s} \sum_{\substack{\ell=1 \\ (\ell, q/w)=1}}^{\infty} \ell^{-2s} \sum_{\substack{rt=\ell w/(w,q/w) \\ t|Dn}} \mu(r) t e\left(-\frac{\overline{\ell r}}{(w, q/w)}(Dn/t)u\right),$$

with absolute convergence of the sum over ℓ here guaranteed by the condition $t|Dn$ (given that $Dn \neq 0$ and $\mathrm{Re}(s) > 1/2$). We may therefore write

$$\varphi_{c\infty}(Dn, s) = \frac{(w, q/w)^s}{q^s w^s} \sum_{f|D} \Phi_f, \tag{3.12}$$

where

$$\Phi_f = \sum_{\substack{\ell=1 \\ (\ell, q/w)=1}}^{\infty} \ell^{-2s} \sum_{\substack{rt=\ell w/(w,q/w) \\ t|Dn \\ (t,D)=f}} \mu(r) t e\left(-\frac{\overline{\ell r}}{(w, q/w)}(Dn/t)u\right). \tag{3.13}$$

Here we write $t = t_1 f$, $D = D_1 f$, so that

$$(t_1, D_1) = 1 \tag{3.14}$$

and the condition $t|Dn$ becomes just $t_1|n$. The condition $rt = \ell w/(w, q/w)$ becomes $rt_1 f = \ell w/(w, q/w)$. As $f|D$ and $w|q$, it follows by (3.9) that one must have $f|\ell$ in this last condition on r and t_1, so that

$$\ell = f\ell_1 \quad \text{(say)} \qquad \text{and} \qquad rt_1 = \ell_1 \frac{w}{(w, q/w)}.$$

In (3.13) we now have

$$-\frac{\overline{\ell r}}{(w, q/w)}(Dn/t)u = -\frac{\overline{f\ell_1 r}}{(w, q/w)}(fD_1 n/ft_1)u = -\frac{\overline{f\ell_1 r}}{(w, q/w)}(n/t_1)D_1 u,$$

so that

$$\Phi_f = f^{1-2s} \sum_{\substack{\ell_1=1 \\ (\ell_1, q/w)=1}}^{\infty} \ell_1^{-2s} \sum_{\substack{rt_1=\ell_1 w/(w,q/w) \\ t_1|n \\ (t_1,D_1)=1}} \mu(r) t_1 \mathrm{e}\left(-\frac{\overline{f\ell_1 r}}{(w, q/w)}(n/t_1)D_1 u\right)$$

(note that $(f\ell_1, q/w) = 1 \Leftrightarrow (\ell_1, q/w) = 1$, since $f|D$ and (3.9) holds).

The next step is to make implicit the unwanted condition (3.14), by attaching to each term of the sum the coefficient $\sum_{g|(t_1,D_1)} \mu(g)$. We then have

$$\Phi_f = f^{1-2s} \sum_{g|D_1} \mu(g) \sum_{\substack{\ell_1=1 \\ (\ell_1,q/w)=1}}^{\infty} \ell_1^{-2s} \sum_{\substack{rt_1=\ell_1 w/(w,q/w) \\ t_1|n \\ t_1\equiv 0 \pmod g}} \mu(r) t_1 e\Big(-\frac{\overline{f\ell_1 r}}{(w,q/w)}(n/t_1)D_1 u\Big).$$

We put $t_1 = gt_2$. Clearly $t_1|n$ implies $g|n$, so we make this a condition upon g and rewrite the condition $t_1|n$ as $t_2|(n/g)$. We also have

$$rt_1 = \ell_1 \frac{w}{(w,\, q/w)} \Rightarrow grt_2 = \ell_1 \frac{w}{(w,\, q/w)},$$

and here $g|D_1$, $D_1|D$ and $w|q$, so that it follows from (3.9) that g must divide ℓ_1. Therefore we write $\ell_1 = g\ell_2$, making the condition become:

$$rt_2 = \ell_2 \frac{w}{(w,q/w)}.$$

As g is aways a factor of D, the assumption of (3.9) means that a condition $(g,\, q/w) = 1$ is superfluous and

$$\Phi_f = \sum_{g|(D/f,\,n)} \mu(g)(fg)^{1-2s} \Phi_{f,g}, \tag{3.15}$$

where, since $\overline{g}g \equiv 1 \pmod{(w,q/w)}$ and $(w,q/w)|w$,

$$\Phi_{f,g} = \sum_{\substack{\ell_2=1 \\ (\ell_2,q/w)=1}}^{\infty} \ell_2^{-2s} \sum_{\substack{rt_2=\ell_2 w/(w,q/w) \\ t_2|(n/g)}} \mu(r) t_2 e\Big(-\frac{\overline{fg\ell_2 r}}{(w,q/w)}\Big(\frac{n/g}{t_2}\Big)(D/f)u\Big) = \tag{3.16}$$

$$= \sum_{\substack{\ell_2=1 \\ (\ell_2,q/w)=1}}^{\infty} \ell_2^{-2s} \sum_{\substack{rt_2=\ell_2 w/(w,q/w) \\ t_2|(n/g)}} \mu(r) t_2 e\Big(-\frac{(\overline{\ell_2 r})u_{f,g}}{(w,q/w)}\Big(\frac{n/g}{t_2}\Big)\Big).$$

Reporting this last result in (3.15), and (thence) in (3.12), we obtain

$$\varphi_{c\infty}(Dn,s) = \sum_{f|D} \sum_{g|(D/f,n)} \mu(g)(fg)^{1-2s} \Psi_{f,g},$$

where, by comparison of (3.16) and (3.8),

$$\Psi_{f,g} = \frac{(w,q/w)^s}{q^s w^s} \Phi_{f,g} = \varphi_{c(f,g)\infty}(n/g,\, s)$$

(note that $\mathfrak{c}(f,g)$ of (3.11) is always a rational of the form (2.1)). Although this completes the proof, we think it interesting to observe that, with regard to (3.11), $\Phi_{f,g}$ in (3.16) only depends on the residue class of $\overline{f}$ modulo $(w, q/w)$. Almost the same conclusion follows from Lemmas 2.1 and 3.1, which show that there is no change in (3.10) when $\overline{f}$ (in (3.11)) is replaced by any integer coprime to w and congruent to $\overline{f}$ modulo $(w, q/w)$. ∎

Proof of Theorem 1.4. As $(q, D) = 1$ is assumed, we may apply (1.24) to the case $\mathfrak{a} = \infty$ of the sum over n in (1.33). This shows

$$\sum_{N/2<n\leqslant N} b_n\psi_{jk}^*(\infty,\, Dn) = \sum_{N/2<n\leqslant N} b_n \sum_{g|(n,D)} \mu(g)\lambda_{jk}^*\left(\frac{D}{g}\right)\psi_{jk}^*\left(\infty,\, \frac{n}{g}\right) =$$
$$= \sum_{g|D}\mu(g)\lambda_{jk}^*\left(\frac{D}{g}\right) \sum_{N/2g<n\leqslant N/g} b_{gn}\psi_{jk}^*(\infty,\, n).$$

Bounding the last sum through the Cauchy-Schwarz Inequality and (1.21) (Deligne's bound), we have

$$\left|\sum_{N/2<n\leqslant N} b_n\psi_{jk}^*(\infty,\, Dn)\right|^2 \leqslant \sum_{h|D}\tau^2(h)\sum_{g|D}\mu^2(g)\left|\sum_{N/2g<n\leqslant N/g} b_{gn}\psi_{jk}^*(\infty,\, n)\right|^2,$$

which, by (1.33) and (1.26), leads us to conclude that

$$\mathcal{S}_{\infty,q,K}^{(0)}(\mathbf{b}^{\langle D\rangle}, DN) \leqslant \tau^3(D)\sum_{g|D}\mu^2(g)\mathcal{S}_{\infty,q,K}^{(0)}(\mathbf{b}^{\{g\}}, N/g), \tag{3.17}$$

where $b_n^{\{g\}} = b_{gn}$ for $n \in \mathbb{N}$. Therefore Theorem 1.3 of Deshouillers and Iwaniec applies, showing (since $\mu(\infty) = 1/q$):

$$\mathcal{S}_{\infty,q,K}^{(0)}(\mathbf{b}^{\langle D\rangle}, DN) \ll_\varepsilon \tau^3(D)\sum_{g|D}\left(K^2 + q^{-1}(N/g)^{1+\varepsilon}\right)\|\mathbf{b}_{N/g}^{\{g\}}\|_2^2.$$

The case $i = 0$ of (1.37) now follows trivially, since $N/g \leqslant N$ and $\|\mathbf{b}_{N/g}^{\{g\}}\|_2 \leqslant \|\mathbf{b}_N\|_2$ (see (1.29) and (1.52)).

The bound (1.36) follows along very similar lines (using (1.23), the Cauchy--Schwarz inequality, (1.20) and the case $i = 1$ of Theorem 1.3). The only novelty is the factor n^{ϑ} in Kim and Sarnak's bound (1.20), which leads to the upper bound being weaker by a factor of $D^{2\vartheta}$ than the corresponding upper bound in (1.37).

We turn now to the remaining case of the bound (1.37), in which $i = 2$. By (1.35) (with $\mathfrak{a} = \infty$), Lemmas 2.1 and 3.1, we are led to consider the sum

$$\mathcal{L}_{\mathfrak{c}}(r) = \sum_{N/2<n\leqslant N} b_n n^{ir}\varphi_{\mathfrak{c}\infty}\left(Dn, \tfrac{1}{2} + ir\right) \tag{3.18}$$

in cases where $\mathfrak{c} = u/w$ is of the form (2.1). Therefore, Lemma 3.4 may be applied to rewrite $\mathcal{L}_{\mathfrak{c}}(r)$ as

$$\sum_{N/2<n\leqslant N} b_n n^{ir} \sum_{g|(D,n)} \sum_{f|(D/g)} \mu(g)(fg)^{-2ir} \varphi_{\mathfrak{c}(f,g)\infty}\left(n/g, \tfrac{1}{2}+ir\right),$$

where $\mathfrak{c}(f,g)$ is given by (3.11). By bringing the summation over n inside the other summations, we find that

$$\mathcal{L}_{\mathfrak{c}}(r) = \sum_{g|D} \mu(g) g^{-ir} \sum_{f|(D/g)} f^{-2ir} \mathcal{L}^{(g)}_{\mathfrak{c}(f,g)}(r),$$

where

$$\mathcal{L}^{(g)}_{\mathfrak{a}}(r) = \sum_{N/2g<n'\leqslant N/g} b_{gn'}(n')^{ir} \varphi_{\mathfrak{a}\infty}\left(n', \tfrac{1}{2}+ir\right). \tag{3.19}$$

By the Cauchy-Schwarz inequality (and some trivial bounds) it follows that

$$|\mathcal{L}_{\mathfrak{c}}(r)|^2 \leqslant \tau^2(D) \sum_{g|D} \mu^2(g) \sum_{f|(D/g)} \left|\mathcal{L}^{(g)}_{\mathfrak{c}(f,g)}\right|^2. \tag{3.20}$$

Given $f, g \in \mathbb{N}$ with $fg|D$, consider, with (3.11) in mind, two cusps ,

$$\mathfrak{a}(f,g) = \frac{(D/fg)\overline{f}u_1}{w_1} \qquad \text{and} \qquad \mathfrak{b}(f,g) = \frac{(D/fg)\overline{f}u_2}{w_2}, \tag{3.21}$$

where $u_1/w_1 = \mathfrak{a}$ and $u_2/w_2 = \mathfrak{b}$ are both cusps of the same form as u/w in Lemma 2.1. Here, by our assumptions, $(u_1, w_1) = (u_2, w_2) = 1$, $w_1, w_2|q$, $(D,q) = 1$ and $f|D$, so that both the rationals shown in (3.21) are, as they stand, well-defined reduced rationals of the same form as the rationals u/w in Lemma 2.1. Therefore (2.2) of Lemma 2.1 shows that $\mathfrak{a}(f,g) \overset{\Gamma}{\sim} \mathfrak{b}(f,g)$ if and only if

$$w_2 = w_1 \qquad \text{and} \qquad (D/fg)\overline{f}u_2 \equiv (D/fg)\overline{f}u_1 \pmod{(w_1,\, q/w_1)}.$$

On multiplying through by $(\overline{D/fg})f$, the latter of these two conditions reduces to just $u_2 \equiv u_1 \pmod{(w_1,\, q/w_1)}$, and so, after referring once more to (2.2) of Lemma 2.1), we have

$$\mathfrak{a}(f,g) \overset{\Gamma}{\sim} \mathfrak{b}(f,g) \Leftrightarrow \mathfrak{a} \overset{\Gamma}{\sim} \mathfrak{b}.$$

By this and Lemma 3.1, it follows that

$$\sum_{\mathfrak{c}}^{\Gamma} \left|\mathcal{L}^{(g)}_{\mathfrak{c}(f,g)}\right|^2 = \sum_{\mathfrak{c}}^{\Gamma} \left|\mathcal{L}^{(g)}_{\mathfrak{c}}\right|^2 \qquad (f, g \in \mathbb{N} \text{ with } fg|D). \tag{3.22}$$

Applying (1.35), (3.18) and (3.20), and making a straightforward change to the order in which the summations and integration are carried out, we deduce from (3.22), (3.19), (1.35) and (1.28) that

$$\mathcal{S}^{(2)}_{\infty,q,K}\Big(\mathbf{b}^{(D)}, DN\Big) \leqslant \tau^2(D)\sum_{g|D}\mu^2(g)\tau(D/g)\mathcal{S}^{(2)}_{\infty,q,K}\Big(\mathbf{b}^{\{g\}}, N/g\Big)\,, \tag{3.23}$$

where $b_n^{\{g\}} = b_{gn}$, for $n \in \mathbb{N}$. Bounding the right-hand side here, by an appeal to Theorem 1.3 (and trivial bounds), we complete our proof of Theorem 1.4. ■

4. Bruggeman-Kuznetsov summation and a form of reduction

In this section we furnish ourselves with several lemmas useful in subsequent sections: the last three of these enabling us to work around the condition $(n,q)=1$ (or $(D,q)=1$) attached to the results (1.24), (1.23) (or Lemma 3.4), which are our means to exploit the multiplicative nature of the relevant Fourier coefficients. As Theorems 2.2 and 2.3 are indispensable for certain proofs, it is expedient not to give separate consideration to each of the sums $\mathcal{S}^{(1)}_{\infty,q,K}(\mathbf{b}^{(D)}, DN)$ and $\mathcal{S}^{(2)}_{\infty,q,K}(\mathbf{b}^{(D)}, DN)$. We work instead with the 'combined' sum $\sigma_{q,K}(\mathbf{b}, N; D, y)$ of (1.42). The surplus parameter y in $\sigma_{q,K}(\mathbf{b}, N; D, y)$ anticipates a technical step in the proof of Lemma 8.2.

Lemma 4.1. *Let $M > 0$, $q \in \mathbb{N}$ and take $\mathbf{a} = (a_n)$ to be any complex sequence. Then, for $H > 0$,*

$$\mathcal{J}_0 = \frac{\cosh(1/H)}{2\sinh^2(1/H)}\|\mathbf{a}_M\|_2^2 + \sum\sum_{M/2<m,n\leqslant M} \overline{a}_m a_n \sum_{\ell=1}^{\infty}\frac{1}{q\ell}E_H\left(\frac{\sqrt{mn}}{q\ell}\right)S(m,n;q\ell),$$

where

$$\mathcal{J}_0 = \sum_{k \text{ even}} \frac{(k-1)!}{(4\pi)^{k-1}}e^{-(k-1)/H}\sum_{j=1}^{\theta_k(q)}\left|\sum_{M/2<n\leqslant M} a_n\psi^*_{jk}(\infty,n)\right|^2$$

and, for $x>0$,

$$E_H(x) = 2\pi\sum_{r=1}^{\infty}(-1)^r(2r-1)e^{-(2r-1)/H}J_{2r-1}(4\pi x).$$

Furthermore, for $H, x > 0$, one has here

$$E_H(x) \in \mathbb{R} \qquad \text{and} \qquad |E_H(x)| \leqslant 4\pi^2 x\cosh(2\pi x)\exp\left((2\pi x)^2\right). \tag{4.1}$$

Proof. The first part of this lemma is a result established within the proof of [7], Proposition 4, essentially by multiplying both sides of the result of Theorem 2.3 by $(k-1)e^{-(k-1)/H}\overline{a}_m a_n$, and then summing over $m, n \in \mathbb{N} \cap (M/2, M]$ and positive even k. Here we need only the case $\mathfrak{a} = \mathfrak{b} = \infty$, so that an appeal to (2.7) shows that the relevant Kloosterman sums are just the classical ones found in our lemma.

For the second part of the lemma we employ the series representation (2.9) for the Bessel functions that appear in the definition of $E_H(x)$. Following that by a change in the order of summations, we obtain

$$E_H(x) = 2\pi \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!}(2\pi x)^{2\ell} \sum_{r=1}^{\infty} \frac{(-1)^r(2r-1)}{(\ell+2r-1)!} e^{-(2r-1)/H}(2\pi x)^{2r-1},$$

which implies (for both choices of sign),

$$\pm E_H(x) \leqslant 4\pi^2 x e^{-1/H} \left(\sum_{\ell=0}^{\infty} \frac{1}{\ell!}(2\pi x)^{2\ell} \right) \left(\sum_{r=1}^{\infty} \frac{1}{(2r-2)!} \left(\frac{2\pi x}{e^{1/H}} \right)^{2r-2} \right)$$

yielding (4.1) as an immediate consequence. ∎

Lemma 4.2. *Let $M > 0$, $q \in \mathbb{N}$ and take $\mathbf{a} = (a_n)$ to be any complex sequence. Then, for $H \geqslant 1$,*

$$\mathcal{I}_1 + \mathcal{I}_2 = \frac{1}{\sqrt{4\pi}} H^3 \|\mathbf{a}_M\|_2^2 + \sqrt{4\pi} \sum\sum_{M/2<m,n\leqslant M} \overline{a}_m a_n \sum_{\ell=1}^{\infty} \frac{1}{q\ell} \Phi_H\!\left(\frac{4\pi\sqrt{mn}}{q\ell} \right) S(m,n;q\ell),$$

where

$$\mathcal{I}_1 = \sum_{j\geqslant 1}^{(q)} R_H(\kappa_j) \frac{1}{\cosh(\pi\kappa_j)} \left| \sum_{M/2<n\leqslant M} a_n \rho_{j\infty}(n) \right|^2,$$

$$\mathcal{I}_2 = \frac{1}{\pi} \sum_{\mathfrak{c}}^{\Gamma} \int_{-\infty}^{\infty} R_H(r) \left| \sum_{M/2<n\leqslant M} a_n n^{ir} \varphi_{\mathfrak{c}\infty}\left(n, \tfrac{1}{2}+ir\right) \right|^2 \mathrm{d}r,$$

and, for $|\mathrm{Im}(r)| < 1/2$ and $x > 0$,

$$R_H(r) = \int_{-\infty}^{\infty} t \sinh(\pi t) e^{-(t/H)^2} \mathcal{H}(r,t) \mathrm{d}t \tag{4.2}$$

and

$$\Phi_H(x) = \frac{x}{\sqrt{4\pi}} \int_{-\infty}^{\infty} \mathcal{D}_{2it}(x) t \sinh(\pi t) e^{-(t/H)^2} \mathrm{d}t, \tag{4.3}$$

with $\mathcal{H}(r,t)$ and $\mathcal{D}_{2it}(x)$ given by (2.17) and (2.18) of Theorem 2.4.

Proof. Repeating the first steps (on [7], page 260) in the proof of [7], Theorem 2, we apply Theorem 2.4 for cusps $\mathfrak{a} = \mathfrak{b} = \infty$, multiply all terms in the result by

$$t \sinh(\pi t) e^{-(t/H)^2} \overline{a}_m a_n,$$

sum over $m, n \in \mathbb{N} \cap (M/2, M]$, and integrate over $t \in (-\infty, \infty)$. The lemma then follows by (2.7) and evaluation of the simplest of the resulting integrals. ■

Lemma 4.3. *Let* $0 \leqslant \delta < 1$. *Then, for* $H \geqslant 1$ *and* $r \in \mathbb{C}$ *satisfying*

$$r^2 \geqslant -(\delta/4)^2, \tag{4.4}$$

one has

$$(|r| + 1)\, e^{-(|r|/H)^2} \ll_\delta R_H(r) \ll_\delta (|r| + 1)\, e^{-\frac{\pi}{2}|r|/H}.$$

Proof. We note first that, by (2.17),

$$\mathcal{H}(r, t) = \frac{2\pi \cosh(\pi r)}{\cosh(2\pi r) + \cosh(2\pi t)}.$$

If $r \in \mathbb{R}$, then it follows immediately that

$$\mathcal{H}(r, t) \asymp e^{-\pi(|t| + ||t| - |r||)}. \tag{4.5}$$

If $r \notin \mathbb{R}$, then (by (4.4)) $ir \in \mathbb{R}$ and $0 < |r| \leqslant \delta/4$, so that

$$\cosh(2\pi r) = \cos(2\pi i r) \in [\cos(\delta\pi/2), 1] \subset (0, 1]$$

and, similarly, $\cosh(\pi r) \in (1/\sqrt{2}, 1]$. Therefore, even if $r \notin \mathbb{R}$, we still obtain (4.5), provided that we allow the implicit constants there to depend upon δ. Using this conclusion in (4.2), we find

$$R_H(r) \asymp_\delta \int_0^\infty t \tanh(\pi t) e^{-(t/H)^2 - \pi|t - |r||} \mathrm{d}t. \tag{4.6}$$

If $|r| \leqslant 2$, then (4.6) shows:

$$R_H(r) \gg_\delta \int_0^1 t^2 \mathrm{d}t \gg 1,$$

confirming the lemma's lower bound on $R_H(r)$. That bound also holds for $|r| > 2$, since (4.6) then implies

$$R_H(r) \gg_\delta \int_{|r|-1}^{|r|} t e^{-(t/H)^2 - \pi(|r| - t)} \mathrm{d}t \geqslant (|r| - 1)\, e^{-(|r|/H)^2 - \pi}.$$

To establish the upper bound on $R_H(r)$, it suffices to note that (4.6) implies

$$R_H(r) \ll_\delta e^{-\pi|r|}\int_0^{|r|} te^{-(t/H)^2+\pi t}\mathrm{d}t + e^{\pi|r|}\int_{|r|}^{\infty} te^{-(t/H)^2-\pi t}\mathrm{d}t <$$
$$< e^{-\pi|r|}|r|\left(\int_0^{|r|/2} e^{\pi t}\mathrm{d}t + e^{-(|r|/2H)^2}\int_{|r|/2}^{|r|} e^{\pi t}\mathrm{d}t\right) + e^{\pi|r|-(|r|/H)^2}\int_{|r|}^{\infty} te^{-\pi t}\mathrm{d}t <$$
$$< e^{-\pi|r|}|r|\left(e^{\frac{\pi}{2}|r|} + e^{-(|r|/2H)^2+\pi|r|}\right) + e^{-(|r|/H)^2}\left(|r|+1\right) \ll$$
$$\ll \left(|r|+1\right)\left(e^{-\frac{\pi}{2}|r|} + e^{-(|r|/2H)^2}\right),$$

where $\frac{\pi}{2}|r| \geqslant \frac{\pi}{2}|r|/H$. ■

Lemma 4.4. *Let $H > 0$. Then, in respect of the domain $(0,\infty)$, Equations (4.3) and (1.48) define the same function, $\Phi_H(x)$, and this function satisfies:*

$$\Phi_H(x) \in \mathbb{R} \qquad \text{and} \qquad |\Phi_H(x)| \leqslant \frac{\sqrt{\pi}}{4}x \qquad (x > 0).$$

Proof. These are results found, and used, in the proof of [7], Theorem 2. Here we shall merely expand on points covered in [7], p. 260. We shall start from (4.3) and deduce (1.48).

The first step is to observe that, by (2.18) and a bound (derived trivially from (2.13)) for $|K_{2it}(e^{i\theta})|$, the definition (4.3) may be rewritten:

$$\Phi_H(x) = \lim_{\delta\to 1-} \frac{-ix}{\sqrt{\pi}} \int_{-\infty}^{\infty} \oint_{\mathcal{C}(\delta)} K_{2it}(x\nu)\frac{\mathrm{d}\nu}{\nu} t^2 e^{-(t/H)^2}\mathrm{d}t,$$

where

$$\mathcal{C}(\delta) = \left\{e^{i\theta} \ : \ -\delta\pi/2 \leqslant \theta \leqslant \delta\pi/2\right\}.$$

The restriction from $\mathcal{C}(1)$ to $\mathcal{C}(\delta)$ (where $\delta < 1$) makes the integral in the expression given for $K_{2it}(x\nu)$ by (2.14) uniformly absolutely convergent, so that, on applying (2.14) in the last expression for $\Phi_H(x)$, we may change the order of integration to obtain:

$$\Phi_H(x) = \lim_{\delta\to 1-} \frac{-ix^2}{2\sqrt{\pi}} \int_0^{\infty} I_\delta(x;\xi) J_H(t)\sinh(\xi)\mathrm{d}\xi,$$

where

$$I_\delta(x;\xi) = \oint_{\mathcal{C}(\delta)} e^{-x\cosh(\xi)\nu}\mathrm{d}\nu, \qquad J_H(\xi) = \int_{-\infty}^{\infty} \sin(2t\xi)te^{-(t/H)^2}\mathrm{d}t.$$

Integrating by parts and applying a result discussed in [2], Exercise 10.22,

$$J_H(\xi) = H^2\xi \int_{-\infty}^{\infty} \cos(2t\xi)e^{-(t/H)^2}\mathrm{d}t = \sqrt{\pi}H^3\xi e^{-(H\xi)^2}.$$

The factor $e^{-(H\xi)^2}$ here helps with convergence, so that the bound

$$|I_\delta(x;\xi) - I_1(x;\xi)| < \oint_{\mathcal{C}(1)-\mathcal{C}(\delta)} |\mathrm{d}\nu| = \pi(1-\delta),$$

allows the limit, in our last expression for $\Phi_H(x)$, to be taken inside the integral there. The proof may then be completed by noting that $2ix^{-1}\mathrm{sech}(\xi)\sin(x\cosh(\xi)) = I_1(x;\xi)$, so that, given the evaluation of $J_H(\xi)$, the last expression we had for $\Phi_H(x)$ reduces to (1.48).

By (1.48), $\Phi_H(x)$ is real-valued for $x > 0$. Using trivial bounds for factors in the integrand in (1.48), one also finds:

$$|\Phi_H(x)| \leqslant H^3 x \int_0^{\infty} \xi^2 e^{-(H\xi)^2}\mathrm{d}\xi,$$

for $x > 0$, which reduces to the bound claimed by the lemma. ∎

Lemma 4.5. *Let* $\mathrm{c}, f \in \mathbb{N}$ *and* $m, n \in \mathbb{Z}$. *Then*

$$S(fm, fn; f\mathrm{c}) = \phi_{\mathrm{c}}(f)S(m, n; \mathrm{c}),$$

where

$$\phi_{\mathrm{c}}(f) = f \prod_{\substack{p \text{ prime} \\ p|f,\ (p,\mathrm{c})=1}} \left(1 - \frac{1}{p}\right) = \sum_{\substack{g|f \\ (g,\mathrm{c})=1}} \mu(g)\frac{f}{g}. \tag{4.7}$$

Proof. By (2.8) and cancellation of common factors,

$$S(fm, fn; f\mathrm{c}) = \sum_{d \bmod f\mathrm{c}}^{*} \mathrm{e}\left(m\frac{\bar{d}}{\mathrm{c}} + n\frac{d}{\mathrm{c}}\right) = \sum_{d \bmod \mathrm{c}}^{*} \mathrm{e}\left(m\frac{\bar{d}}{\mathrm{c}} + n\frac{d}{\mathrm{c}}\right) \sum_{\substack{x \bmod f\mathrm{c} \\ x \equiv d \ (\bmod \mathrm{c})}}^{*} 1.$$

The sum over x is, for d coprime to c,

$$\sum_{\substack{x \bmod f\mathrm{c} \\ x\equiv d \ (\bmod \mathrm{c})}} \sum_{g|(f,x)} \mu(g) = \sum_{\substack{g|f \\ (g,\mathrm{c})=1}} \mu(g) \sum_{\substack{x \bmod f\mathrm{c} \\ x\equiv d \ (\bmod \mathrm{c}) \\ x\equiv 0 \ (\bmod g)}} 1.$$

This last sum is $\phi_{\mathrm{c}}(f)$ (see (4.7)), which does not depend on d. The lemma therefore follows, using (2.8) for the sum over d. ∎

Lemma 4.6. *Let $\Lambda \in \mathbb{R}$, $\delta, N_1 > 0$ and take $\mathbf{c} = (c_n)$ to be any complex sequence. Suppose also that $E(x)$ is a real-valued function with domain $(0, \infty)$, and is such that*

$$x^{-\frac{1}{2}-\delta}E(x) \longrightarrow 0 \qquad \text{as} \quad x \to 0+\,. \tag{4.8}$$

Then the equation,

$$\alpha^+(x,y) = \Lambda + \sum\sum_{N_1/2<m,n\leqslant N_1} \bar{c}_m c_n \sum_{\ell=1}^{\infty} \frac{1}{x\ell} E\left(\frac{y\sqrt{mn}}{x\ell}\right) S(ym, yn; x\ell), \tag{4.9}$$

defines a real-valued function $\alpha^+ : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{R}$ such that, for $q, D \in \mathbb{N}$,

$$\alpha^+(q,D) = \sum_{\substack{g|(q,D)\\(g,q')=1}} \frac{\mu(g)}{g} \sum_{f|g} \mu(f)\alpha^+(q'f, D') = \tag{4.10}$$

$$= \sum_{\substack{f|(q,D)\\(f,q')=1}} \alpha^+(q'f, D')\frac{\mu^2(f)}{f} \prod_{\substack{p\text{ prime}\\p|(q,D)\\(p,fq')=1}} \left(1 - \frac{1}{p}\right), \tag{4.11}$$

where $q' = q/(q,D)$, $D' = D/(q,D)$, and it is moreover the case that

$$\alpha^+(q,D) \leqslant \alpha^+(q_*, D_*), \tag{4.12}$$

for some pair $q_*, D_* \in \mathbb{N}$ satisfying

$$\frac{q}{(q,D)}\Big|q_*\Big|q, \qquad D_*\Big|\frac{D}{(q,D)} \qquad \text{and} \qquad (q_*, D_*) = 1. \tag{4.13}$$

Proof. From (4.8) and the Weil bound, Theorem 2.1, one easily finds that, for given $x, y, m, n \in \mathbb{N}$,

$$\ell^{\delta/2}E\left(\frac{y\sqrt{mn}}{x\ell}\right) S(ym, yn; x\ell) \longrightarrow 0 \qquad \text{as} \quad \ell \to \infty,$$

which shows, by a comparison with the series $\sum_{\ell=1}^{\infty} \ell^{-1-\delta/2}$, that the sum over ℓ in (4.9) converges. As the sum over m and n there is finite, it follows that the right-hand side of (4.9) is defined. As the coefficients $\bar{c}_m, c_n$ are the only factors on the right-hand side of (4.9) that might not be real, one can immediately observe (using $S(a,b;c) = S(b,a;c)$) that the expression there is invariant under complex-conjugation, so that (4.9) does make $\alpha^+(x,y)$ a real valued function on the domain $\mathbb{N} \times \mathbb{N}$.

Suppose now that $q, D \in \mathbb{N}$. Lemma 4.5 shows that, for $m, n, \ell \in \mathbb{N}$,

$$S(Dm, Dn; q\ell) = S(D'm, D'n; q'\ell)(q, D) \sum_{\substack{g \mid (q,D) \\ (g, q'\ell)=1}} \frac{\mu(g)}{g},$$

where q', D' are as under (4.10)-(4.11). Using this last result in (4.9) and cancelling some common factors, we may rewrite $\alpha^+(q, D)$ as:

$$\Lambda + \sum_{\substack{g \mid (q,D) \\ (g,q')=1}} \frac{\mu(g)}{g} \sum_{N_1/2<m,n\leqslant N_1}\sum \bar{c}_m c_n \sum_{\substack{\ell=1 \\ (\ell,g)=1}}^{\infty} \frac{1}{q'\ell} E\left(\frac{D'\sqrt{mn}}{q'\ell}\right) S(D'm, D'n; q'\ell).$$

We next attach an extra coefficient $\sum_{f \mid (\ell,g)} \mu(f)$ to each term of the last sum over ℓ, which does not change the value of the sum, but does make the explicit condition $(\ell, g) = 1$ superfluous. Since

$$\sum_{\substack{g \mid (q,D) \\ (g,q')=1}} \frac{\mu(g)}{g} \sum_{f \mid g} \mu(f)\Lambda = \Lambda \frac{\mu(1)}{1} = \Lambda, \tag{4.14}$$

one obtains (4.10) on bringing the summations over m, n and ℓ inside the summation over f and recognising that $q'f$, D' and ℓ/f take the places of x, y and ℓ in (4.9).

To show that (4.11) follows from (4.10) we first observe that, for any function $H : \mathbb{N} \longrightarrow \mathbb{C}$,

$$\begin{aligned}
\sum_{\substack{g \mid (q,D) \\ (g,q')=1}} \frac{\mu(g)}{g} \sum_{f \mid g} \mu(f) H(f) &= \sum_{\substack{f \mid (q,D) \\ (f,q')=1}} \mu(f) H(f) \sum_{\substack{g \mid (q,D) \\ (g,q')=1 \\ g \equiv 0 \ (\mathrm{mod}\ f)}} \frac{\mu(g)}{g} = \\
&= \sum_{\substack{f \mid (q,D) \\ (f,q')=1}} \mu(f) H(f) \sum_{\substack{g' \mid (q/f, D/f) \\ (g',q')=1}} \frac{\mu(fg')}{fg'} = \\
&= \sum_{\substack{f \mid (q,D) \\ (f,q')=1}} \frac{\mu^2(f)}{f} H(f) \sum_{\substack{g' \mid (q,D) \\ (g',fq')=1}} \frac{\mu(g')}{g'}.
\end{aligned} \tag{4.15}$$

Observing that the last sum over g' is equal to the product over primes p in (4.11), we obtain (4.11) on choosing $H(f) = \alpha^+(q'f, D')$. The last observation also informs us that, in the last summation over f of (4.15), the non-zero coefficients of $H(f)$ are all positive. One can evaluate the sum of these positive coefficients by applying first (4.15), for $H(f)$ identically equal to 1 (say), and then the case $\Lambda = 1$ of (4.14). This reveals that the non-zero coefficients sum to 1, so that, given their positivity and the fact that $\alpha^+(x,y)$ is real valued, we deduce from (4.10) and the case $H(f)=\alpha^+(q'f,D')$ of (4.15) (or from (4.11)) that

$$\alpha^+(q,D)\leqslant \max_{\substack{f\text{ squarefree}\\ f|(q,D)\\ (f,q')=1}} \alpha^+(q'f,D'),$$

so that

$$\alpha^+(q,D)\leqslant \alpha^+(q_1,D_1), \tag{4.16}$$

for some pair $q_1,D_1\in\mathbb{N}$ satisfying

$$\frac{q}{(q,D)}\Big| q_1|q \qquad\text{and}\qquad D_1=\frac{D}{(q,D)}. \tag{4.17}$$

Applying (4.16)-(4.17) in an iterative fashion, we augment (4.16) by a sequence of inequalities,

$$\alpha^+(q_1,D_1)\leqslant \ldots \leqslant \alpha^+(q_k,D_k)\leqslant \alpha^+(q_{k+1},D_{k+1})\leqslant \ldots \tag{4.18}$$

(say), where

$$\frac{q_k}{(q_k,D_k)}\Big| q_{k+1}|q_k \quad\text{and}\quad D_{k+1}=\frac{D_k}{(q_k,D_k)} \qquad (k\in\mathbb{N}). \tag{4.19}$$

By (4.19) and (4.17) we have (for $k\in\mathbb{N}$) $D_k|D_{k-1}|\ldots|D_2|D_1=D/(q,D)$, so that

$$D_k\Big|\frac{D}{(q,D)} \qquad (k\in\mathbb{N}). \tag{4.20}$$

By this and by (4.19),

$$\mathbb{N}\ni\frac{q_k}{q_{k+1}}\Big|(q_k,D_k)\,\Big|D_k\Big|\frac{D}{(q,D)} \qquad (k\in\mathbb{N}),$$

so that, as $(q/(q,D),D/(q,D))=1$, we must have

$$\left(\frac{q_k}{q_{k+1}},\frac{q}{(q,D)}\right)=1 \qquad (k\in\mathbb{N}).$$

Therefore, starting from the premise that $q/(q,D)$ is a factor of q_1 (for which see (4.17)), one can show by induction that $(q/(q,D))|(q_1,\ldots,q_n)$. This, together with (4.17), (4.19) and (4.20), permits us to conclude that

$$\frac{q}{(q,D)}\Big|q_k|q \qquad\text{and}\qquad D_k\Big|\frac{D}{(q,D)} \qquad (k\in\mathbb{N}).$$

We search through the sequence of pairs, q_1, D_1, q_2, D_2, ... , for a pair q_*, D_* satisfying (4.13). By our last result above, all we need to look for is any pair q_j, D_j with $(q_j, D_j) = 1$. If $(q_j, D_j) > 1$ for $j = 1, \ldots, k$ (say), then we move on to examine the next pair, q_{k+1}, D_{k+1}. Since (4.19) then implies $D_{k+1} \leqslant D_k/2$, this search procedure cannot be repeated indefinitely without success, since that would contradict 1 being a lower bound for $\{D_1, D_2, \ldots\} \subset \mathbb{N}$. Therefore our search must eventually succeed. By (4.18) and (4.16), whatever pair (q_*, D_*) we find will satisfy (4.12) (as well as (4.13)), so that all the claims of the lemma have now been shown true. ∎

Lemma 4.7. *Let $N > 0$, $q, D \in \mathbb{N}$ and take $\mathbf{b} = (b_n)$ to be any complex sequence. Then, given $H \geqslant 2$, there exist $q_*, D_* \in \mathbb{N}$ satisfying* (4.13) *and such that*

$$\mathcal{S}^{(0)}_{\infty,q,H}\left(\mathbf{b}^{\langle D\rangle}, DN\right) \leqslant \frac{1}{H}\int_2^\infty \mathcal{S}^{(0)}_{\infty,q_*,G}\left(\mathbf{b}^{\langle D_*\rangle}, D_*N\right) e^{1-G/H} \mathrm{d}G.$$

Proof. We first apply Lemma 4.1 with $\mathbf{a} = \mathbf{b}^{\langle D\rangle}$ and $M = DN$, so that the sum $\mathcal{J}_0$ (defined in that lemma) must satisfy

$$\mathcal{J}_0 \geqslant e^{-(H-1)/H} \mathcal{S}^{(0)}_{\infty,q,H}\left(\mathbf{b}^{\langle D\rangle}, DN\right) \tag{4.21}$$

(see (1.26)). With $N_1 = N$, $\mathbf{c} = \mathbf{b}$, $E(x) = E_H(x)$ and

$$\Lambda = \frac{\cosh(1/H)}{2\sinh^2(1/H)} \|\mathbf{b}_N\|_2^2, \tag{4.22}$$

Lemma 4.1 and (1.31) show $\mathcal{J}_0 = \alpha^+(q, D)$, where $\alpha^+(x, y)$ is the function defined in (4.9) of Lemma 4.6. As (4.1) shows that (4.8) holds with $\delta = 1/3$, we may appeal to (4.12)-(4.13) and (4.9) of Lemma 4.6 for the bound:

$$\mathcal{J}_0 \leqslant \Lambda + \sum\sum_{N/2<m,n\leqslant N} \overline{b}_m b_n \sum_{\ell=1}^\infty \frac{1}{q_*\ell} E_H\left(\frac{D_*\sqrt{mn}}{q_*\ell}\right) S(D_*m, D_*n; q_*\ell),$$

where $q_*, D_* \in \mathbb{N}$ satisfy (4.13). From this, (4.22), (1.31) and Lemma 4.1 (once more), we deduce that the term $\mathcal{J}_0$ above satisfies

$$\mathcal{J}_0 \leqslant \sum_{k \text{ even}} \frac{(k-1)!}{(4\pi)^{k-1}} e^{-(k-1)/H} \sum_{j=1}^{\theta_k(q_*)} \left| \sum_{D_*N/2<n\leqslant D_*N} b_n^{\langle D_*\rangle} \psi^*_{jk}(\infty, n)\right|^2.$$

As

$$\frac{1}{H}\int_k^\infty e^{1-G/H} \mathrm{d}G = e^{1-k/H} = e^{(H-1)/H-(k-1)/H},$$

it follows from (1.26) that the result we are seeking to prove is just what follows directly from (4.21) and our upper bound for $\mathfrak{I}_0$. ■

Lemma 4.8. *Let $N > 0$, $q, D \in \mathbb{N}$, $y \in \mathbb{R}$, and take $\mathbf{b} = (b_n)$ to be any complex sequence. Suppose that $D = D_1 D'$, where $D_1, D' \in \mathbb{N}$. Then, given $H, H_1 \in \mathbb{R}$ satisfying*

$$H \geqslant H_1 \geqslant 1, \tag{4.23}$$

there exist $q_*, D'_* \in \mathbb{N}$ satisfying

$$\frac{q}{(q,D')}|q_*|q, \qquad D'_*|\frac{D'}{(q,D')} \qquad \text{and} \qquad (q_*, D'_*) = 1, \tag{4.24}$$

and such that

$$\breve{\sigma}_q(H_1, H) \ll \int_0^\infty \sigma_{q_*,G}(\mathbf{b}, N; D'_* D_1, y) \frac{G}{H} e^{-(\pi/2)G/H} \mathrm{d}G \ll \tag{4.25}$$

$$\ll \sigma_{q_*,H}(\mathbf{b}, N; D'_* D_1, y) H + \int_H^\infty \frac{\sigma_{q_*,G}(\mathbf{b}, N; D'_* D_1, y)}{e^{G/H}} \mathrm{d}G, \tag{4.26}$$

where

$$\breve{\sigma}_q(H_1, H) = \sigma_{q,H_1}(\mathbf{b}, N; D, y) + (\sigma_{q,H}(\mathbf{b}, N; D, y) - \sigma_{q,H_1}(\mathbf{b}, N; D, y))H_1. \tag{4.27}$$

Proof. We shall consider only the case $y = 0$, since the first line of (1.44) shows that the general case follows (no special properties that the complex sequence $\mathbf{b}$ might have being required here). We begin by considering the results yielded by Lemmas 4.2 and 4.3 when the former lemma is applied for

$$\mathbf{a} = \mathbf{b}^{\langle D \rangle} \qquad \text{and} \qquad M = DN,$$

where $\mathbf{b}^{\langle D \rangle}$ is the sequence given by (1.31). In the sums defining $\mathfrak{I}_1$, $\mathfrak{I}_2$ of Lemma 4.2 one always has

$$\kappa_j^2 \geqslant -(7/64)^2 \qquad \text{and} \qquad r^2 \geqslant 0$$

(see (1.13)-(1.15)), so that, with $\delta = 7/16 \in (0,1]$ (for example) all the relevant factors of the form $R_H(r)$ (where, in this instance, r may represent κ_j) will satisfy the condition (4.4) sufficient for the application of Lemma 4.3. By that lemma's lower bound for $R_H(r)$ (or $R_H(\kappa_j)$), and by our choice of $\mathbf{a}$ and M, the sums $\mathfrak{I}_1$, $\mathfrak{I}_2$ must satisfy:

$$\mathfrak{I}_1 \gg \sum_{j \geqslant 1}^{(q)} \frac{(|\kappa_j| + 1)\, e^{-(|\kappa_j|/H)^2}}{\cosh(\pi \kappa_j)} \left| \sum_{DN/2 < n \leqslant DN} b_n^{\langle D \rangle} \rho_{j\infty}(n) \right|^2,$$

$$\mathcal{J}_2 \gg \sum_{\mathfrak{c}}^{\Gamma_0(q)} \int_{-\infty}^{\infty} (|r|+1)\, e^{-(r/H)^2} \left| \sum_{DN/2<n\leqslant DN} b_n^{\langle D\rangle} n^{ir} \varphi_{\mathfrak{c}\infty}\left(n,\, \tfrac{1}{2}+ir\right) \right|^2 dr.$$

Therefore, and by (1.27), (1.28) and (4.23), we find that, for $j = 1, 2$,

$$\mathcal{J}_j \gg \mathcal{S}^{(j)}_{\infty,q,H_1}\left(\mathbf{b}^{\langle D\rangle}, DN\right) + \left(\mathcal{S}^{(j)}_{\infty,q,H}\left(\mathbf{b}^{\langle D\rangle}, DN\right) - \mathcal{S}^{(j)}_{\infty,q,H_1}\left(\mathbf{b}^{\langle D\rangle}, DN\right)\right) H_1.$$

By (4.27), (1.44) and our non-negative lower bounds for $\mathcal{J}_1$ and $\mathcal{J}_2$,

$$\check{\sigma}_q\left(H_1, H\right) = O\left(\mathcal{J}_1\right) + O\left(\mathcal{J}_2\right) \ll \mathcal{J}_1 + \mathcal{J}_2. \tag{4.28}$$

Still considering the application of Lemma 4.2, with a and M as indicated above, we observe that, by (1.31), it yields:

$$\mathcal{J}_1 + \mathcal{J}_2 = \Lambda + \sum_{N/2<m,n\leqslant N}\sum \bar{b}_m b_n \sum_{\ell=1}^{\infty} \frac{1}{q\ell}\, E\!\left(\frac{D\sqrt{mn}}{q\ell}\right) S(Dm, Dn; q\ell),$$

where

$$\Lambda = \frac{1}{\sqrt{4\pi}}\, H^3 \left\|\mathbf{b}_N\right\|_2^2, \qquad E(x) = \sqrt{4\pi}\, \Phi_H(4\pi x). \tag{4.29}$$

We may rewrite the last sum of Kloosterman sums so as to conclude that

$$\mathcal{J}_1 + \mathcal{J}_2 = \alpha^+\left(q,\, D'\right), \tag{4.30}$$

where $\alpha^+(x, y)$ is the function given by (4.9) of Lemma 4.6, with Λ and $E(x)$ as in (4.29), and $\mathbf{c}$, N_1 chosen to satisfy:

$$\mathbf{c} = \mathbf{b}^{\langle D_1\rangle}, \quad \text{and} \qquad N_1 = D_1 N, \tag{4.31}$$

where $\mathbf{b}^{\langle D_1\rangle}$ is defined as in (1.31). By (4.29) and Lemma 4.4 the function $E(x)$ here satisfies the condition (4.8) with $\delta = 1/3 > 0$ (for example). Therefore it follows from (4.30), (4.31) and (4.12), (4.9) and (4.13) of Lemma 4.6 that

$$\begin{aligned}\mathcal{J}_1 + \mathcal{J}_2 &\leqslant \Lambda + \sum_{N_1/2<m,n\leqslant N_1}\sum \bar{c}_m c_n \sum_{\ell=1}^{\infty} \frac{1}{q_*\ell}\, E\!\left(\frac{D'_*\sqrt{mn}}{q_*\ell}\right) S\left(D'_* m, D'_* n; q_*\ell\right) = \\ &= \Lambda + \sum_{D_*N/2<m,n\leqslant D_*N}\sum \overline{b_m^{\langle D_*\rangle}}\, b_n^{\langle D_*\rangle} \sum_{\ell=1}^{\infty} \frac{1}{q_*\ell}\, E\!\left(\frac{\sqrt{mn}}{q_*\ell}\right) S(m, n; q_*\ell),\end{aligned}$$

where $D_* = D_1 D'_*$, $\mathbf{b}^{\langle D_*\rangle}$ is as in (1.31), and q_*, D'_* are some pair of natural number satisfying (4.24). As $\|\mathbf{b}_N\|_2 = \|\mathbf{b}^{\langle g\rangle}_{gN}\|_2$ for $g \in \mathbb{N}$ (see (1.29), (1.31)), it follows from (4.29) that the form of the last bound on $\mathcal{J}_1 + \mathcal{J}_2$ invites us to make a second application of Lemma 4.2 (with $\mathbf{a} = \mathbf{b}^{\langle D_* \rangle}$ and $M = D_* N$ now), following which we have:

$$\mathfrak{I}_1 + \mathfrak{I}_2 \leqslant \mathfrak{I}_1^{(*)} + \mathfrak{I}_2^{(*)}, \tag{4.32}$$

where

$$\mathfrak{I}_1^{(*)} = \sum_{j \geqslant 1}^{(q_*)} R_H(\kappa_j) \frac{1}{\cosh(\pi\kappa_j)} \left| \sum_{D_* N/2 < n \leqslant D_* N} b_n^{\langle D_* \rangle} \rho_{j\infty}(n) \right|^2,$$

$$\mathfrak{I}_2^{(*)} = \frac{1}{\pi} \sum_{\mathfrak{c}}^{\Gamma_0(q_*)} \int_{-\infty}^{\infty} R_H(r) \left| \sum_{D_* N/2 < n \leqslant D_* N} b_n^{\langle D_* \rangle} n^{ir} \varphi_{\mathfrak{c}\infty}\left(n, \tfrac{1}{2} + ir\right) \right|^2 \mathrm{d}r.$$

The upper bound on $R_H(r)$ of Lemma 4.3 shows that, for r satisfying the condition (4.4) of that lemma,

$$\int_{|r|}^{\infty} \frac{G}{H} e^{-\frac{\pi}{2} G/H} \, \mathrm{d}G \gg R_H(r).$$

This enables us to deduce from (1.44), (1.27) and (1.28) that we have

$$\int_0^{\infty} \sigma_{q_*,G}(\mathbf{b}, N; D_*, 0) \frac{G}{H} e^{-(\pi/2)G/H} \, \mathrm{d}G \gg \mathfrak{I}_1^{(*)} + \mathfrak{I}_2^{(*)}$$

here. As $D_* = D_1 D_*'$, this last result, together with (4.28) and (4.32), yields (4.24)-(4.25) for $y = 0$. The case $y = 0$ of the second bound, (4.26), follows trivially from (4.25), using both the bound $xe^{(1-\pi/2)x} < 2/(\pi - 2)$ for $x = G/H \geqslant 1$, and the fact that, by its definition in (1.42), the term $\sigma_{q_*,G}(\mathbf{b}, N; D_*, 0)$ represents a non-decreasing, non-negative valued, real function of G. ∎

Lemma 4.9. *Let $U > 1$, $N > 0$, $q, D \in \mathbb{N}$, $y \in \mathbb{R}$ and take $\mathbf{b} = (b_n)$ to be any complex sequence. Then, for $K \geqslant 1$,*

$$\sigma_{q,K}(\mathbf{b}, N; D, y) \leqslant \sum_{\substack{h=0 \\ U^h \leqslant K}}^{\infty} U^{-h} \breve{\sigma}_q\left(U^h, U^{h+1}\right),$$

where $\breve{\sigma}_q(H_1, H)$ is as defined in (4.27) of Lemma 4.8.

Proof. By (4.27),

$$U^{-0} \breve{\sigma}_q\left(U^0, U^{0+1}\right) = \breve{\sigma}_q(1, U) = \sigma_{q,U_0}(\mathbf{b}, N; D, y),$$

where $U_0 = U$, and, for $h \in \mathbb{N}$,

$$U^{-h} \breve{\sigma}_q\left(U^h, U^{h+1}\right) \geqslant \sigma_{q,U_h}(\mathbf{b}, N; D, y) - \sigma_{q,U_{h-1}}(\mathbf{b}, N; D, y),$$

where $U_j = U^{j+1}$ for $j \in \mathbb{N}$. These bounds enable a proof by induction that, for $k \in \mathbb{N}$,

$$\sum_{h=0}^{k-1} U^{-h} \breve{\sigma}_q \left(U^h, U^{h+1}\right) \geqslant \sigma_{q,U_{k-1}}(\mathbf{b}, N; D, y).$$

The lemma follows from the case $k-1 = [\log_U K]$ of this bound, since in this case one has $U_{k-1} = U^k > K \geqslant 1$, so that $k \in \mathbb{N}$ and (see (1.42)) $\sigma_{q,U_{k-1}}(\mathbf{b}, N; D, y) \geqslant \geqslant \sigma_{q,K}(\mathbf{b}, N; D, y)$. ∎

5. Multiplicativity revisited

By making use of Lemmas 4.7, 4.8 and 4.9 are able to deal with the sums $\sigma_{q,K}(\mathbf{b}, N; D, y)$ in cases where $(q, D) > 1$. Results worth noting in their own right are Lemmas 5.1 and 5.3. Proposition 1.1 is proved at the end of the section, as a simple corollary of Lemma 5.3.

Lemma 5.1. *Let $\varepsilon > 0$ and $\vartheta = 7/64$. Then, for $N > 0$, $q, D \in \mathbb{N}$, $K \geqslant 1$, and any complex sequence $\mathbf{b} = (b_n)$, one has:*

$$\mathcal{S}^{(0)}_{\infty,q,K}\left(\mathbf{b}^{(D)}, DN\right) \ll_\varepsilon \tau^4\!\left(\frac{D}{(q,D)}\right)\left(K^2 + \frac{(q,D)}{q} N^{1+\varepsilon}\right) \|\mathbf{b}_N\|_2^2 \tag{5.1}$$

and, for $y \in \mathbb{R}$,

$$\sigma_{q,K}(\mathbf{b}, N; D, y) \ll_\varepsilon \left(\frac{D}{(q,D)}\right)^{2\vartheta} \tau^4\!\left(\frac{D}{(q,D)}\right)\left(K^2 + \frac{(q,D)}{q} N^{1+\varepsilon}\right) \|\mathbf{b}_N\|_2^2. \tag{5.2}$$

Proof. By the coprimality condition of (4.13), Theorem 1.4 applies to bound the integrand on the right-hand side of the bound given by Lemma 4.7, so we are able to conclude that

$$\mathcal{S}^{(0)}_{\infty,q,K}\left(\mathbf{b}^{(D)}, DN\right) \ll \frac{1}{H}\int_2^\infty O_\varepsilon\left(\tau^4(D_*)\left(G^2 + \frac{1}{q_*}N^{1+\varepsilon}\right)\|\mathbf{b}_N\|_2^2\right) e^{-G/H}\, \mathrm{d}G.$$

Since (4.13) also shows we have

$$D_* \left| \frac{D}{(q,D)} \right. \qquad \text{and} \qquad \left. \frac{q}{(q,D)} \right| q_*, \tag{5.3}$$

here, the bound (5.1) follows trivially on noting that

$$\frac{1}{H}\int_2^\infty G^j e^{-G/H}\, \mathrm{d}G = H^j \int_{2/H}^\infty t^j e^{-t} \mathrm{d}t < j! H^j \qquad (j = 0, 1, \ldots). \tag{5.4}$$

It now only remains to prove (5.2), which is a task that the case $U = 2$ of Lemma 4.9 enables us to approach through consideration of terms $\breve{\sigma}_q(H_1, H)$ where $2 \leqslant H \leqslant 2K$ and $H_1 = H/2$. Given such a pair, H, H_1, we apply Lemma 4.8 with $D_1 = 1$, $D' = D$, obtaining (from (4.26)) a bound,

$$\breve{\sigma}_q(H/2, H) \ll H\sigma_{q_*,H}(\mathbf{b}, N; D_*, y) + \int_H^\infty \sigma_{q_*,G}(\mathbf{b}, N; D_*, y)\, e^{-G/H}\, \mathrm{d}G, \qquad (5.5)$$

where $q_*, D_* \in \mathbb{N}$ satisfy (5.3). Therefore Theorem 1.4 now applies (via (1.44), (1.45) and (1.31)), so we have:

$$\sigma_{q_*,G}(\mathbf{b}, N; D_*, y) \ll_\varepsilon D_*^{2\vartheta} \tau^4(D_*) \left(G^2 + q_*^{-1} N^{1+\varepsilon/2}\right) \|\mathbf{b}_N\|_2^2,$$

for $G \geqslant H$ (given that $H \geqslant 1$). From this, (5.3), (5.4) and (5.5), we conclude that

$$\frac{\breve{\sigma}_q(H/2, H)}{H/2} \ll_\varepsilon \left(\frac{D}{(q, D)}\right)^{2\vartheta} \tau^4\left(\frac{D}{(q, D)}\right) \left(H^2 + \frac{(q, D)}{q} N^{1+\varepsilon/2}\right) \|\mathbf{b}_N\|_2^2 .$$

Summing this last bound over $H = 2, 4, 8, \dots 2^{[\log_2 K]+1}$, gives a bound for the sum in Lemma 4.9, and so also for $\sigma_{q,K}(\mathbf{b}, N; D, y)$. To see that (5.2) is implied, note one either has

$$K \leqslant N \qquad \text{and} \qquad [\log_2 K] + 1 \ll_\varepsilon N^{\varepsilon/2},$$

or (given that $0 < \varepsilon < 1$ and $N \geqslant 1$),

$$K > N \qquad \text{and} \qquad \frac{(q, D)}{q} N^{1+\varepsilon/2} ([\log_2 K] + 1) \ll N^{3/2} K^{1/2} < K^2. \qquad \blacksquare$$

Lemma 5.2. *Let $\vartheta = 7/64$. Then, for $M > 0$, $G > 0$, $y \in \mathbb{R}$, any complex sequence $\mathbf{a} = (a_n)$, and $D_1, D', q \in \mathbb{N}$ with*

$$(D', q) = 1, \qquad (5.6)$$

we have:

$$\begin{aligned} &\sigma_{q,G}(\mathbf{a}, M; D'D_1, y) \leqslant \qquad (5.7)\\ &\leqslant (D')^{2\vartheta} \tau^3(D') \sum_{\substack{g_0 \mid D' \\ (g_0, D_1) = 1}} \sum_{g_1 \mid (D', D_1)} \mu^2(g_0 g_1)\, \sigma_{q,G}\left(\mathbf{a}^{\{g_0\}}, \frac{M}{g_0}; \frac{D_1}{g_1}, y\right), \end{aligned}$$

where $a_n^{\{g_0\}} = a_{g_0 n}$ for $n \in \mathbb{N}$.

Proof. As in the proof of Lemma 4.8, we need only to establish the case $y = 0$ here, since all other cases will then follow by the first identity of (1.44) and the observations made under (1.45). By (1.44) and (1.31),

$$\sigma_{q,G}(\mathbf{a}, M; D'D_1, y) = \mathcal{S}^{(1)}_{\infty,q,G}\left(\mathbf{b}^{\langle D' \rangle}, D'N\right) + \frac{1}{\pi} \mathcal{S}^{(2)}_{\infty,q,G}\left(\mathbf{b}^{\langle D' \rangle}, D'N\right), \qquad (5.8)$$

where

$$\mathbf{b} = \mathbf{a}^{\langle D_1 \rangle} \qquad \text{and} \qquad N = D_1 M. \tag{5.9}$$

Now referring back to the proof of Theorem 1.4, the coprimality condition (5.6) permits an appeal to both (3.23) and the non-holomorphic analog of (3.17) (a simple consequence of (1.23) and (1.20)). We find that, for $i = 1, 2$,

$$\mathcal{S}^{(i)}_{\infty,q,G}\Big(\mathbf{b}^{(D')},\, D'N\Big) \leqslant (D')^{2\vartheta}\tau^3(D') \sum_{g|D'} \mu^2(g)\mathcal{S}^{(i)}_{\infty,q,G}\Big(\mathbf{b}^{\{g\}},\, \frac{N}{g}\Big), \tag{5.10}$$

where $b_n^{\{g\}} = b_{gn}$ for $n \in \mathbb{N}$ (and where 0 may replace ϑ for $i = 2$). Given $g \in \mathbb{N}$, squarefree and satisfying $g|D'$, we define

$$g_1 = (g,\, D_1) \qquad \text{and} \qquad g_0 = g/g_1. \tag{5.11}$$

By (5.9), (1.31), (1.45) and (5.11), we have in (5.10):

$$\mathcal{S}^{(i)}_{\infty,q,G}\left(\mathbf{b}^{\{g\}},\, N/g\right) = \mathcal{S}^{(i)}_{\infty,q,G}\left(\mathbf{c}^{(D_1/g_1)},\, (D_1/g_1)(M/g_0)\right) \qquad (i = 1, 2),$$

where $\mathbf{c} = \mathbf{a}^{\{g_0\}}$. The bound (5.7) therefore follows by an application of (1.44) after both cases ($i = 1, 2$) of (5.10) are used with (5.8): note that each pair g_0, g_1 given by (5.11) occurs for only one g (i.e. $g = g_0 g_1$) and, given what is assumed about g above (5.11), the pair will satisfy all the conditions of summation in (5.7). ∎

Lemma 5.3. *Let $\varepsilon > 0$ and $\vartheta = 7/64$. Suppose that $N > 0$, $y \in \mathbb{R}$, $\mathbf{b} = (b_n)$ is a complex sequence and $D = D_1 D'$ with $D_1, D' \in \mathbb{N}$. Then, given $Q > 0$ and $K \geqslant 1$, one has*

$$S_{Q,K}(\mathbf{b}, N; D, y) \ll_\varepsilon (D')^{2\vartheta+\varepsilon} \int_1^\infty e^{-\frac{G}{2K}} S_{Q_1,G}\Big(\mathbf{b}^{\{g_0\}},\, \frac{N}{g_0};\, \frac{D_1}{g_1},\, y\Big)\, \frac{dG}{G}$$

(with $b_n^{\{g_0\}} = b_{g_0 n}$ for $n \in \mathbb{N}$), where g_0, g_1 are some pair of natural numbers satisfying

$$g_0|D', \quad (g_0,\, D_1) = 1 \qquad \text{and} \qquad g_1|(D',\, D_1), \tag{5.12}$$

and Q_1 is some real number with

$$\frac{Q}{2D'} < Q_1 \leqslant Q. \tag{5.13}$$

Proof. In view of (1.43) and the case $U = 2$ of Lemma 4.9, it is reasonable to begin by considering the sums $\breve{\sigma}_q(H_1, H)$ (given by (4.27) of Lemma 4.8) in cases where $Q < q \leqslant 2Q$ and $1 \leqslant H_1 \leqslant H \leqslant 2K$. In such cases Lemma 4.8 gives us the bound (4.25) for some pair $q_*, D'_* \in \mathbb{N}$ satisfying (4.24). As (4.24) implies that the hypothesis (5.6) of Lemma 5.2 will hold if one replaces q and D' there by q_* and D'_*, so Lemma 5.2 may be applied in the context of (4.25), showing:

$$\begin{aligned} \sigma_{q_*,G}(\mathbf{b},N;D'_*D_1,y) &\leqslant \qquad (5.14)\\ &\leqslant (D'_*)^{2\vartheta}\tau^3(D'_*)\sum_{\substack{g_0|D'_*\\(g_0,D_1)=1}}\ \sum_{g_1|(D'_*,D_1)}\mu^2(g_0g_1)\,\sigma_{q_*,G}\left(\mathbf{b}^{\{g_0\}},\frac{N}{g_0};\frac{D_1}{g_1},y\right)\leqslant\\ &\leqslant (D')^{2\vartheta}\tau^3(D')\sum_{\substack{g_0|D'\\(g_0,D_1)=1}}\ \sum_{g_1|(D',D_1)}\ \sum_{\substack{q_1|q\\(q/(q,D'))|q_1}}\sigma_{q_1,G}\left(\mathbf{b}^{\{g_0\}},\frac{N}{g_0};\frac{D_1}{g_1},y\right)\end{aligned}$$

(where we appeal to (4.24) for the last inequality). Given G, g_0 and g_1,

$$\begin{aligned}\sum_{Q/2<q\leqslant Q}\ \sum_{\substack{q_1|q\\(q/(q,D'))|q_1}}\sigma_{q_1,G}\left(\mathbf{b}^{\{g_0\}},\frac{N}{g_0};\frac{D_1}{g_1},y\right)&\leqslant\\ \leqslant\sum_{Q/2D'<q_1\leqslant Q}\sigma_{q_1,G}\left(\mathbf{b}^{\{g_0\}},\frac{N}{g_0};\frac{D_1}{g_1},y\right)\tau(D'),\end{aligned}$$

so it follows from (4.25), (5.14) and (1.43) that

$$\sum_{Q/2<q\leqslant Q}\breve{\sigma}_q(H_1,H)\ll(D')^{2\vartheta}\tau^4(D')\sum_{\substack{g_0|D'\\(g_0,D_1)=1}}\ \sum_{g_1|(D',D_1)}\ \sum_{\substack{r=0\\2^{r-1}<D'}}^{\infty}\mathcal{U}\left(\frac{Q}{2^r},H;g_0,g_1\right),$$

where

$$\mathcal{U}(R,H;g_0,g_1)=\int_0^\infty S_{R,G}\left(\mathbf{b}^{\{g_0\}},\frac{N}{g_0};\frac{D_1}{g_1},y\right)\frac{G}{H}e^{-\frac{\pi}{2}G/H}\,\mathrm{d}G.$$

Using this last bound with the case $U=2$ of Lemma 4.9 and (1.43) (again), we deduce:

$$S_{Q,K}(\mathbf{b},N;D,y)\ll(D')^{2\vartheta}\tau^4(D')\sum_{\substack{g_0|D'\\(g_0,D_1)=1}}\ \sum_{g_1|(D',D_1)}\ \sum_{\substack{r=0\\2^{r-1}<D'}}^{\infty}\mathcal{V}\left(\frac{Q}{2^r};g_0,g_1\right),\qquad(5.15)$$

where

$$\mathcal{V}(R;g_0,g_1)=\int_0^\infty S_{R,G}\left(\mathbf{b}^{\{g_0\}},\frac{N}{g_0};\frac{D_1}{g_1},y\right)\mathcal{W}_K(G)\frac{\mathrm{d}G}{1+G}\qquad(5.16)$$

and

$$\mathcal{W}_K(G)=(1+G)G\sum_{\substack{h=0\\2^h\leqslant K}}^{\infty}2^{-2h}e^{-\frac{\pi}{4}G/2^h}.$$

Here $\mathcal{W}_K(G) < 2/(1-1/4)$ if $0 < G \leqslant 1$, while, for $1 < G < K$ one has,

$$\mathcal{W}_K(G) \leqslant 2G^2 \left(\sum_{\substack{h=0\\ 2^h \leqslant G}}^{\infty} \frac{3!}{2^{2h}\left(\frac{\pi}{4}G/2^h\right)^3} + \sum_{\substack{h=0\\ 2^h > G}}^{\infty} \frac{1}{2^{2h}} \right) \ll 1,$$

and, for $G \geqslant K \geqslant 1$, there is the bound

$$\mathcal{W}_K(G) \leqslant 2G^2 \sum_{\substack{h=0\\ 2^h \leqslant K}}^{\infty} \frac{3!}{2^{2h}\left(\left(\frac{\pi}{4}-\frac{1}{2}\right)G/2^h\right)^3} e^{-G/2K} \ll G^{-1}Ke^{-\frac{1}{2}G/K}.$$

These bounds show $\mathcal{W}_K(G) \ll e^{-G/2K}$ for $G > 0$, so it follows from (5.15) and (5.16) that, for some $g_0, g_1 \in \mathbb{N}$ and $Q_1 \in \mathbb{R}$ satisfying (5.12) and (5.13), there is the bound

$$S_{Q,K}(\mathbf{b}, N; D, y) \ll (D')^{2\vartheta}\, \tau^5(D') \log(2D') \int_0^\infty S_{Q_1,G}\left(\mathbf{b}^{\{g_0\}}, \frac{N}{g_0}; \frac{D_1}{g_1}, y\right) \frac{e^{-\frac{G}{2K}}\, \mathrm{d}G}{1+G}.$$

The result of the lemma now follows quite directly, by virtue of the bounds $\tau(D') \ll_\varepsilon (D')^{\varepsilon/6}$ and $\log(2D') \ll_\varepsilon (D')^{\varepsilon/6}$, and the observations that $e^{-G/2K}/(1+ +G) \asymp 1$ for $0 < G \leqslant 2$ (and $K \geqslant 1$), and that $S_{Q_1,G}(\mathbf{b}^{\{g_0\}}, N/g_0; D_1/g_1, y)$ is a non-decreasing non-negative real valued function of G (see (1.42), (1.43)). ■

Proof of Proposition 1.1. Proposition 1.1 is a corollary of Lemma 5.3. To see this we note that, by (1.42), (1.43) and (5.2) of Lemma 5.1,

$$f(G) = \frac{1}{G^2} S_{Q_1,G}\left(\mathbf{b}^{\{g_0\}}, \frac{N}{g_0}; \frac{D_1}{g_1}, y\right)$$

is (when $\mathbf{b}$, N, D_1, y, Q_1, g_0 and g_1 are given) a bounded function from $[1,\infty)$ into $[0,\infty)$. Therefore, from the result of Lemma 5.3 we have:

$$S_{Q,K}(\mathbf{b}, N; D, y) \ll_\varepsilon (D/D_1)^{2\vartheta+\varepsilon} \int_1^\infty f(G) G e^{-G/2K}\, \mathrm{d}G \leqslant$$

$$\leqslant 2f(G)\,(D/D_1)^{2\vartheta+\varepsilon} \int_1^\infty g e^{-g/2K} \mathrm{d}g,$$

for some $G \geqslant 1$. Since the last integral here does not exceed $O(K^2)$, and since (5.12) and (5.13) (with $D' = D/D_1$) are just the conditions (1.51) and (1.50), this proves the proposition.

6. Preparation for the swapping of levels

In this section the primary object of concern is the term $\check{\sigma}_q(H_1, H)$ defined in (4.27) of Lemma 4.8. This is first bounded in terms of sums of Kloosterman sums. We then work to show that certain of these sums have no significant influence on the final outcome (our treatment of the remaining, less tractable, sums being postponed until Section 8). The first two lemmas provide certain prerequisite information about the function $\Phi_H(x)$ from (1.48).

Lemma 6.1. *Let $H \geqslant 1$. Then*

$$\Phi_H^{(j)}(x) \ll_j x^{2-j} \qquad (0 < x \leqslant 1 \text{ and } j = 0, 1, \ldots),$$

where $\Phi_H(x)$ is the real function given by (1.48).

Proof. In view of the general identity

$$\frac{\mathrm{d}^j}{\mathrm{d}x^j}(xf(x)) = xf^{(j)}(x) + jf^{(j-1)}(x),$$

and the restriction to $x \in (0,1]$, it will suffice here to establish that $f(x) = x^{-1}\Phi_H(x)$ satisfies

$$f(x) \ll x \qquad (x > 0) \tag{6.1}$$

and

$$f^{(j)}(x) \ll_j 1 \qquad (x > 0 \text{ and } j = 1, 2, \ldots). \tag{6.2}$$

For (6.1) we employ the bounds $|\sin(x\cosh(\xi))| \leqslant x\cosh(\xi) \leqslant xe^{\xi}$ and $0 \leqslant \tanh(\xi) \leqslant \xi$ for factors of the integrand in (1.48). This shows that, for $x > 0$,

$$|f(x)| \leqslant H^3 \int_0^\infty xe^{\xi}\xi^2 e^{-(H\xi)^2}\mathrm{d}\xi = \int_0^\infty xt^2 e^{t/H - t^2}\mathrm{d}t,$$

which implies (6.1) (given that $H \geqslant 1$).

For (6.2) we note that, by differentiating (with respect to x) inside the integral of (1.48), we have the identity:

$$f^{(j)}(x) = H^3 \int_0^\infty \mathcal{T}_j(x\cosh(\xi))\,\xi\tanh(\xi)\cosh^j(\xi)e^{-(H\xi)^2}\mathrm{d}\xi,$$

where $\mathcal{T}_j(u) = (\mathrm{d}^j/\mathrm{d}u^j)\sin(u)$, so that $|\mathcal{T}_j(u)| \leqslant 1$ here. Using this, together with the bounds $0 \leqslant \tanh(\xi) \leqslant \xi$ and $\cosh(\xi) \leqslant e^{\xi}$, we find:

$$\left|f^{(j)}(x)\right| \leqslant H^3 \int_0^\infty \xi^2 e^{j\xi - (H\xi)^2}\mathrm{d}\xi = \int_0^\infty t^2 e^{jt/H - t^2}\mathrm{d}t,$$

which (as $H \geqslant 1$) does yield (6.2), so completing the proof. ■

Lemma 6.2. *Let* $0<\delta<1/2$, $0<X\leqslant 1/2$ *and* $H\geqslant 1$. *Suppose that* $\Phi_{H,X}(x)$ *is the function given by* (1.56), (1.48) *and* (1.71) *(where the function* $\Omega(x)$ *is an infinitely differentiable real function satisfying* (1.69)-(1.70)*) and denote by* $\tilde{\Phi}_{H,X}(\ell)$ *and* $\hat{\Phi}_{H,X}(r)$ *the corresponding Bessel transforms defined in* (2.15) *and* (2.16) *of Theorem* 2.2. *Then* $\Phi_{H,X}(x)$ *is an infinitely differentiable real function of* x *satisfying*

$$\Phi_{H,X}(x)=0 \qquad (x\notin(X/2,2X)\subset(0,1)), \tag{6.3}$$

and one has:

$$\tilde{\Phi}_{H,X}(k-1)\ll X^2k^{-5/2} \quad (k=2,4,6,\ldots), \tag{6.4}$$
$$\hat{\Phi}_{H,X}(r)\ll_\delta X^{1+2\delta} \quad (r\in\mathbb{C} \text{ and } -(1/2-\delta)^2\leqslant r^2\leqslant 1), \tag{6.5}$$
$$\hat{\Phi}_{H,X}(r)\ll_j X^2|r|^{-j} \quad (r\in\mathbb{R}, |r|\geqslant 1 \text{ and } j=0,1,\ldots). \tag{6.6}$$

Proof. As $0<X\leqslant 1/2$, the result (6.3) follows from (1.57) and (1.56). Given that $H\geqslant 1$, the infinite differentiability of $\Omega_0(x)$, together with (1.56), permit us to conclude from Lemma 6.1 that, for $j=0,1,\ldots$,

$$\mathbb{R}\ni\Phi_{H,X}^{(j)}(x)\ll_j X^{2-j} \qquad (x\in[X/2,2X]\subset(0,1]). \tag{6.7}$$

By (6.3) and (6.7) the conditions for Lemma 2.4 (up to and including (2.19)) do hold, with $\phi(x)=\Phi_{H,X}(x)$, $F=O(X^2)$, $Y=1$, and the 'X' there equal to $X/2\in(0,1/4]$. Therefore (6.4) follows on applying (2.21) with $r=k-1\geqslant 1$. Moreover, for $-(1/2-\delta)^2\leqslant r^2<0$ one has $r=ir'$, where $r'\in\mathbb{R}$ and $0<|r'|\leqslant$ $\leqslant 1/2-\delta$, so that (2.22) of Lemma 2.4 is satisfied with the 'r' there equal to r'. Therefore, for such r, we may apply (2.23) to obtain:

$$\hat{\Phi}_{H,X}(r)\ll X^{2-2|r|}\left(\delta^{-1}+\min\left(|r|^{-1},\log(2/X)\right)\right)\ll_\delta$$
$$\ll_\delta\begin{cases} X^{2-(\frac12-\delta)}\left(1+(2/X)^{\frac12-\delta}\right), & \text{if } 0<|r|<\frac12\left(\frac12-\delta\right),\\ X^{2-2(\frac12-\delta)}, & \text{if } \frac12\left(\frac12-\delta\right)\leqslant|r|\leqslant\frac12-\delta,\end{cases}$$

enabling us to conclude that (6.5) holds in the cases where $r\notin\mathbb{R}$. The remaining cases of (6.5) follow from the bound (2.20) of Lemma 2.4, since (given that $X\leqslant 1/2$) one has $1\ll\log(2/X)\ll_\delta(1/X)^{1-2\delta}$.

It now only remains to prove (6.6). We begin by using the power-series expansion (2.9) for the Bessel functions $J_{\pm 2ir}(x)$ in (2.16). This shows that, for $r\in\mathbb{C}-\{0,\pm i,\pm 2i,\ldots\}$,

$$\hat{\Phi}_{H,X}(r)=\frac{\pi}{\sinh(\pi r)}\int_0^\infty\left(\sum_{\ell=0}^\infty\frac{(-1)^\ell x^{2\ell}}{\ell!2^{2\ell}}\operatorname{Im}\left(\frac{(x/2)^{2ir}}{\Gamma(\ell+1+2ir)}\right)\right)\Phi_{H,X}(x)\frac{\mathrm{d}x}{x}= \tag{6.8}$$
$$=\sum_{\ell=0}^\infty\frac{(-1)^\ell}{\ell!2^{2\ell}}\mathfrak{F}_\ell(r),$$

where

$$\mathfrak{F}_\ell(r) = \frac{\pi}{\sinh(\pi r)} \int_0^\infty \frac{1}{2i} \left(\frac{2^{-2ir} x^{2\ell-1+2ir}}{\Gamma(\ell+1+2ir)} - \frac{2^{2ir} x^{2\ell-1-2ir}}{\Gamma(\ell+1-2ir)} \right) \Phi_{H,X}(x) \mathrm{d}x$$

(with interchange of integration being justified by the uniform absolute convergence of the series for $x \in [X/2, 2X]$). Integration by parts j times here yields:

$$\mathfrak{F}_\ell(r) = \frac{(-1)^j \pi}{\sinh(\pi r)} \int_0^\infty \operatorname{Im}\left(\frac{2^{-2ir} x^{2\ell+j-1+2ir} \Gamma(2\ell+2ir)}{\Gamma(\ell+1+2ir)\Gamma(2\ell+j+2ir)} \right) \Phi_{H,X}^{(j)}(x) \mathrm{d}x.$$

We bound the imaginary part here by its absolute value, which Lemma 2.5 shows to be:

$$x^{2\ell+j-1} \left(\prod_{n=2\ell}^{2\ell+j-1} \frac{1}{|n+2ir|} \right) \sqrt{\frac{\sinh(2\pi r)}{2\pi r}} \left(\prod_{m=1}^{\ell} \frac{1}{|m+2ir|} \right) \leqslant$$
$$\leqslant x^{2\ell+j-1} |2r|^{-\ell-j} \sqrt{\frac{\sinh(2\pi r)}{2\pi r}}.$$

From this, (6.3) and (6.7), it follows that

$$\mathfrak{F}_\ell(r) \ll_j \frac{1}{\sinh(\pi r)} \sqrt{\frac{\sinh(2\pi r)}{2\pi r}} |2r|^{-\ell-j} \int_{X/2}^{2X} x^{2\ell+j-1} \cdot x^{2-j} \mathrm{d}x,$$

so that, for $j = 0, 1, \ldots$,

$$\mathfrak{F}_\ell(r) \ll_j |2r|^{-\ell-\frac{1}{2}-k} X^{2+2\ell} \qquad (|r| \geqslant 1 \text{ and } \ell = 0, 1, \ldots).$$

Using this last bound in (6.8) shows that

$$\hat{\Phi}_{H,X}(r) \ll_j |r|^{-(j+\frac{1}{2})} X^2 \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left(\frac{X^2}{|8r|} \right)^\ell \ll_j$$
$$\ll_j |r|^{-(j+\frac{1}{2})} X^2 \exp\left(X^2/8|r| \right),$$

which contains (6.6) (given that $0 < X \leqslant 1/2$). This completes the proof. ■

Lemma 6.3. *Let $N > 0$, $q, D \in \mathbb{N}$, $y \in \mathbb{R}$ and take $\mathbf{b} = (b_n)$ to be any complex sequence. Then, given $H, H_1 \in \mathbb{R}$ satisfying*

$$H \geqslant H_1 \geqslant 1, \tag{6.9}$$

we have the bound

$$\breve{\sigma}_q(H_1, H) \ll \frac{1}{4\pi} H^3 \|\mathbf{b}_N\|_2^2 + \alpha_{q,H}(\mathbf{b}, N; D, y), \tag{6.10}$$

where the terms on the right and left sides are defined through (1.46), (1.48), (4.27) *and* (1.42). *Moreover, for* $W = 2^u$ *and* $Y = 2^v$ *with* $u, v \in \mathbb{Z}$ *such that* $W \leqslant 1/2$ *and*

$$4\pi DN/q \leqslant Y \leqslant 32\pi DN/q, \tag{6.11}$$

we have here

$$\alpha_{q,H}(\mathbf{b}, N; D, y) = \sum_{\substack{r \in \mathbb{Z} \\ W < X_r = 2^r \leqslant Y}} \alpha_{q,H,X_r}(\mathbf{b}, N; D, y) + O\left(\frac{W}{q} N^2 D \|\mathbf{b}_N\|_2^2\right), \tag{6.12}$$

where $\alpha_{q,H,X}(\mathbf{b}, N; D, y)$ *is given by* (1.54), (1.56) *and* (1.71) (*see also* (1.69)-(1.70)).

Proof. In view of Lemma 4.4, the case $y = 0$ of (6.10) was already established in the course of our proof of Lemma 4.8 (see (4.28)-(4.29) and note that the assumptions made there match up with our current assumptions). The case $y = 0$ of (6.10) implies (6.10) for any $y \in \mathbb{R}$, as the substitution of $\mathbf{b}(y) = (b_n n^{iy})$ for $\mathbf{b}$ transforms the former result into the latter (see (1.42) and (1.46)).

In deducing (6.12) (which will complete our proof of the lemma) we must keep in mind the pertinent definitions: (1.46), (1.69)-(1.70), (1.71), (1.54) and (1.56). Given (6.11), it follows from (1.69) that one may think of the terms summed in (1.46) as having an unseen coefficient, $\Omega(4\pi D\sqrt{mn}/Yq\ell)$, which equates to 1 for the relevant values of m, n, q and ℓ. We can then replace that coefficient by another, $\Omega(4\pi D\sqrt{mn}/Wq\ell)$, through the identity:

$$\Omega(x/Y) - \Omega(x/W) = \sum_{r=u+1}^{v} \left(\Omega\left(2^{-r}x\right) - \Omega\left(2^{-(r-1)}x\right)\right) = \sum_{\substack{r \in \mathbb{Z} \\ W < X_r = 2^r \leqslant Y}} \Omega_0(x/X_r),$$

where the function $\Omega_0(x)$ is given by (1.71). This identity is valid for $x > 0$ if $W \leqslant Y$, which is the only case we need consider (the case $W \in (Y, 1/2]$ of (6.12) being an immediate consequence of (6.12) for $W = Y$).

Following the application of the above identity, and a change in the order of summation, one finds (see (1.54) and (1.56)) that the sum over r on the right of (6.12) equals $\alpha_{q,H}(\mathbf{b}, N; D, y) - \alpha(W)$, where

$$\alpha(W) = \sum\sum_{N/2 < m,n \leqslant N} \bar{b}_m b_n \left(\frac{m}{n}\right)^{-iy} \sum_{\ell=1}^{\infty} \frac{1}{q\ell} F\left(\frac{4\pi D\sqrt{mn}}{q\ell}\right) S(Dm, Dn; q\ell),$$

with $F(x) = \Omega(x/W)\Phi_H(x)$. Therefore (6.12) will follow if we can show

$$\alpha(W) \ll \frac{W}{q} N^2 D \|\mathbf{b}_N\|_2^2. \tag{6.13}$$

By (1.69)-(1.70) and Lemma 6.1 we have here (given that $W \leqslant 1/2$),

$$F(x) = \begin{cases} O(x^2), & \text{if } 0 < x < 2W, \\ 0, & \text{if } x \geqslant 2W, \end{cases}$$

which, together with the trivial bound $|S(a,b;c)| \leqslant c$, allows us to conclude that

$$\alpha(W) \ll \sum_{N/2<m,n\leqslant N}\sum |b_m b_n| \sum_{\ell \geqslant D\sqrt{mn}/2qW} \left(\frac{D\sqrt{mn}}{q\ell}\right)^2 \ll \frac{DW}{q} \sum_{N/2<m,n\leqslant N}\sum |b_m b_n| \sqrt{mn}.$$

Since (6.13) follows from this by the arithmetic-geometric mean inequality (and some trivial bounds) our proof of Lemma 6.3 is complete. ■

Lemma 6.4. *Let* $\vartheta = 7/64$, $\varepsilon > 0$ *and* $j \in \mathbb{N}$. *Then, for* $N > 0$, $q, D \in \mathbb{N}$, $H, G \geqslant 1$, $0 < X \leqslant 1/2$, $y \in \mathbb{R}$ *and any complex sequence* $\mathbf{b} = (b_n)$, *one has*

$$\alpha_{q,H,X}(\mathbf{b}, N; D, y) \ll X^{2-2\vartheta} \sigma_{q,G}(\mathbf{b}, N; D, y) + \\ + O_\varepsilon\left(\frac{\mathcal{A}_\varepsilon(q) X^2}{q}\right)\left(1 + O_j\left(\frac{(D/X)^{2\vartheta}}{G^j}\right)\right),$$

where

$$\mathcal{A}_\varepsilon(q) = \tau^4(D)\left(q + (q,D)N^{1+\varepsilon}\right) \|\mathbf{b}_N\|_2^2, \tag{6.14}$$

and where the other terms are as in (1.54), (1.56), (1.71), (1.69)-(1.70) *and* (1.42).

Proof. We will treat only the case $y = 0$, since the other cases follow on substitution of $\mathbf{b}(y) = (b_n n^{iy})$ in place of $\mathbf{b}$ (see the definitions in (1.42) and (1.54)).

By (1.54) and (2.7),

$$\alpha_{q,H,X}(\mathbf{b}, N; D, 0) = \sum_{N/2<m,n\leqslant N}\sum \bar{b}_m b_n \sigma_{mn},$$

where

$$\sigma_{mn} = \sum_{\gamma}^{\Gamma_0(q)} \frac{1}{\gamma} S_{\infty\infty}(Dm, Dn; \gamma) \Phi_{H,X}\left(\frac{4\pi\sqrt{(Dm)(Dn)}}{\gamma}\right).$$

By Lemma 6.2 (with any $\delta \in (0, 1/2)$) we see that $\phi(x) = \Phi_{H,X}(x)$ satisfies the (few) hypotheses of Theorem 2.2. The case $\mathfrak{a} = \mathfrak{b} = \infty$ of that theorem therefore applies, showing that the sum σ_{mn} above is equal to another sum, $\mathcal{K}_0' + \mathcal{K}_1' + \mathcal{K}_2'$, where, for $i = 0, 1, 2$, the expression $\mathcal{K}_i'$ differs from the corresponding expression $\mathcal{K}_i$ (in Theorem 2.2) only in that where $\mathcal{K}_i$ has 'm', or 'n', the expression $\mathcal{K}_i'$ has instead 'Dm', or 'Dn' (respectively). By (6.4)-(6.6) of Lemma 6.2, Theorem 1.3, and Weyl's law (for the discrete spectrum), the sums and integrals defining $\mathcal{K}_0'$, $\mathcal{K}_1'$ and $\mathcal{K}_2'$ are absolutely convergent, so that we may bring the (finite) summation over m, n inside the other summations, to obtain:

$$\alpha_{q,H,X}(\mathbf{b}, N; D, 0) = \mathcal{R}_0 + \mathcal{R}_1 + \mathcal{R}_2, \tag{6.15}$$

where

$$\mathcal{R}_0 = \frac{1}{2\pi} \sum_{k \text{ even}} \sum_{j=1}^{\theta_k(q)} \frac{i^k (k-1)!}{(4\pi)^{k-1}} \tilde{\Phi}_{H,X}(k-1) \left| \sum_{N/2<n\leqslant N} b_n \psi_{jk}^*(\infty, Dn) \right|^2 ,$$

$$\mathcal{R}_1 = \sum_{j\geqslant 1}^{(q)} \frac{1}{\cosh(\pi\kappa_j)} \hat{\Phi}_{H,X}(\kappa_j) \left| \sum_{N/2<n\leqslant N} b_n \rho_{j\infty}(Dn) \right|^2 ,$$

$$\mathcal{R}_2 = \frac{1}{\pi} \sum_{\mathfrak{c}}^{\Gamma_0(q)} \int_{-\infty}^{\infty} \hat{\Phi}_{H,X}(r) \left| \sum_{N/2<n\leqslant N} b_n n^{ir} \varphi_{\mathfrak{c}\infty}\left(Dn, \tfrac{1}{2}+ir\right) \right|^2 \mathrm{d}r.$$

By (1.14)-(1.15) we may apply Lemma 6.2 here, with $\delta = 1/2 - \vartheta = 25/64 \in (0, 1/2)$ in (6.5). Then, by (6.4)-(6.6) of that lemma, we have

$$\tilde{\Phi}_{H,X}(k-1) \ll X^2 \int_k^\infty K^{-7/2} \mathrm{d}K \qquad (k = 2, 4, 6, \ldots)$$

and (given that $0 < X \leqslant 1/2$),

$$\hat{\Phi}_{H,X}(r) \ll X^{2-2\vartheta} \left(1 + \mathrm{c}_j |r|^j\right)^{-1} = X^{2-2\vartheta} \int_{|r|}^\infty \frac{j\mathrm{c}_j K^{j-1}}{\left(1 + \mathrm{c}_j K^j\right)^2} \mathrm{d}K,$$

where c_j is some small positive constant (depending only on j), and r can be any real number, or any value taken by κ_j in the summation of $\mathcal{R}_1$. Applying these bounds to obtain upper bounds for the absolute values of $\mathcal{R}_0$, $\mathcal{R}_1$ and $\mathcal{R}_2$, and then rewriting the results by combining the outcomes for $\mathcal{R}_1$ and $\mathcal{R}_2$, and bringing all other summations, or integrations, inside the integration over K, we find (see (1.26), (1.31) and (1.42)):

$$\mathcal{R}_0 \ll X^2 \int_2^\infty K^{-7/2} \mathcal{S}^{(0)}_{\infty,q,K}\left(\mathbf{b}^{\langle D\rangle}, DN\right) \mathrm{d}K,$$

$$\mathcal{R}_1 + \mathcal{R}_2 \ll X^{2-2\vartheta} \int_0^\infty \frac{j\mathrm{c}_j K^{j-1}}{\left(1 + c_j K^j\right)^2} \sigma_{q,K}(\mathbf{b}, N; D, 0)\, \mathrm{d}K.$$

By (5.1) and (5.2) of Lemma 5.1, it now follows that

$$\mathcal{R}_0 \ll_\varepsilon X^2 \tau^4(D) \left\|\mathbf{b}_N\right\|_2^2 \int_2^\infty \left(K^{-3/2} + \frac{(q,D)}{q} N^{1+\varepsilon} K^{-7/2}\right) \mathrm{d}K \ll$$

$$\ll q^{-1} X^2 \tau^4(D) \left\|\mathbf{b}_N\right\|_2^2 \left(q + (q,D) N^{1+\varepsilon}\right)$$

and that, since $G \geqslant 1$, $j \in \{3,4,5,\ldots\}$, and $\sigma_{q,K}(\mathbf{b},N;D,0)$ is a non-decreasing, non-negative real function of K (see (1.42)),

$$\begin{aligned}
\mathcal{R}_1 + \mathcal{R}_2 &\ll X^{2-2\vartheta}\Bigg(\int_0^G \frac{jc_jK^{j-1}}{(1+c_jK^j)^2}\,\sigma_{q,K}(\mathbf{b},N;D,0)\,\mathrm{d}K + \\
&\quad + \int_G^\infty \frac{j}{c_jK^{j+1}} O_\varepsilon\Big(D^{2\vartheta}\tau^4(D)\Big(K^2 + \frac{(q,D)}{q}N^{1+\varepsilon}\Big)\,\|\mathbf{b}_N\|_2^2\Big)\mathrm{d}K\Bigg) \ll \\
&\ll X^{2-2\vartheta}\sigma_{q,G}(\mathbf{b},N;D,0)\int_0^\infty \frac{jc_jK^{j-1}}{(1+c_jK^j)^2}\,\mathrm{d}K + \\
&\quad + O_\varepsilon\Big(\tau^4(D)\Big(1+\frac{(q,D)}{q}N^{1+\varepsilon}\Big)\,\|\mathbf{b}_N\|_2^2\Big)\frac{jX^2D^{2\vartheta}}{c_jX^{2\vartheta}}\int_G^\infty \frac{\mathrm{d}K}{K^{j-1}} = \\
&= X^{2-2\vartheta}\sigma_{q,G}(\mathbf{b},N;D,0) + \\
&\quad + O_\varepsilon\Big(\tau^4(D)\left(q+(q,D)N^{1+\varepsilon}\right)\|\mathbf{b}_N\|_2^2\Big)\frac{X^2}{q}\left(\frac{D}{X}\right)^{2\vartheta}\frac{jG^{2-j}}{(j-2)c_j}.
\end{aligned}$$

As j may be replaced by $j+2$ throughout (with the new j lying in $\mathbb{N}$), so this bound for $\mathcal{R}_1+\mathcal{R}_2$, the last bound for $\mathcal{R}_0$, and the identity (6.15), together yield the result claimed by the lemma. ∎

Lemma 6.5. *Let $\vartheta = 7/64$ and suppose that $\Delta \in (0,1/2]$ is sufficiently small in absolute terms. Then, for $\varepsilon > 0$, $j \in \mathbb{N}$, $N > 0$, $q, D \in \mathbb{N}$, $y \in \mathbb{R}$, any complex sequence $\mathbf{b} = (b_n)$, $H, H_1 \in \mathbb{R}$ satisfying (6.9), and $Y = 2^v$ (with $v \in \mathbb{Z}$) satisfying (6.11), one has an inequality of the form:*

$$\breve{\sigma}_q(H_1,H) \ll O_\varepsilon\Big(\frac{A_\varepsilon(q)}{q}\Big)\left(1 + O_j\left(\frac{D^{2\vartheta}}{H^j}\right)\right) + H^3\|\mathbf{b}_N\|_2^2 + \sum_{\substack{r\in\mathbb{Z}\\ \Delta < X_r = 2^r \leqslant Y}} \alpha_{q,H,X_r}(\mathbf{b},N;D,y),$$

where the terms on the left and right sides are defined through (4.27), (1.42), (6.14), (1.54), (1.56), (1.48) and (1.71) (and see also (1.69)-(1.70)).

Proof. By (6.10) and (6.12) of Lemma 6.3, we obtain

$$\breve{\sigma}_q(H_1,H) \ll H^3\|\mathbf{b}_N\|_2^2 + \sum_{\substack{r\in\mathbb{Z}\\ W < X_r = 2^r \leqslant Y}} \alpha_{q,H,X_r}(\mathbf{b},N;D,y), \tag{6.16}$$

on choosing $W = 2^u$ with $W < \Delta$ and W sufficiently small in terms of N and D: the O-term from (6.12) being properly accounted for here, as (6.16) omits the factor $1/4\pi$ shown in (6.10).

Given $\Delta \in (0, 1/2]$, the case $G = H$ of Lemma 6.4 shows:

$$\sum_{\substack{r\in\mathbb{Z} \\ X_r = 2^r \leqslant \Delta}} |\alpha_{q,H,X_r}(\mathbf{b}, N; D, y)| \ll \Delta^{2-2\vartheta} \sigma_{q,H}(\mathbf{b}, N; D, y) + \\ + O_\varepsilon\!\left(\frac{\mathcal{A}_\varepsilon(q)}{q}\right)\left(1 + O_j\left(\frac{D^{2\vartheta}}{H^j}\right)\right),$$

where $\mathcal{A}_\varepsilon(q)$ is given by (6.14). As W was chosen with $W < \Delta$, this last bound and (6.16) together imply an inequality,

$$\breve{\sigma}_q(H_1, H) \leqslant C_1 \Bigg(H^3 \|\mathbf{b}_N\|_2^2 + C_2 \Delta^{2-2\vartheta} \sigma_{q,H}(\mathbf{b}, N; D, y) + \\ + O_\varepsilon\!\left(\frac{\mathcal{A}_\varepsilon(q)}{q}\right)\left(1 + O_j\left(\frac{D^{2\vartheta}}{H^j}\right)\right) + \sum_{\substack{r\in\mathbb{Z} \\ \Delta < X_r = 2^r \leqslant Y}} \alpha_{q,H,X_r}(\mathbf{b}, N; D, y) \Bigg), \tag{6.17}$$

where $C_1, C_2 \geqslant 1$ are certain absolute constants. Given that Δ is chosen sufficiently small, and that $H \geqslant H_1 \geqslant 1$, we will have here:

$$0 \leqslant C_1 C_2 \Delta^{2-2\vartheta} \sigma_{q,H}(\mathbf{b}, N; D, y) \leqslant \frac{1}{2}\sigma_{q,H}(\mathbf{b}, N; D, y) \leqslant \frac{1}{2}\breve{\sigma}_q(H_1, H)$$

(see (4.27) and (1.42)), so that the lemma follows from (6.17). ∎

7. Bessel transforms: a special case

In Lemmas 7.1, 7.3 and 7.6, we obtain bounds for Bessel transforms of the function

$$\phi(x) = x^{it}\Phi_{H,X}(x), \tag{7.1}$$

where $t \in \mathbb{R}$, $H \geqslant 1$ and $X > \Delta$ are given (see (1.56), (1.57) and (1.48) regarding $\Phi_{H,X}(x)$, and (2.15), (2.16) for the transforms). These bounds are needed for our work in the next section.

Lemma 7.1. *Let $0 < \delta < 1/2$, $H \geqslant 1$, $t \in \mathbb{R}$ and $X > \Delta$, where Δ is a positive absolute constant. Suppose that $\phi(x)$ is given by (7.1). Then*

$$\phi(x) = 0 \qquad (x \notin (X/2, 2X)), \tag{7.2}$$

and ϕ is an infinitely differentiable function from $\mathbb{R}$ into $\mathbb{C}$. Moreover,

$$\hat{\phi}(r), \tilde{\phi}(n) \ll (1 + |t|)\, X\, (1 + |\log(X)|) \qquad (r \in \mathbb{R},\ n \in \mathbb{N}), \tag{7.3}$$

$$\hat{\phi}(r) \ll_\delta (1 + |t|)\, X \qquad (r \in \mathbb{C} \text{ and } -\left(\tfrac{1}{2} - \delta\right)^2 \leqslant r^2 < 0), \tag{7.4}$$

and, for $r \in \mathbb{R}$,

$$(1+|t|)^2 X^{9/2} |r|^{-5/2} \gg \begin{cases} \hat{\phi}(r) & \text{if } |r| \geqslant 1, \\ \tilde{\phi}(r) & \text{if } r \in \mathbb{N}. \end{cases} \tag{7.5}$$

Proof. The property (7.2) follows by (7.1), (1.56) and (1.57) (the integral in (1.48) being convergent for all real x). An argument already used in the proof of Lemma 6.1 shows that $x^{-1}\Phi_H(x)$ is an infinitely differentiable real function, so that the infinite differentiability of $\phi(x)$ follows by (1.56), (7.1), (7.2) and the infinite differentiability of $\Omega_0(x)$ and x^{it+1} on the interval $[X/3, 3X]$ (for example).

By (7.1), (1.56), the infinite differentiability of $\Omega_0(x)$, and (1.48), one finds that

$$\begin{aligned} \phi(x) &\ll X|f(x)|, \\ \phi'(x) &\ll X\,|f'(x)| + (1+|t|)\,|f(x)|, \\ \phi''(x) &\ll X\,|f''(x)| + (1+|t|)\,|f'(x)| + (1+|t|)^2 X^{-1} |f(x)|, \end{aligned}$$

where $f(x) = x^{-1}\Phi_H(x)$. For $H \geqslant 1$ and $j = 0, 1, 2$, one has

$$\int_0^\infty \cosh^j(\xi)\xi \tanh(\xi) e^{-(H\xi)^2} d\xi \leqslant \int_0^\infty \xi^2 e^{j\xi - H^2\xi^2} d\xi \ll_j H^{-3},$$

allowing us to deduce from (1.48) and the last set of bounds for ϕ, ϕ' and ϕ'' that

$$\|\phi\|_\infty \ll X, \quad \|\phi'\|_1 \ll (1+|t|+X)\,X \quad \text{and} \quad \|\phi''\|_1 \ll (1+|t|+X)^2 .$$

Therefore Lemma 2.4 applies, with some pair $F, Y > 0$ satisfying

$$F \ll (1+|t|+X)\,X, \qquad Y \ll 1+|t|+X,$$

and with 'X' there equal to $X/2$. By this application of Lemma 2.4 we obtain results, (2.20), (2.21) and (2.22)-(2.23), which trivially imply (7.3), (7.5) and (7.4) (respectively), by virtue of the conditions $0 < \delta < 1/2$ and $X > \Delta$ (where $\Delta \gg 1$).

Lemma 7.2. *Let* $a < b$, $V > 0$ *and* $0 < M \leqslant 1$. *Suppose that* $f(x)$ *and* $g(x)$ *are infinitely differentiable real functions such that*

$$g(x) = 0 \qquad (x \notin (a,b))$$

and, for $a < x < b$,

$$g^{(j)}(x) \ll_j M^{-j} \qquad (j = 0, 1, \ldots), \tag{7.6}$$

$$f^{(j)}(x) \ll_j V/M \qquad (j = 2, 3, \ldots) \tag{7.7}$$

and

$$|f'(x)| \geqslant V. \tag{7.8}$$

Then

$$\int_{-\infty}^{\infty} g(x)e^{if(x)}\mathrm{d}x \ll_j (b-a)(VM)^{-j} \qquad (j=0,1,\ldots).$$

Proof. Substituting $u = f(x)$, and then integrating by parts j times, we find:

$$\int_{-\infty}^{\infty} g(x)e^{if(x)}\mathrm{d}x = \int_{f(a)}^{f(b)} \frac{g(x)}{f'(x)}e^{iu}\mathrm{d}u = i^j \int_c^d \left(\frac{\mathrm{d}^j}{\mathrm{d}u^j}\frac{g(x)}{f'(x)}\right) e^{iu}\mathrm{d}u \tag{7.9}$$

(say), since the relevant integrand at each stage is a function that vanishes at the boundary points (where $x = a$ or $x = b$). Here we can observe that, for $k = 0,1,\ldots,$ $\ell = 1,2,\ldots$ and $m = 2,3,\ldots,$

$$\frac{\mathrm{d}}{\mathrm{d}u}g^{(k)}(x) = g^{(k+1)}(x)\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{g^{(k+1)}(x)}{f'(x)},$$
$$\frac{\mathrm{d}}{\mathrm{d}u}\left(f'(x)\right)^{-\ell} = -\ell\left(f'(x)\right)^{-\ell-1} f''(x)\frac{\mathrm{d}x}{\mathrm{d}u} = -\ell\frac{f^{(2)}(x)}{\left(f'(x)\right)^{\ell+2}}$$
$$\frac{\mathrm{d}}{\mathrm{d}u}f^{(m)}(x) = f^{(m+1)}(x)\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{f^{(m+1)}(x)}{f'(x)}.$$

Therefore it follows by induction that, for $j = 0,1,\ldots$ and $a < x < b$,

$$\frac{\mathrm{d}^j}{\mathrm{d}u^j}\frac{g(x)}{f'(x)} = \sum_{r=0}^{j}\sum_{k_0\geqslant 0}\sum_{k_1\geqslant}\sum_{k_2\geqslant\ldots}\cdots\sum_{\substack{\geqslant k_r\geqslant 2\\ k_0+\ldots+k_r=j+r}} C_r(\mathbf{k})\frac{g^{(k_0)}(x)f^{(k_1)}(x)\ldots f^{(k_r)}(x)}{\left(f'(x)\right)^{1+k_0+\ldots+k_r}},$$

for certain coefficients $C_r(\mathbf{k}) \in \mathbb{Z}$ depending only on r and $\mathbf{k} = (k_0,\ldots,k_r)$. By (7.6)-(7.8), we have here (in the above sum):

$$\frac{g^{(k_0)}(x)f^{(k_1)}(x)\ldots f^{(k_r)}(x)}{\left(f'(x)\right)^{1+k_0+\ldots+k_r}} \ll_j \frac{M^{-k_0}(V/M)^r}{|f'(x)|\,V^{k_0+\ldots+k_r}} =$$
$$= \frac{1}{|f'(x)|\,M^{k_0+r}V^j} = \frac{M^{j-(k_0+r)}}{|f'(x)|\,M^jV^j},$$

where $j + r = k_0 + \ldots k_r \geqslant k_0 + 2 + \ldots + 2 = k_0 + 2r$, so that $j - (r + k_0) \geqslant 0$. On using our bound (for all the terms of the above sum), and then recalling that $0 < M \leqslant 1$ (so that $M^{j-(r+k_0)} \leqslant 1$), we arrive, by way of (7.9) and the related observation that $|\mathrm{d}u|/|f'(x)| = |\mathrm{d}x|$, at the result stated in the lemma. Note that $C_r(\mathbf{k}) = O_j(1)$ in all relevant cases. ■

Lemma 7.3. *Let $0 < \varepsilon < 1/4$, $H \geqslant 1$, $t \in \mathbb{R}$ and $X > \Delta$, where Δ is a positive absolute constant. Suppose that $\phi(x)$ is given by (7.1). Then,*

$$\tilde{\phi}(k-1) \ll_\varepsilon (1+|t|)^{1/\varepsilon} \min\left(X^{\varepsilon+1/2}, \frac{H^3}{X^{1-4\varepsilon}}\right) \qquad (k = 2,4,6,\ldots) \tag{7.10}$$

and, for even integers k satisfying

$$k \geqslant 10X^{\varepsilon+1/2}, \tag{7.11}$$

we have:

$$\tilde{\phi}(k-1) \ll_j (1+|t|)^j X^{1-2j\varepsilon} \qquad (j = 0,1,\ldots). \tag{7.12}$$

Proof. Let $k/2 \in \mathbb{N}$. By (2.15), (7.1) and (1.56),

$$\tilde{\phi}(k-1) = \int_0^\infty J_{k-1}(x) x^{it-1} \Phi_H(x) \Omega_0(x/X) \mathrm{d}x, \tag{7.13}$$

where $J_{k-1}(x)$ can be represented in the form (2.12) (see Lemma 2.3) and where $\Phi_H(x)$ is given by (1.48). By (1.57) the integrand here is a function of x with support in the interval $[X/2, 2X]$. Since (2.12) involves a proper integral over $[-\pi/2, \pi/2]$, and since (1.48) involves an integral that is uniformly absolutely convergent (for all $x > 0$), we are justified in appealing to (7.13), (2.12), (1.48), and then changing the order of integration so as to obtain:

$$\tilde{\phi}(k-1) = -\frac{i^k H^3}{\pi} \int_{-\pi/2}^{\pi/2} \int_0^\infty D_{X,t}(\eta,\xi) \frac{\xi \tanh(\xi)}{e^{(H\xi)^2}} e^{-(k-1)i\eta} \mathrm{d}\xi \mathrm{d}\eta. \tag{7.14}$$

with

$$D_{X,t}(\eta,\xi) = \int_{X/2}^{2X} \sin(x\cos(\eta)) \sin(x\cosh(\xi)) x^{it} \Omega_0(x/X) \mathrm{d}x, \tag{7.15}$$

so that

$$|D_{X,t}(\eta,\xi)| \leqslant \max_{\alpha,\beta=\pm1} |E_{X,\beta t}(\cosh(\xi) + \alpha\cos(\eta))|, \tag{7.16}$$

where

$$E_{X,t'}(y) = \int_{X/2}^{2X} e^{ixy} x^{it'} \Omega_0(x/X) \mathrm{d}x.$$

Integrating by parts j times (and using the bound $\Omega_0^{(j)}(x) \ll_j 1$), we find here:

$$E_{X,t'}(y) \ll_j (1+|t'|)^j X^{1-j}|y|^{-j} \qquad (j=0,1,\ldots). \tag{7.17}$$

For $\alpha = \pm 1$, $\eta \in [-\pi/2, \pi/2]$ and $\xi > 0$, we have

$$\cosh(\xi) + \alpha\cos(\eta) \geqslant \cosh(\xi) - \cos(\eta) =$$
$$= \int_0^\xi \sinh(u)\mathrm{d}u + \int_0^\eta \sin(\theta)\mathrm{d}\theta \geqslant \int_0^\xi u\mathrm{d}u + \int_0^\eta \frac{2}{\pi}\theta\mathrm{d}\theta = \frac{\xi^2}{2} + \frac{\eta^2}{\pi},$$

so that, by (7.16) and (7.17),

$$D_{X,t}(\eta,\xi) \ll_j (1+|t|)^j X^{1-j} \left|\xi^2+\eta^2\right|^{-j} \qquad (j=0,1,\ldots), \tag{7.18}$$

whenever $-\pi/2 \leqslant \eta \leqslant \pi/2$ and $\xi > 0$.

Taking now $\Omega(x)$ to be the function of (1.69)-(1.70), and setting

$$Y = X^{\varepsilon - 1/2}, \tag{7.19}$$

we find that, by (1.69) and (7.18),

$$\int_{-\pi/2}^{\pi/2}\int_0^\infty D_{X,t}(\eta,\xi)\frac{\xi\tanh(\xi)}{e^{(H\xi)^2}}e^{-(k-1)i\eta}\left(1-\Omega(\xi/Y)\Omega(|\eta|/Y)\right)\mathrm{d}\xi\mathrm{d}\eta \ll_j$$
$$\ll_j (1+|t|)^j X^{1-j}Y^{-2j}\int_0^\infty \xi\tanh(\xi)e^{-(H\xi)^2}\mathrm{d}\xi \ll$$
$$\ll H^{-3}(1+|t|)^j X^{1-2j\varepsilon},$$

for $j = 0,1,2,\ldots$. Therefore, and by (7.14), (7.15) and (1.69),

$$\tilde{\phi}(k-1) = F_Y(k-1) + O_j\left((1+|t|)^j X^{1-2j\varepsilon}\right), \tag{7.20}$$

where

$$F_Y(k-1) = \frac{-i^k H^3}{\pi}\int_{-\pi/2}^{\pi/2}\int_0^\infty D_{X,t}(\eta,\xi)\frac{\xi\tanh(\xi)\Omega(\xi/Y)\Omega(|\eta|/Y)}{e^{(H\xi)^2+(k-1)i\eta}}\mathrm{d}\xi\mathrm{d}\eta = \tag{7.21}$$
$$= \int_{X/2}^{2X} J_{k-1}^*(x)\Phi_H^*(x)x^{it-1}\Omega_0(x/X)\mathrm{d}x,$$

with

$$J_{k-1}^*(x) = \frac{-i^k}{\pi} \int_{-\pi/2}^{\pi/2} e^{-(k-1)i\eta} \sin(x\cos(\eta))\, \Omega(|\eta|/Y) \mathrm{d}\eta, \tag{7.22}$$

$$\Phi_H^*(x) = H^3 x \int_0^{2Y} \xi \tanh(\xi) e^{-(H\xi)^2} \sin(x\cosh(\xi))\, \Omega(\xi/Y) \mathrm{d}\xi.$$

By (1.69)-(1.70), we have here (trivially),

$$J_{k-1}^*(x) \ll Y \qquad (x > 0), \tag{7.23}$$

$$\frac{|\Phi_H^*(x)|}{x} \leqslant H^3 \min\left(\int_0^{2Y} \xi^2 \mathrm{d}\xi,\ \int_0^{\infty} \frac{\xi^2}{e^{(H\xi)^2}} \mathrm{d}\xi \right) \ll \min(H^3Y^3, 1) \quad (x > 0). \tag{7.24}$$

As $\Omega_0(x) \ll 1$, it follows from (7.21), (7.23), (7.24) and (7.19) that

$$F_Y(k-1) \ll XY \min\left(H^3Y^3,\ 1\right) = \min\left(H^3X^{4\varepsilon-1},\ X^{\varepsilon+1/2}\right). \tag{7.25}$$

Since $0 < \varepsilon < 1/4$, $X > \Delta$ and $H \geqslant 1$, the bound (7.10) follows from this bound for $F_Y(k-1)$, by way of the case $j = [1/\varepsilon]$ of (7.20).

For the remaining result, (7.11)-(7.12), we shall need to improve on the bound (7.23). Note first that, if $\Delta < X < 16$, then the bound (7.25) and the case $j = 0$ of (7.20) together imply $\tilde{\phi}(k-1) \ll X^{\varepsilon+1/2} + X \ll 1$, which contains (7.12) (in cases where $X < 16$ and $0 < \varepsilon < 1/4$). Therefore we may assume henceforth that $X \geqslant 16$, so that (by (7.19)),

$$2Y = 2X^{\varepsilon-1/2} < 2X^{-1/4} \leqslant 1.$$

As this shows $2Y < \pi/2$, it now follows from (7.22) and (1.69) that

$$\left|J_{k-1}^*(x)\right| \leqslant \frac{1}{\pi} \left| \int_{-\infty}^{\infty} g(\eta) e^{if(\eta)} \mathrm{d}\eta \right|, \tag{7.26}$$

where $g(\eta) = \Omega(|\eta|/Y)$ (an infinitely differentiable real function with support in $[-2Y, 2Y]$) and $f(\eta) = \pm x\cos(\eta) - (k-1)\eta$ (with a choice of sign maximising the right-hand side of (7.26)). Assuming (7.11), we have it by (7.19) that, for $x \in [X/2, 2X]$ and $\eta \in [-2Y, 2Y]$,

$$f'(\eta) = \mp x \sin(\eta) - (k-1) \leqslant |x\eta| - k/2 \leqslant 4XY - 5X^{\varepsilon+1/2} = -XY$$

and $|f^{(j)}(\eta)| \leqslant |x| \leqslant 2X$, for $j = 2, 3, \ldots$. Lemma 7.2 applies, with $[a, b] = [-2Y, 2Y]$, $V = XY$ and $M = Y$, showing that, for $x \in [X/2, 2X]$,

$$\int_{-\infty}^{\infty} g(\eta) e^{if(\eta)} \mathrm{d}\eta \ll_j Y(XY^2)^{-j} = YX^{-2j\varepsilon} \qquad (j = 0, 1, \ldots)$$

(see (7.19)). By this, (7.26), (7.24), (7.21) and (7.19), we obtain in place of (7.25) the bound,

$$F_Y(k-1) \ll_j X^{-2j\varepsilon} \min\left(H^3 X^{4\varepsilon-1},\, X^{\varepsilon+1/2}\right) \leqslant X^{\varepsilon+1/2-2j\varepsilon},$$

so that (7.12) follows by (7.20) (given that $X \geqslant 16$ and $0 < \varepsilon < 1/4$). ∎

Lemma 7.4. *Let* $0 < \varepsilon < 1/2$, $H \geqslant 1$ *and* $X > 0$. *Suppose that*

$$Y = X^{\varepsilon-1/2}. \tag{7.27}$$

Then, for $X/2 \leqslant x \leqslant 2X$ *and* $j = 0, 1, \ldots,$

$$\Phi_H(x) = \Phi_H^*(x) + O_j\left(X^{1-2j\varepsilon}\right),$$

where

$$\Phi_H^*(x) = H^3 x \int_0^Y \sin\left(x\cosh(\xi)\right) \xi \tanh(\xi) e^{-(H\xi)^2} \Omega(2\xi/Y) \mathrm{d}\xi \tag{7.28}$$

(with $\Omega(x)$ *as in* (1.69)-(1.70)*).*

Proof. Suppose that $X/2 \leqslant x \leqslant 2X$ and $j \in \{0, 1, \ldots\}$. By (1.48), (1.69) and (7.28),

$$\Phi_H(x) - \Phi_H^*(x) = H^3 x \int_{Y/2}^{\infty} \sin\left(x\cosh(\xi)\right) \frac{\xi \tanh(\xi)}{e^{(H\xi)^2}} \alpha(\xi/Y) \mathrm{d}\xi, \tag{7.29}$$

where

$$\alpha(u) = 1 - \Omega(2u), \tag{7.30}$$

so that (see (1.69)) the function α is real, infinitely differentiable, and satisfies:

$$\alpha(u) = \begin{cases} 0, & \text{if } u \leqslant 1/2, \\ 1, & \text{if } u \geqslant 1. \end{cases} \tag{7.31}$$

Therefore, after one integration by parts we find:

$$\Phi_H(x) - \Phi_H^*(x) = -H^3 x \int_{Y/2}^{\infty} \frac{\xi \tanh(\xi) \alpha(\xi/Y)}{e^{(H\xi)^2} x \sinh(\xi)} \mathrm{d}(\cos\left(x\cosh(\xi)\right)) =$$

$$= H^3 x \int_{Y/2}^{\infty} \left(\frac{\mathrm{d}}{\mathrm{d}\xi} \frac{\xi \alpha(\xi/Y)}{e^{(H\xi)^2} x \cosh(\xi)}\right) \cos\left(x\cosh(\xi)\right) \mathrm{d}\xi.$$

We may repeat this step (integration by parts) any given number of times. Each iteration involves a division by $\pm x\sinh(\xi)$, followed by a differentiation with respect to the variable ξ. With regard to the differentiation, one should note that, for $\xi > 0$, we have:

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\log(\xi) = \frac{1}{\xi}, \qquad \frac{\mathrm{d}}{\mathrm{d}\xi}\log\left(e^{(H\xi)^2}\right) = 2H^2\xi = \frac{2(H\xi)^2}{\xi},$$
$$\frac{\mathrm{d}}{\mathrm{d}\xi}\log(\sinh(\xi)) = \frac{\cosh(\xi)}{\sinh(\xi)} \leqslant \frac{e^\xi}{\xi}, \qquad \frac{\mathrm{d}}{\mathrm{d}\xi}\log(\cosh(\xi)) = \frac{\sinh(\xi)}{\cosh(\xi)} \leqslant 1 \leqslant \frac{e^\xi}{\xi}$$

and, by (7.31),

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\alpha^{(k)}(\xi/Y) = \frac{1}{Y}\alpha^{(k+1)}(\xi/Y) = O\left(\frac{1}{\xi}\left|\alpha^{(k+1)}(\xi/Y)\right|\right) \qquad (k = 0, 1, \ldots).$$

At any point we can stop the iteration and apply the bounds

$$\alpha^{(k)}(\xi/Y) \ll_k 1 \qquad (\xi > 0,\ k = 0, 1, \ldots),$$

which hold by virtue of (7.30). The result will be an upper bound for $|\Phi_H(x) - \Phi_H^*(x)|$ in terms of an integral involving an integrand that is some non-negative real-valued function of ξ. The bounds we have given for logarithmic derivatives show (in respect of our last point) that the k-th iteration alone reduces the final bound's integrand by a factor

$$F_k \gg_k \left(\frac{e^\xi + (H\xi)^2}{x\sinh(\xi)\xi}\right)^{-1} \gg x\xi^2 e^{-H\xi}.$$

Recalling the starting point, (7.29), we conclude that just j integrations by parts are sufficient to show:

$$\Phi_H(x) - \Phi_H^*(x) \ll_j H^3 x \int_{Y/2}^{\infty} \xi^2 e^{-(H\xi)^2}\left(x\xi^2 e^{-H\xi}\right)^{-j} \mathrm{d}\xi \ll_j$$
$$\ll_j H^3 x \left(xY^2\right)^{-j} \int_0^\infty \xi^2 e^{jH\xi - (H\xi)^2} \mathrm{d}\xi \ll_j X\left(XY^2\right)^{-j},$$

which (see (7.27)) is the result given by the lemma. ■

Lemma 7.5. *Let $0 < \delta < 1/4$, $0 < \varepsilon < 1/2$, $H \geqslant 1$, $t \in \mathbb{R}$ and $X > \Delta$, where Δ is a positive absolute constant. Suppose moreover that $\phi(x)$ is given by (7.1) and that $r \in \mathbb{C}$ satisfies*

$$r^2 \geqslant -\left(\tfrac{1}{4} - \delta\right)^2. \qquad (7.32)$$

Let

$$F_Y(r) = \int_{X/2}^{2X} G_r^*(x)\Phi_H^*(x)x^{it-1}\Omega_0(x/X)\mathrm{d}x, \tag{7.33}$$

where Y *and* $\Phi_H^*(x)$ *are as in* (7.27)-(7.28), $\Omega_0(x)$ *is as in* (1.71), *and*

$$G_r^*(x) = -2\int_0^{4Y} \cos(x\cosh(\eta))\cos(2r\eta)\,\Omega(\eta/2Y)\mathrm{d}\eta \tag{7.34}$$

(*with* $\Omega(x)$ *as in* (1.69)-(1.70)). *Then*

$$\hat{\phi}(r) = F_Y(r) + O_{\delta,j}\left((1+|t|)^j X^{1-2j\varepsilon}\right) \qquad (j\in\mathbb{N}).$$

Proof. By (2.16), (2.11) of Lemma 2.3, (7.1), (1.56) and (1.57), we have

$$\hat{\phi}(r) = \int_{X/2}^{2X} G_r(x)\Phi_H(x)x^{it-1}\Omega_0(x/X)\mathrm{d}x, \tag{7.35}$$

where

$$G_r(x) = -2\int_0^{\infty} \cos(x\cosh(\eta))\cos(2r\eta)\,\mathrm{d}\eta \qquad (x>0). \tag{7.36}$$

Applying the second derivative test [23], Lemmas 4.4 and 4.5, one obtains, with help from (7.32), the bound:

$$\int_n^{n+\theta} \cos(x\cosh(\eta))\cos(2r\eta)\,\mathrm{d}\eta \ll x^{-1/2}e^{-2n\delta},$$

for $x>0$, $n=0,1,\ldots$ and $0<\theta\leqslant 1$. From this and (7.36), we find that

$$G_r(x) = G_{r,n}(x) + O\left(\delta^{-1}e^{-2n\delta}x^{-1/2}\right) \qquad (n=0,1,\ldots), \tag{7.37}$$

where

$$G_{r,n}(x) = -2\int_0^{n} \cos(x\cosh(\eta))\cos(2r\eta)\,\mathrm{d}\eta. \tag{7.38}$$

By Lemma 7.4, together with the case $n=0$ of (7.37)-(7.38), we find by (7.35) that, for $j=0,1,\ldots,$

$$\hat{\phi}(r) - \int_{X/2}^{2X} G_r(x)\Phi_H^*(x)x^{it-1}\Omega_0(x/X)\mathrm{d}x = O_j\left(X^{1-2j\varepsilon}\int_{X/2}^{2X}\delta^{-1}x^{-3/2}\mathrm{d}x\right) =$$

$$= O_{\delta,j}\left(X^{1/2-2j\varepsilon}\right).$$

We take N to be any integer satisfying

$$N \geqslant 4Y. \tag{7.39}$$

By our last result, and the case $n = N$ of (7.37), and the trivial bound, $\Phi_H^*(x) \ll x$ (for which see (7.28)), we have now:

$$\hat{\phi}(r) = \int_{X/2}^{2X} G_{r,N}(x)\Phi_H^*(x)x^{it-1}\Omega_0(x/X)\mathrm{d}x + O_{\delta,j}\left(X^{1/2-2j\varepsilon}\right) + O_\delta\left(X^{1/2}e^{-2N\delta}\right).$$

We may rewrite this in the form

$$\hat{\phi}(r) = F_Y(r) + I_N(r) + O_{\delta,j}\left(X^{1/2-2j\varepsilon}\right) + O_\delta\left(X^{1/2}e^{-2N\delta}\right), \tag{7.40}$$

where $F_Y(r)$ is given by (7.33)-(7.34) and (7.28), while

$$I_N(r) = \int_{X/2}^{2X} \left(G_{r,N}(x) - G_r^*(x)\right)\Phi_H^*(x)x^{it-1}\Omega_0(x/X)\mathrm{d}x. \tag{7.41}$$

By (7.34), (7.38), (7.39) and (1.69), we have here:

$$G_{r,N}(x) - G_r^*(x) = \int_{2Y}^{N} \cos\left(x\cosh(\eta)\right)\cos\left(2r\eta\right)\beta(\eta/Y)\mathrm{d}\eta,$$

where

$$\beta(u) = 2\Omega(u/2) - 2, \tag{7.42}$$

which (by (1.69)) is an infinitely differentiable real function satisfying

$$\beta(u) = \begin{cases} 0, & \text{if } u \leqslant 2, \\ -2, & \text{if } u \geqslant 4. \end{cases} \tag{7.43}$$

Using this and (7.28) with (7.41), we find (after a change in the order of integration),

$$I_N(r) = \int_{2Y}^{N}\int_{0}^{Y} D_{X,t}(\eta,\xi)\cos\left(2r\eta\right)\frac{\xi\tanh(\xi)}{H^{-3}e^{(H\xi)^2}}\beta(\eta/Y)\Omega(2\xi/Y)\mathrm{d}\xi\mathrm{d}\eta, \tag{7.44}$$

where

$$D_{X,t}(\eta,\xi) = \int_{X/2}^{2X} \cos\left(x\cosh(\eta)\right)\sin\left(x\cosh(\xi)\right)x^{it}\Omega_0(x/X)\mathrm{d}x.$$

Despite this being not quite the same function $D_{X,t}(\eta,\xi)$ found in the proof of Lemma 7.3, the argument from (7.15) to (7.17) adapts (on substitution of $\cosh(\eta)$ for $\cos(\eta)$), and shows that, for $j = 0, 1, \ldots$,

$$D_{X,t}(\eta,\xi) \ll_j (1+|t|)^j X^{1-j} |\cosh(\eta) - \cosh(\xi)|^{-j}. \tag{7.45}$$

For $\eta \geqslant 2Y$ and $0 \leqslant \xi \leqslant Y$, one has

$$\frac{\cosh(\xi)}{\cosh(\eta)} \leqslant \frac{\cosh(Y)}{\cosh(2Y)} = 1 - \frac{1}{\cosh(2Y)} \int_Y^{2Y} \sinh(u) du,$$

so that

$$1 - \frac{\cosh(\xi)}{\cosh(\eta)} \geqslant \frac{1}{\cosh(2Y)} \int_Y^{2Y} u\, du = \frac{3Y^2}{2\cosh(2Y)},$$

which implies

$$\cosh(\eta) - \cosh(\xi) \geqslant \frac{3\cosh(\eta)}{2\cosh(2Y)} Y^2 \geqslant \frac{3}{4} e^{\eta - 2Y} Y^2.$$

Therefore, and by (7.32), (7.27), (7.42), (7.43) and (1.69)-(1.70), use of (7.45) in (7.44) shows that, for $j = 1, 2, 3, \ldots$,

$$\begin{aligned} I_N(r) &\ll_j (1+|t|)^j X^{1-j} Y^{-2j} H^3 \int_{2Y}^{N} \int_0^{Y} |\cos(2r\eta)|\, \xi^2 e^{-(H\xi)^2 - j(\eta - 2Y)} d\xi d\eta \ll \\ &\ll (1+|t|)^j X^{1-2j\varepsilon} \int_{2Y}^{N} e^{(1/2-2\delta)\eta - j(\eta-2Y)} d\eta < \\ &< (1+|t|)^j X^{1-2j\varepsilon} e^Y \int_0^\infty e^{-(j-1/2)\lambda} d\lambda \ll (1+|t|)^j X^{1-2j\varepsilon} \exp\left(\sqrt{1/X}\right). \end{aligned}$$

From this and (7.40), we conclude that, for $j \in \mathbb{N}$, $N \in \mathbb{N}$ satisfying (7.39), and $X > \Delta$, one has

$$\hat{\phi}(r) = F_Y(r) + O_{\delta,j}\left((1+|t|)^j X^{1-2j\varepsilon}\right) + O_\delta\left(X^{1/2} e^{-2N\delta}\right).$$

The implicit constants here do not depend on N, so that the result claimed by the lemma follows on passing to the limit as $N \to \infty$. ■

Lemma 7.6. *Suppose that the hypotheses of Lemma 7.5 up to and including (7.32) hold. Then*

$$\hat{\phi}(r) \ll_{\delta,\varepsilon} (1+|t|)^{1/\varepsilon} \min\left(X^{\varepsilon+1/2}, \frac{H^3}{X^{1-4\varepsilon}}\right), \tag{7.46}$$

and, if it is the case that

$$X \geqslant 1 \qquad \textit{and} \qquad |r| \geqslant 5e^4 X^{\varepsilon+1/2}, \tag{7.47}$$

then

$$\hat{\phi}(r) \ll_j (1+|t|)^j X^{1-2j\varepsilon} \qquad (j \in \mathbb{N}). \tag{7.48}$$

Proof. For both (7.46) and (7.48) the first step is to apply Lemma 7.5. Following that one has only to establish suitable bounds for the functions $G_r^*(x)$ and $\Phi_H^*(x)$ in (7.33)-(7.34) and (7.28). With regard to the latter function, we note that substitution of $Y/2$ for Y brings the $\Phi_H^*(x)$ defined (under (7.22)) while proving Lemma 7.3 into conformity with (7.28) of Lemma 7.4. Therefore, and by (7.27), the bound (7.24) applies to show:

$$\Phi_H^*(x) \ll \min\left(H^3 X^{3\varepsilon-1/2}, X\right) \qquad (X/2 \leqslant x \leqslant 2X). \tag{7.49}$$

By (7.32), (7.34), (1.69)-(1.70) and (7.27), we also have (trivially):

$$|G_r^*(x)| \leqslant 2\int_0^{4Y} e^{(1/2-2\delta)\eta} d\eta \leqslant 8Ye^{2Y} \ll X^{\varepsilon-1/2} \exp\left(\sqrt{4/X}\right) \ll X^{\varepsilon-1/2}, \tag{7.50}$$

for $X/2 \leqslant x \leqslant 2X$ (with $X > \Delta$).

By (7.33) and the result given by Lemma 7.5, our bounds, (7.49) and (7.50), show:

$$\hat{\phi}(r) \ll X^{\varepsilon-1/2} \min\left(H^3 X^{3\varepsilon-1/2}, X\right) + O_{\delta,j}\left((1+|t|)^j X^{1-2j\varepsilon}\right).$$

As $H \geqslant 1$, $0 < \varepsilon < 1/2$ and $X > \Delta$, the bound (7.46) follows on taking $j = [1/\varepsilon] \in \{2, 3, \ldots\}$ here.

The proof of (7.48) requires a non-trivial bound for $G_r^*(x)$ in (7.33). As $\cosh(\eta)$ and $\cos(2r\eta)$ are even functions of η, the definition (7.34) may be rewritten as

$$G_r^*(x) = -\int_{-4Y}^{4Y} \cos(x\cosh(\eta))\cos(2r\eta)\Omega_1(\eta/2Y)\, d\eta,$$

where

$$\Omega_1(u) = \Omega(|u|), \tag{7.51}$$

so that, by (1.69)-(1.70), $\Omega_1(u)$ is an infinitely differentiable real even function, satisfying

$$\Omega_1(u) = \begin{cases} 1, & \text{if } |u| \leqslant 1, \\ 0, & \text{if } |u| \geqslant 2. \end{cases} \tag{7.52}$$

Assuming (7.47), it follows from the above that

$$|G_r^*(x)| \leqslant \max_{\alpha=\pm1} \left| \int_{-\infty}^{\infty} g(\eta) e^{if_\alpha(\eta)} \mathrm{d}\eta \right|, \tag{7.53}$$

where $g(\eta)$ and $f_{\pm1}(\eta)$ are given by

$$g(\eta) = \Omega_1(\eta/2Y) \qquad \text{and} \qquad f_\alpha(\eta) = x\cosh(\eta) - 2r\alpha\eta. \tag{7.54}$$

Note that, by (7.32) (where $0 < \delta < 1/4$), the condition (7.47) (with $\varepsilon > 0$) can only hold in cases where r (and therefore $f_\alpha(\eta)$) is real. This frees us to assume that $\delta = 1/8$ (say). As $\varepsilon < 1/2$ in (7.27), it also follows from (7.47) that $0 < Y \leqslant 1$.

For $\alpha = \pm1$, $x \in [X/2, 2X]$ and $\eta \in [-4Y, 4Y]$, one has, as a consequence of (7.47) and (7.27), the bounds:

$$\begin{aligned} |f_\alpha'(\eta)| = |x\sinh(\eta) - 2\alpha r| &\geqslant |2r| - \left| x \int_0^\eta \cosh(u)\mathrm{d}u \right| \geqslant \\ &\geqslant 2|r| - x|\eta|\cosh(\eta) \geqslant \left(10e^4 - 8e^{4Y}\right)XY \geqslant 2e^4XY, \end{aligned}$$

and

$$\left|f_\alpha^{(j)}(\eta)\right| \leqslant x\cosh(\eta) \leqslant 2e^4X \qquad (j = 2, 3, \ldots).$$

Therefore, and by (7.54) and (7.51)-(7.52), our Lemma 7.2 applies, with $[a, b] = [-4Y, 4Y]$, $V = XY$ and $M = Y$. Through (7.53) and (7.27), we find that Lemma 7.2 shows

$$G_r^*(x) \ll_j Y\left(XY^2\right)^{-j} = X^{\varepsilon-1/2-2j\varepsilon} \qquad (j = 0, 1, \ldots).$$

Using this and (7.49) in (7.33), we obtain the bounds

$$F_Y(r) \ll_j X^{1/2+\varepsilon-2j\varepsilon} \qquad (j = 0, 1, \ldots),$$

which directly imply (7.48), by virtue of Lemma 7.5 (with $\delta = 1/8$) and the assumptions about ε and X. ∎

8. The swapping of levels

We give 5 lemmas leading up to a proof of Proposition 1.2 (at the very end). Lemma 8.5 is stronger than Proposition 1.2, but the latter has a useful simplicity (and suffices for what we seek to achieve in this paper).

Lemma 8.1. *Let* $\vartheta = 7/64$ *and suppose that* $\Delta \in (0, 1/2]$ *is sufficiently small (in absolute terms). Then, for* $\varepsilon > 0$, $j \in \mathbb{N}$, $Q, N > 0$, $K \geqslant 1$, $D \in \mathbb{N}$, $y \in \mathbb{R}$ *and any complex sequence* $\mathbf{b} = (b_n)$, *one has*

$$S_{Q,K}(\mathbf{b}, N; D, y) \ll$$

$$\ll O_{\varepsilon,j}\left(D^{1/j}\left(QK^2 + N^{1+\varepsilon}\right)\|\mathbf{b}_N\|_2^2\right) + \sum_{\substack{h=0 \\ U^h \leqslant K}}^{\infty} U^{-h} \sum_{\substack{r \in \mathbb{Z} \\ \Delta < X_r = 2^r \leqslant Y}} A_{Q,U_h,X_r}(\mathbf{b}, N; D, y),$$

where $S_{Q,K}(\mathbf{b}, N; D, y)$ *is as in* (1.42)-(1.43),

$$U = 2D^{2\vartheta/j} \quad \textit{and} \quad U_h = U^{h+1} \quad (h = 0, 1, \ldots), \tag{8.1}$$

$$Y = 16\pi Q^{-1} DN, \tag{8.2}$$

and $A_{Q,H,X}(\mathbf{b}, N; D, y)$ *is defined by* (1.54)-(1.56), (1.48) *and* (1.69)-(1.71).

Proof. Assume (in addition to the above hypotheses) that

$$q \in [Q/4,\, 2Q] \cap \mathbb{N}. \tag{8.3}$$

Then (6.11) holds (with Y given by (8.2)) and, for $h = 0, 1, \ldots$, the pair $H_1 = U^h$, $H = U^{h+1}$ satisfies (6.9). Therefore Lemma 6.5 applies, showing that, for $h = 0, 1, \ldots$,

$$U^{-h}\breve{\sigma}_q\left(U^h, U^{h+1}\right) \ll O_\varepsilon\left(\frac{\mathcal{A}_\varepsilon(q)}{U^h q}\right)\left(1 + O_j\left(\frac{D^{2\vartheta}}{U^{(h+1)j}}\right)\right) +$$

$$+ U^{2h+3}\|\mathbf{b}_N\|_2^2 + U^{-h} \sum_{\substack{r \in \mathbb{Z} \\ \Delta < X_r = 2^r \leqslant Y}} \alpha_{q,U_h,X_r}(\mathbf{b}, N; D, y).$$

Note that here we have $D^{2\vartheta}/U^{(h+1)j} \leqslant D^{2\vartheta}/U^j = 2^{-j} \ll 1$ (for $h \geqslant 0$ and $j \in \mathbb{N}$), so that we may apply the above results with Lemma 4.9 in order to obtain:

$$\sigma_{q,K}(\mathbf{b}, N; D, y) \ll O_{\varepsilon,j}\left(q^{-1}\mathcal{A}_\varepsilon(q)\right) + O\left(K^2 U^3 \|\mathbf{b}_N\|_2^2\right) +$$

$$+ \sum_{\substack{h=0 \\ U^h \leqslant K}}^{\infty} U^{-h} \sum_{\substack{r \in \mathbb{Z} \\ \Delta < X_r = 2^r \leqslant Y}} \alpha_{q,U_h,X_r}(\mathbf{b}, N; D, y).$$

As this holds for any q satisfying (8.3), we may now conclude from (1.43), (1.49), (1.55) and the above, that

$$S_{Q,K}(\mathbf{b}, N; D, y) \leqslant \sum_q \omega(q/Q)\sigma_{q,K}(\mathbf{b}, N; D, y) \ll$$

$$\ll \sum_{Q/4 \leqslant q \leqslant 2Q} O_{\varepsilon,j}\left(q^{-1}\mathcal{A}_\varepsilon(q)\right) + O\left(QK^2 U^3 \|\mathbf{b}_N\|_2^2\right) +$$

$$+ \sum_{\substack{h=0 \\ U^h \leqslant K}}^{\infty} U^{-h} \sum_{\substack{r \in \mathbb{Z} \\ \Delta < X_r = 2^r \leqslant Y}} A_{Q,U_h,X_r}(\mathbf{b}, N; D, y).$$

This proves the lemma, as it follows by (6.14), (8.1) and our other hypotheses that we have here

$$\sum_{Q/4\leqslant q\leqslant 2Q} q^{-1}\mathcal{A}_\varepsilon(q) = \tau^4(D)\,\|\mathbf{b}_N\|_2^2 \sum_{Q/4\leqslant q\leqslant 2Q}\left(1+\frac{(q,D)}{q}N^{1+\varepsilon}\right) \ll$$
$$\ll \tau^4(D)\,\|\mathbf{b}_N\|_2^2\left(Q+\tau(D)N^{1+\varepsilon}\right) \ll_j D^{1/j}\left(Q+N^{1+\varepsilon}\right)\|\mathbf{b}_N\|_2^2,$$

while $U^3 \ll D^{6\vartheta/j} \leqslant D^{1/j}$ and $K \geqslant 1$. ∎

Lemma 8.2. *Suppose that the hypotheses of Lemma* 8.1 *hold, and that* $\varepsilon > 0$, $j \in \mathbb{N}$, $Q, N > 0$, $K \geqslant 1$, $D \in \mathbb{N}$, $y \in \mathbb{R}$ *and* $\mathbf{b} = (b_n)$ *is some complex sequence. Let* Y, U, U_0 *and the sequence* (U_h) *be as defined in* (8.2) *and* (8.1). *Then one either has the bound*

$$S_{Q,K}(\mathbf{b}, N; D, y) \ll_{\varepsilon,j} D^{1/j}\left(QK^2 + N^{1+\varepsilon}\right)\|\mathbf{b}_N\|_2^2, \tag{8.4}$$

or there exist

$$X \in (\Delta, Y] \qquad \text{and} \qquad L \in \left[2^{-4}Y/X,\, 2Y/X\right] \tag{8.5}$$

such that, for $k = 4, 5, 6, \ldots$,

$$S_{Q,K}(\mathbf{b}, N; D, y) \ll_k \log\left(\tfrac{2}{\Delta}Y\right)\int_{-\infty}^{\infty}\sum_{\substack{h=0\\ U^h\leqslant K}}^{\infty} \mathcal{P}_{L,X}(U_h, t)\,\frac{\mathrm{d}t}{U^h(1+|t|)^k}, \tag{8.6}$$

where

$$\mathcal{P}_{L,X}(H,t) = \sum_{L/2<\ell\leqslant L}\sum_{i=0}^{2}\mathcal{Q}_X^{(i)}(\ell; H, t), \tag{8.7}$$

with

$$\mathcal{Q}_X^{(0)}(\ell; H, t) = \sum_{k\ \text{even}}\sum_{j=1}^{\theta_k(\ell)}\frac{(k-1)!}{(4\pi)^{k-1}}\left|\tilde{\phi}(k-1)\right|\left|\mathcal{L}_{jk}^*(t)\right|^2, \tag{8.8}$$

$$\mathcal{Q}_X^{(1)}(\ell; H, t) = \sum_{j\geqslant 1}^{(\ell)}\frac{1}{\cosh(\pi\kappa_j)}\left|\hat{\phi}(\kappa_j)\right|\left|\mathcal{L}_j(t)\right|^2, \tag{8.9}$$

and

$$\mathcal{Q}_X^{(2)}(\ell; H, t) = \sum_{\mathfrak{c}}^{\Gamma_0(\ell)}\int_{-\infty}^{\infty}\left|\hat{\phi}(r)\right|\left|\mathcal{L}_{\mathfrak{c}}(r;t)\right|^2\mathrm{d}r, \tag{8.10}$$

in which the transforms $\tilde{\phi}$ *and* $\hat{\phi}$ *are those of* (2.15) *and* (2.16), *applied to the function*

$$\phi(x) = \phi_{X,H,t}(x) = x^{2it}\Phi_{H,X}(x), \tag{8.11}$$

while

$$\mathcal{L}_{jk}^{*}(t) = \sum_{N/2<n\leqslant N} b_n n^{i(y+t)} \psi_{jk}^{*}(\infty, Dn), \tag{8.12}$$

$$\mathcal{L}_{j}(t) = \sum_{N/2<n\leqslant N} b_n n^{i(y+t)} \rho_{j\infty}(Dn) \tag{8.13}$$

$$\mathcal{L}_{c}(r;t) = \sum_{N/2<n\leqslant N} b_n n^{i(y+t)+ir} \varphi_{c\infty}\left(Dn, \tfrac{1}{2}+ir\right) . \tag{8.14}$$

Proof. By Lemma 8.1 we find that either (8.4) holds, or it must be the case that, for some $X \in (\Delta, Y]$, one has

$$S_{Q,K}(\mathbf{b}, N; D, y) \ll_k \log(\tfrac{2}{\Delta}Y) \left| \sum_{\substack{h=0 \\ U^h \leqslant K}}^{\infty} U^{-h} A_{Q,U_h,X}(\mathbf{b}, N; D, y) \right| , \tag{8.15}$$

where $A_{Q,H,X}(\mathbf{b}, N; D, y)$ is defined by (1.54)-(1.56), (1.48) and (1.69)-(1.71). Therefore we may assume henceforth that (8.15) holds.

By (1.49), (1.57) and (1.56), the variables m, n, ℓ and q, that index the summations defining $A_{Q,H,X}(\mathbf{b}, N; D, y)$, are effectively constrained to satisfy $m,n \in (N/2, N]$, $q \in (Q/4, 2Q)$ and $X/2 < 4\pi D\sqrt{mn}/q\ell < 2X$. These conditions imply that $\ell \in (2^{-5}Y/X, 2Y/X)$. Therefore it follows from (8.15) that, for some $L \in \{2^{-4}Y/X, 2^{-3}Y/X, \ldots, 2Y/X\}$, one has

$$S_{Q,K}(\mathbf{b}, N; D, y) \ll \log(\tfrac{2}{\Delta}Y) \left| \sum_{\substack{h=0 \\ U^h \leqslant K}}^{\infty} U^{-h} A_{Q,L,X}^{\times}(U_h) \right| , \tag{8.16}$$

where (see (1.54)-(1.55)):

$$A_{Q,L,X}^{\times}(H) = \sum_{L/2<\ell\leqslant L} \alpha_{Q,X}^{\times}(\ell; H), \tag{8.17}$$

$$\alpha_{Q,X}^{\times}(\ell; H) = \sum\sum_{N/2<m,n\leqslant N} \bar{b}_m b_n \left(\frac{m}{n}\right)^{-iy} \sigma_{mn}^{\times} \tag{8.18}$$

and

$$\sigma_{mn}^{\times} = \sum_{q=1}^{\infty} \omega(q/Q) \frac{1}{q\ell} \Phi_{H,X}\left(\frac{4\pi D\sqrt{mn}}{q\ell}\right) S(Dm, Dn; q\ell). \tag{8.19}$$

In order to replace the factor $\omega(q/Q)$ in (8.19) with a function of $x = 4\pi D\sqrt{mn}/q\ell$, we shall follow the procedure of [7], page 272, in our use of the Mellin transform

$$\psi(s) = \int_0^{\infty} x^{s-1} \omega(x) \mathrm{d}x \qquad (s \in \mathbb{C}).$$

First we note the identity,

$$\omega(y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \psi(2it) y^{-2it} \mathrm{d}t \qquad (y > 0), \tag{8.20}$$

where, by (1.49) (or (1.69)-(1.71)),

$$|\psi(2it)| \leqslant \int_{0}^{\infty} x^{-1} \omega(x) \mathrm{d}x \ll \int_{1/4}^{2} \frac{\mathrm{d}x}{x} \ll 1$$

and (through repeated integrations by parts)

$$\psi(2it) = (-1)^k \int_{0}^{\infty} \frac{x^{it-1+k} \omega^{(k)}(x)}{(it-1+k)\dots(it-1+1)} \mathrm{d}x \ll_k |t|^{-k} \int_{1/4}^{2} x^{k-1} \mathrm{d}x \ll_k |t|^{-k},$$

for $t \neq 0$ and $k \in \mathbb{N}$, so that

$$\psi(2it) \ll_k (1 + |t|)^{-k} \qquad (t \in \mathbb{R} \text{ and } k = 0, 1, \dots). \tag{8.21}$$

Then, with $x = 4\pi D\sqrt{mn}/q\ell$, we have $q/Q = 4\pi(D/Q)(\sqrt{mn}/x\ell)$, so that by using (8.20), for $y = q/Q$, with (8.16)-(8.19), we can obtain:

$$S_{Q,K}(\mathbf{b}, N; D, y) \ll \log\left(\tfrac{2}{\Delta}Y\right) \int_{-\infty}^{\infty} \left| \psi(2it) \sum_{\substack{h=0 \\ U^h \leqslant K}}^{\infty} U^{-h} A^*_{L,X}(U_h, t) \right| \mathrm{d}t, \tag{8.22}$$

where

$$A^*_{L,X}(H,t) = \sum_{L/2 < \ell \leqslant L} \ell^{2it} \alpha^*_X(\ell; H, t), \tag{8.23}$$

$$\alpha^*_X(\ell; H, t) = \sum\sum_{N/2 < m,n \leqslant N} \bar{b}_m m^{-i(y+t)} b_n n^{i(y-t)} \sigma^*_{mn} \tag{8.24}$$

and (see (2.7))

$$\sigma^*_{mn} = \sum_{q=1}^{\infty} \phi\left(\frac{4\pi D\sqrt{mn}}{q\ell}\right) \frac{S(Dm, Dn; q\ell)}{q\ell} = \sum_{\gamma}^{\Gamma_0(\ell)} \frac{1}{\gamma} S_{\infty\infty}(Dm, Dn; \gamma)\, \phi\left(\frac{4\pi D\sqrt{mn}}{q\ell}\right),$$

with $\phi(x) = \phi_{X,H,t}(x)$ given by (8.11). In view of (7.1) and Lemma 7.1 (with $2t$ substituted for t), the above makes it possible to apply Theorem 2.2 (with m and n there replaced by Dm and Dn, and with cusps $\mathfrak{a} = \mathfrak{b} = \infty$), so as to obtain the identity, $\sigma^*_{mn} = \mathcal{K}_0 + \mathcal{K}_1 + \mathcal{K}_2$, stated there. Bringing the summations of (8.24) inside the summations and integrations occurring in the definitions of the $\mathcal{K}_i$'s, we find:

$$\alpha_X^*(\ell; H, t) = \sum_{i=0}^{2} \mathcal{R}_X^{(i)}(\ell; H, t), \tag{8.25}$$

where

$$\mathcal{R}_X^{(0)}(\ell; H, t) = \frac{1}{2\pi} \sum_{k \text{ even}} \sum_{j=1}^{\theta_k(\ell)} \frac{i^k (k-1)!}{(4\pi)^{k-1}} \tilde{\phi}(k-1) \overline{\mathcal{L}_{jk}^*(t)} \mathcal{L}_{jk}^*(-t),$$

$$\mathcal{R}_X^{(1)}(\ell; H, t) = \sum_{j \geqslant 1}^{(\ell)} \frac{1}{\cosh(\pi \kappa_j)} \hat{\phi}(\kappa_j) \overline{\mathcal{L}_j(t)} \mathcal{L}_j(-t),$$

$$\mathcal{R}_X^{(2)}(\ell; H, t) = \frac{1}{\pi} \sum_{\mathfrak{c}}^{\Gamma_0(\ell)} \int_{-\infty}^{\infty} \hat{\phi}(r) \overline{\mathcal{L}_{\mathfrak{c}}(r; t)} \mathcal{L}_{\mathfrak{c}}(r; -t) \mathrm{d}r,$$

with $\phi(x)$, $\mathcal{L}_{jk}^*(t)$, $\mathcal{L}_j(t)$ and $\mathcal{L}_{\mathfrak{c}}(r; t)$ given by (8.11)-(8.14). By the arithmetic-geometric mean inequality and (8.7)-(8.10), we have here

$$\sum_{L/2 < \ell \leqslant L} \sum_{i=0}^{2} \left| \mathcal{R}_X^{(i)}(\ell; H, t) \right| \leqslant \mathcal{P}_{L,X}(H, t) + \mathcal{P}_{L,X}(H, -t),$$

so that (8.6) is a direct consequence of (8.21)-(8.23) and (8.25). ■

Lemma 8.3. *Suppose that the hypotheses of Lemma* 8.1 *hold, and that* $\varepsilon > 0$, $X > \Delta$, $L, N > 0$, $H \geqslant 1$, $D \in \mathbb{N}$, $y, t \in \mathbb{R}$ *and* $\mathbf{b} = (b_n)$ *is some complex sequence. Let* $\mathcal{P}_{L,X}(H,t)$ *be given by* (8.7)-(8.14) *of Lemma* 8.2. *Then*

$$\mathcal{P}_{L,X}(H,t) \ll (1 + |t|)^2 X^{9/2} \Big(\int_1^{\infty} S_{L,G}(\mathbf{b}, N; D, y + t) G^{-7/2} \, \mathrm{d}G + \tag{8.26}$$
$$+ O_\varepsilon\Big(\tau^4(D)\left(L + \tau(D) N^{1+\varepsilon}\right) \|\mathbf{b}_N\|_2^2\Big)\Big)$$

and, for $F \geqslant 0$,

$$\mathcal{P}_{L,X}(H,t) = \mathcal{P}_{L,X,F}(H,t) + \tag{8.27}$$
$$+ O_\varepsilon\Big((1+|t|)^2 \frac{X^{9/2} D^{2\vartheta+\varepsilon}}{\sqrt{1+F}} \left(L + \frac{N^{1+\varepsilon}}{1+F^2} \right) \|\mathbf{b}_N\|_2^2 \Big),$$

where

$$\mathcal{P}_{L,X,F}(H,t) = \sum_{L/2 < \ell \leqslant L} \sum_{i=0}^{2} \mathcal{Q}_{X,F}^{(i)}(\ell; H, t) \tag{8.28}$$

and

$$\mathcal{Q}^{(0)}_{X,F}(\ell;H,t) = \sum_{\substack{k \text{ even} \\ k<F}} \sum_{j=1}^{\theta_k(\ell)} \frac{(k-1)!}{(4\pi)^{k-1}} \left|\tilde{\phi}(k-1)\right| \left|\mathcal{L}^*_{jk}(t)\right|^2, \tag{8.29}$$

$$\mathcal{Q}^{(1)}_{X,F}(\ell;H,t) = \sum_{\substack{j\geqslant 1 \\ |\kappa_j|<F}}^{(\ell)} \frac{1}{\cosh(\pi\kappa_j)} \left|\hat{\phi}(\kappa_j)\right| \left|\mathcal{L}_j(t)\right|^2, \tag{8.30}$$

$$\mathcal{Q}^{(2)}_{X,F}(\ell;H,t) = \sum_{\mathfrak{c}}^{\Gamma_0(\ell)} \int_{-F}^{F} \left|\hat{\phi}(r)\right| \left|\mathcal{L}_{\mathfrak{c}}(r;t)\right|^2 \mathrm{d}r, \tag{8.31}$$

with $\phi(x) = \phi_{X,H,t}(x)$ *as in* (8.11) *and* $\mathcal{L}^*_{jh}(t)$, $\mathcal{L}_j(t)$ *and* $\mathcal{L}_{\mathfrak{c}}(r;t)$ *as in* (8.12)-(8.14).

Proof. Suppose that $F \geqslant 0$ and $\ell \in \mathbb{N}$. By (8.11) and (1.13)-(1.15), we may apply the case $\delta = \frac{1}{2} - \vartheta$ of Lemma 7.1 (with 't' there replaced throughout by '$2t$'). Given that $X > \Delta$, it then follows from the results (7.3)-(7.5) that one has, in (8.8)-(8.10):

$$\tilde{\phi}(k-1) \ll (1+|t|)^2 X^{9/2} k^{-5/2}, \tag{8.32}$$

$$\hat{\phi}(\kappa_j) \ll (1+|t|)^2 X^{9/2} (\max(1, |\kappa_j|))^{-5/2} \tag{8.33}$$

and

$$\hat{\phi}(r) \ll (1+|t|)^2 X^{9/2} (\max(1, |r|))^{-5/2}. \tag{8.34}$$

Clearly these bounds are similar in shape (especially when one notes that $k = \max(1,k)$ for positive even k). We shall focus on how (8.34) is used, since the corresponding applications of (8.32) and (8.33) are very similar.

Comparing (8.31), (8.10) one finds that the difference $\mathcal{Q}^{(2)}_X(\ell;H,t) - \mathcal{Q}^{(2)}_{X,F}(\ell;H,t)$ can be written as a sum of integrals, similar to that in (8.31), but with r (the variable of integration) running over $(-\infty,-F] \cup [F,\infty)$ instead of $[-F,F]$. After applying (8.34) to bound this sum, we may rewrite our bound using

$$(\max(1, |r|))^{-5/2} = \frac{5}{2} \int_{\max(1,|r|)}^{\infty} G^{-7/2}\,\mathrm{d}G.$$

The result is a bound involving a summation over cusps $\mathfrak{c}$ (as in (8.31)), an integration over $r \in (-\infty,-F] \cup [F,\infty)$ and the integration over $G \geqslant \max(1,|r|)$. Bringing the summation and the other integration inside the integration with respect to G, we obtain (see (8.14)) a result containing the case $i = 2$ of the bound

$$\mathcal{Q}^{(i)}_X(\ell;H,t) - \mathcal{Q}^{(i)}_{X,F}(\ell;H,t) \ll \tag{8.35}$$

$$\ll (1+|t|)^2 X^{9/2} \int_{\max(1,F)}^{\infty} \mathcal{S}^{(i)}_{\infty,\ell,G}\left(\mathbf{b}^{(D)}(y+t), DN\right) \frac{\mathrm{d}G}{G^{7/2}},$$

where the term in the integrand is as in (1.28), with $\mathbf{b}^{(D)}$ being the sequence given by (1.31) and $b_n^{(D)}(u) = b_n^{(D)} n^{iu}$ for $n \in \mathbb{N}$ and $u \in \mathbb{R}$. The cases $i = 0$ and $i = 1$ of (8.35) follow by arguments similar to those used above, but involving the bounds (8.32), (8.33) and the definitions (8.8), (8.9), (8.12), (8.13), (8.29), (8.30), (1.26) and (1.27). As this is easily verified, we do not give further details.

Combining the cases $i = 1$ and $i = 2$ of (8.35) through (1.44), and recalling the definitions made in (8.7) and (8.28), we conclude that

$$\mathcal{P}_{L,X}(H,t) - \mathcal{P}_{L,X,F}(H,t) \ll \tag{8.36}$$
$$\ll (1+|t|)^2 X^{9/2} \int_{\max(1,F)}^{\infty} G^{-7/2} \times$$
$$\times \left(\sum_{L/2<\ell\leqslant L} \left(\mathcal{S}^{(0)}_{\infty,\ell,G}\left(\mathbf{b}^{(D)}(y+t), DN\right) + \sigma_{\ell,G}(\mathbf{b}, N; D, y+t) \right) \right) \mathrm{d}G.$$

For (8.27) we apply both (5.1) and (5.2) of Lemma 5.1 here (the result then follows trivially). To obtain (8.26) from (8.36) we first note that, in view of the definitions (8.28)-(8.31), one has $\mathcal{P}_{L,X,F}(H,t) = 0$ if $F = 0$. The result (8.26) is therefore a straightforward consequence of the case $F = 0$ of (8.36), the bound (5.1) of Lemma 5.1, and the definition of $S_{Q,K}(\mathbf{b}, N; D, y)$ given in (1.43). ∎

Lemma 8.4. *Suppose that the hypotheses of Lemma 8.1 hold, and that* $0 < \varepsilon < 1/4$, $L, N > 0$, $H \geqslant 1$, $D \in \mathbb{N}$, $y, t \in \mathbb{R}$ *and* $\mathbf{b} = (b_n)$ *is some complex sequence. Suppose moreover that*

$$X \geqslant D^{\varepsilon}. \tag{8.37}$$

Then

$$\mathcal{P}_{L,X}(H,t) \ll_{\varepsilon} (1+|t|)^{4/\varepsilon^2} \min\left(E, \frac{H^3}{X^{1-4\varepsilon}}\right) \times$$
$$\times \left(S_{L,E}(\mathbf{b}, N; D, y+t) + \tau^4(D)\left(LE^2 + \tau(D)N^{1+\varepsilon}\right)\|\mathbf{b}_N\|_2^2\right),$$

where

$$E = 5\mathrm{e}^4 X^{\varepsilon+1/2}, \tag{8.38}$$

and $\mathcal{P}_{L,X}(H,t)$ *is as given by (8.7)-(8.14) of Lemma 8.2.*

Proof. By applying (8.27) of Lemma 8.3 with

$$F = D^{4\vartheta+2\varepsilon} X^9 \tag{8.39}$$

we obtain:

$$\mathcal{P}_{L,X}(H,t) = \mathcal{P}_{L,X,F}(H,t) + O_{\varepsilon}\left((1+|t|)^2\left(L + \frac{N^{1+\varepsilon}}{F^2}\right)\|\mathbf{b}_N\|_2^2\right),$$

with $\mathcal{P}_{L,X,F}(H,t)$ given by (8.28) (see also (8.29)-(8.31) and (8.11)-(8.14)). By (8.37)-(8.39) and our hypotheses,

$$\min\left(E, \frac{H^3}{X^{1-4\varepsilon}}\right)\left(LE^2 + \tau(D)N^{1+\varepsilon}\right) \geqslant L + \frac{\tau(D)N^{1+\varepsilon}}{X^{1-2\varepsilon}} \geqslant L + \frac{N^{1+\varepsilon}}{F^2}. \quad (8.40)$$

Therefore, comparing the last O-term above with the bound for $\mathcal{P}_{L,X}(H,t)$ given by the lemma, we conclude (given that $4/\varepsilon^2 > 64$) that either the bound given by the lemma holds, or one must have

$$\mathcal{P}_{L,X}(H,t) \ll \mathcal{P}_{L,X,F}(H,t). \quad (8.41)$$

Accordingly we assume (8.41) for the remainder of the proof.

Suppose now that $\ell \in \mathbb{N}$. Applying Lemmas 7.3 and 7.6 to bound the factors $\tilde{\phi}(k-1)$, $\hat{\phi}(\kappa_j)$, $\hat{\phi}(r)$ in the sums $\mathcal{Q}^{(i)}_{X,F}(\ell; H,t)$ of (8.28)-(8.31), we find that, for $i = 0,1,2$ and $j \in \mathbb{N}$,

$$\mathcal{Q}^{(i)}_{X,F}(\ell; H,t) \ll \quad (8.42)$$
$$\ll O_\varepsilon\left((1+|t|)^{1/\varepsilon} \min\left(E, \frac{H^3}{X^{1-4\varepsilon}}\right) \mathcal{S}^{(i)}_{\infty,\ell,E}\left(\mathbf{b}^{(D)}(y+t), DN\right)\right) +$$
$$+ O_j\left((1+|t|)^j X^{1-2j\varepsilon} \mathcal{S}^{(i)}_{\infty,\ell,F}\left(\mathbf{b}^{(D)}(y+t), DN\right)\right),$$

(see (8.11), (8.12)-(8.14), (1.26)-(1.28), (1.31) and (1.45), and note that, by (1.15), (1.12), we may take $\delta = 1/4 - 7/64 = 9/64$ in (7.32) and (7.46)). By (1.44) and the cases $i = 1,2$ of the above,

$$\mathcal{Q}^{(1)}_{X,F}(\ell; H,t) + \mathcal{Q}^{(2)}_{X,F}(\ell; H,t) \ll$$
$$\ll O_\varepsilon\left((1+|t|)^{1/\varepsilon} \min\left(E, \frac{H^3}{X^{1-4\varepsilon}}\right) \sigma_{\ell,E}(\mathbf{b}, N; D, y+t)\right) +$$
$$+ O_j\left((1+|t|)^j X^{1-2j\varepsilon} \sigma_{\ell,F}(\mathbf{b}, N; D, y+t)\right).$$

Therefore, it follows by the bound (5.2) of Lemma 5.1 that

$$\mathcal{Q}^{(1)}_{X,F}(\ell; H,t) + \mathcal{Q}^{(2)}_{X,F}(\ell; H,t) \ll_{j,\varepsilon}$$
$$\ll_{j,\varepsilon} (1+|t|)^{1/\varepsilon} \min\left(E, \frac{H^3}{X^{1-4\varepsilon}}\right) \sigma_{\ell,E}(\mathbf{b}, N; D, y+t) +$$
$$+ (1+|t|)^j \frac{D^{2\vartheta}\tau^4(D)}{X^{2j\varepsilon-1}} \left(F^2 + \frac{(\ell,D)}{\ell} N^{1+\varepsilon}\right) \|\mathbf{b}_N\|_2^2.$$

Applying the bound (5.1) of Lemma 5.1 to both the O-terms on the right of (8.42) (in the case $i = 0$), we find that

$$\mathcal{Q}^{(0)}_{X,F}(\ell; H,t) \ll_{j,\varepsilon}$$
$$\ll_{j,\varepsilon} \left((1+|t|)^{1/\varepsilon} \min\left(E, \frac{H^3}{X^{1-4\varepsilon}}\right) \left(E^2 + \frac{(\ell,D)}{\ell} N^{1+\varepsilon}\right) +\right.$$
$$\left. + (1+|t|)^j X^{1-2j\varepsilon} \left(F^2 + \frac{(\ell,D)}{\ell} N^{1+\varepsilon}\right)\right) \tau^4(D) \|\mathbf{b}_N\|_2^2.$$

By (8.28), the last two bounds above, and (1.43), we conclude that

$$\mathcal{P}_{L,X,F}(H,t) \ll_{j,\varepsilon} (1+|t|)^{1/\varepsilon} \min\left(E, \frac{H^3}{X^{1-4\varepsilon}}\right) S_{L,E}(\mathbf{b}, N; D, y+t) + \\ + \left((1+|t|)^{1/\varepsilon} \min\left(E, \frac{H^3}{X^{1-4\varepsilon}}\right) \left(LE^2 + \tau(D)N^{1+\varepsilon}\right) + \right. \\ \left. + (1+|t|)^j \frac{D^{2\vartheta}\left(LF^2 + \tau(D)N^{1+\varepsilon}\right)}{X^{2j\varepsilon-1}} \right) \tau^4(D) \left\|\mathbf{b}_N\right\|_2^2 .$$

Taking $j = [4/\varepsilon^2]$, we will have here

$$2j\varepsilon - 1 > 2\varepsilon\left(4/\varepsilon^2 - 1\right) - 1 = 8/\varepsilon - 2\varepsilon - 1 > 15/(2\varepsilon) + 1 - 2\varepsilon. \tag{8.43}$$

Moreover, by (8.37) and (8.39),

$$D^{2\vartheta}\left(LF^2 + \tau(D)N^{1+\varepsilon}\right) \leqslant D^{2\vartheta}F^2\left(L + \tau(D)N^{1+\varepsilon}\right),$$

where (given that $0 < \varepsilon < 1/4$),

$$D^{2\vartheta}F^2 = D^{10\vartheta+4\varepsilon}X^{18} \leqslant D^3X^{18} \leqslant X^{18+3/\varepsilon} \leqslant X^{15/(2\varepsilon)},$$

so that, by (8.43) and (8.37),

$$D^{2\vartheta}F^2/X^{2j\varepsilon-1} \leqslant X^{-(1-2\varepsilon)} \leqslant 1.$$

This, with (8.40), shows that the lemma follows from (8.41) and the case $j = [4/\varepsilon^2]$ of the bound found for $\mathcal{P}_{L,X,F}(H,t)$. ■

Lemma 8.5. *Suppose that $\Delta \in (0, 1/2]$ is sufficiently small (in absolute terms). Let $0 < \varepsilon < 1/4$, $Q, N > 0$, $K \geqslant 1$, $D \in \mathbb{N}$, $y \in \mathbb{R}$. Suppose also that $\mathbf{b} = (b_n)$ is a complex sequence, and that $Y > 0$ is given by* (8.2). *Then either*

$$S_{Q,K}(\mathbf{b}, N; D, y) \ll_\varepsilon (DN)^\varepsilon \left(QK^2 + N\right) \left\|\mathbf{b}_N\right\|_2^2 , \tag{8.44}$$

or there exists

$$L \in \left[2^{-4}D^{-\varepsilon}Y, 2\Delta^{-1}Y\right] \tag{8.45}$$

such that, for $k = 2, 3, 4, \ldots$,

$$S_{Q,K}(\mathbf{b}, N; D, y) \ll_{\varepsilon,k} (DN)^{5\varepsilon} \int_1^\infty \int_{-\infty}^\infty \frac{S_{L,G}(\mathbf{b}, N; D, y+t)}{(1+|t|)^k} \, \mathrm{d}t \, \frac{\mathrm{d}G}{G^{7/2}} + \\ + (DN)^{6\varepsilon}(Y+N)\left\|\mathbf{b}_N\right\|_2^2 , \tag{8.46}$$

or there exist X, L satisfying (8.37) *and* (8.5) *such that, for* $k = 2, 3, 4, \ldots$,

$$S_{Q,K}(\mathbf{b}, N; D, y) \ll_{\varepsilon,k} (DN)^{6\varepsilon}\left(1 + \frac{X}{K^2}\right)^{-1} \int_{-\infty}^{\infty} \frac{S_{L,E}(\mathbf{b}, N; D, y+t)}{(1+|t|)^k}\,\mathrm{d}t + \tag{8.47}$$

$$+ (DN)^{9\varepsilon}\left(1 + \frac{X}{K^2}\right)^{-1} (Y + N)\,\|\mathbf{b}_N\|_2^2,$$

where

$$E = 5e^4 X^{\varepsilon+1/2} \ll (DN)^{\varepsilon}\sqrt{X}. \tag{8.48}$$

Proof. We may assume $Q, N \geqslant 1$, since otherwise it follows trivially from (1.42) and (1.43) that $S_{Q,K}(\mathbf{b}, N; D, y) = 0$. Therefore, and by (8.2),

$$Y \leqslant 16\pi DN \qquad \text{and} \qquad \log\left(\tfrac{2}{\Delta}Y\right) \ll_{\varepsilon} (DN)^{\varepsilon/2}. \tag{8.49}$$

We apply Lemma 8.2 with

$$j = [1/\varepsilon] + 1. \tag{8.50}$$

As (8.4) with this j would imply (8.44), it follows by Lemma 8.2 that we may henceforth take as given a pair X, L satisfying (8.5) for which (8.6) holds.

Now suppose the pair X, L has $X < D^{\varepsilon}$. Then (8.45) follows by (8.5), while the bound (8.26) of Lemma 8.3 shows that

$$\mathcal{P}_{L,X}(U_h, t) \ll$$

$$\ll (1+|t|)^2 D^{(9/2)\varepsilon}\left(\int_1^{\infty} S_{L,G}(\mathbf{b}, N; D, y+t)\frac{\mathrm{d}G}{G^{7/2}} + O_{\varepsilon}\Big((DN)^{\varepsilon}(L+N)\,\|\mathbf{b}_N\|_2^2\Big)\right),$$

for $h = 0, 1, \ldots$ (with U_h given by (8.1)). Since the bound here is independent of h, and since $\sum_{h=0}^{\infty} U^{-h} \leqslant 2$, the bound (8.46) follows, by (8.45), (8.49) and substitution of '$k+2$' for 'k', after using the bound for $\mathcal{P}_{L,X}(U_h, t)$ with (8.6) of Lemma 8.2.

It remains for us to consider the cases where (8.37) holds (that is, where the pair X, L has $X \geqslant D^{\varepsilon}$). In such cases Lemma 8.4 applies, showing that

$$\mathcal{P}_{L,X}(U_h, t) \ll_{\varepsilon} (1+|t|)^{4/\varepsilon^2} \min\left(E, \frac{U_h^3}{X^{1-4\varepsilon}}\right) \times \tag{8.51}$$

$$\times \left(S_{L,E}(\mathbf{b}, N; D, y+t) + (DN)^{\varepsilon}\left(LE^2 + N\right)\|\mathbf{b}_N\|_2^2\right),$$

for $h = 0, 1, \ldots$. The only factor here dependent on h is the 'U_h' in the minimum, and, by (8.1) and (8.38),

$$U^{-h}\min\left(E, \frac{U_h^3}{X^{1-4\varepsilon}}\right) \ll \min\left(\frac{X^{\varepsilon+1/2}}{U^h}, \frac{U^{2h+3}}{X^{1-4\varepsilon}}\right) \leqslant \left(\frac{X^{2\varepsilon+1}U^{2h+3}}{U^{2h}X^{1-4\varepsilon}}\right)^{1/3} = X^{2\varepsilon}U,$$

where $U \geqslant 2$, so that

$$\sum_{h \in \mathbb{Z}} U^{-h} \min\left(E, \frac{U_h^3}{X^{1-4\varepsilon}}\right) \ll X^{2\varepsilon} U.$$

We also have here

$$\sum_{\substack{h=0 \\ U^h \leqslant K}}^{\infty} U^{-h} \min\left(E, \frac{U_h^3}{X^{1-4\varepsilon}}\right) \leqslant \sum_{\substack{h=0 \\ U^h \leqslant K}}^{\infty} \frac{U^{2h+3}}{X^{1-4\varepsilon}} \ll \frac{U^3 K^2}{X^{1-4\varepsilon}},$$

so it may be concluded that

$$\sum_{\substack{h=0 \\ U^h \leqslant K}}^{\infty} U^{-h} \min\left(E, \frac{U_h^3}{X^{1-4\varepsilon}}\right) \ll U^3 X^{4\varepsilon} \left(1 + \frac{X}{K^2}\right)^{-1},$$

where, by (8.1), (8.5), (8.49) and (8.50),

$$U^3 X^{4\varepsilon} \leqslant 8 D^{6\vartheta/j} Y^{4\varepsilon} \ll D^{1/j} (DN)^{4\varepsilon} \leqslant (DN)^{5\varepsilon}.$$

Therefore, by using (8.51) with (8.6) and (8.49), we obtain a result from which (8.47) follows directly, on noting that (8.48) is implied by (8.38), (8.5) and (8.49), and itself implies $LE^2 \ll (DN)^{2\varepsilon} Y$ (given (8.5)). ■

Proof of Proposition 1.2. In view of (1.42)-(1.43) and the statement of Proposition 1.2, we need only complete the proof for those cases where $Q, N \geqslant 1$ and $0 < \varepsilon < 1$. We shall show that in such cases the proposition is a corollary of Lemma 8.5, applied with $\Delta = 32\pi/C$, and with $\varepsilon/9$ substituted for ε.

Note first that, if (8.44) holds, then it follows trivially from (1.42) and (1.43) that the bound given by the proposition holds with any choice of $G, L \in (0, \infty)$. As the choice $G = 1$, $L = DN/Q$ makes (1.53) and the inequality $G \geqslant 1$ hold, we therefore are left only needing to consider the second and third of the three cases described in Lemma 8.5. Before moving on to consider these two cases in isolation from one another, we may note here that they both involve a parameter L satisfying (1.53) (for this see (8.45), (8.5) and (8.2), while recalling that $\Delta = 32\pi/C$). Moreover, as k, in (8.46) or (8.47), is in either case an arbitrary element of the set $\{2, 3, 4, \ldots\}$, we may now commit ourselves to the choice $k = j$.

In the former of the two cases we find, by (8.46) and (8.2),

$$S_{Q,K}(\mathbf{b}, N; D, y) \ll_{\varepsilon,j} (DN)^{\varepsilon} \left(\int_1^{\infty} F_j(G) \frac{\mathrm{d}G}{G^{3/2}} + \left(\frac{DN}{Q} + N\right) \|\mathbf{b}_N\|_2^2 \right), \quad (8.52)$$

where

$$F_j(G) = \int_{-\infty}^{\infty} \frac{1}{G^2} S_{L,G}(\mathbf{b}, N; D, y + t) \frac{\mathrm{d}t}{(1 + |t|)^j}. \quad (8.53)$$

Since $j \geqslant 2$, this function $F_j(G)$ is (like $f(G)$ in the proof of Proposition 1.1 at the end of Section 5) a bounded function from $[1,\infty)$ into $[0,\infty)$. Therefore

$$\int_1^\infty F_j(G)\frac{\mathrm{d}G}{G^{3/2}} \leqslant 2F_j(G)\int_1^\infty g^{-3/2}\mathrm{d}g \ll F_j(G),$$

for some $G \geqslant 1$. As $K \geqslant 1$ this shows that (8.52)-(8.53) is at least as strong as the result given by Proposition 1.2.

Finally, in the last of the cases described in Lemma 8.5, we deduce from (8.47)-(8.48), (8.2) and (8.37) (with $\varepsilon/9$ substituted for ε and j for k) the bound:

$$S_{Q,K}(\mathbf{b},N;D,y) \ll_{\varepsilon,j} (DN)^{2\varepsilon/3}K^2X^{-1}E^2F_j(E) + (DN)^{\varepsilon}\left(\frac{DN}{Q}+N\right)\|\mathbf{b}_N\|_2^2,$$

where $F_j(E)$ is as in (8.53)(with E substituted for G) and $5e^4 \leqslant E \ll (DN)^{\varepsilon/9}\sqrt{X}$. Since we have here $E \geqslant 1$ (as well as $K \geqslant 1$) and

$$(DN)^{2\varepsilon/3}X^{-1}E^2 \ll (DN)^{8\varepsilon/9} \leqslant (DN)^{\varepsilon},$$

it therefore follows that the bound just given for $S_{Q,K}(\mathbf{b},N;D,y)$ implies that the bound stated in the proposition holds with $G = E \geqslant 1$ (see (8.53) and recall our previous observation that L there satisfies (1.53)). As we have reached this same conclusion in all the three cases allowed by Lemma 8.5, this completes the proof of Proposition 1.2.

9. Proving Theorem 1.6

By Lemma 5.1 and (1.43) it follows that, for $\varepsilon, N, Q > 0$, $D \in \mathbb{N}$, $K \geqslant 1$ and any complex sequence $\mathbf{b} = (b_n)$, one has

$$\begin{aligned} S_{Q,K}(\mathbf{b},N;D,y) &\ll_\varepsilon D^{2\vartheta}\tau^4(D)\|\mathbf{b}_N\|_2^2 \sum_{Q/2<q\leqslant Q}\left(K^2+\frac{(q,D)}{q}N^{1+\varepsilon}\right) \ll_\varepsilon \\ &\ll_\varepsilon D^{2\vartheta+\varepsilon}\left(QK^2+N^{1+\varepsilon}\right)\|\mathbf{b}_N\|_2^2 . \end{aligned} \tag{9.1}$$

We take the rest of this section to prove Theorem 1.6. The proof uses Propositions 1.1 and 1.2, and (as an 'initial result') the bound (9.1).

Proof of Theorem 1.6. Since the case $\varepsilon = \varepsilon_0$ (say) directly implies all cases with $\varepsilon \geqslant \varepsilon_0$ (cases with $QDN < 1$ being trivial, by (1.42), (1.43)), we may assume $\varepsilon \in (0,1/2)$ henceforth. There are several points in this proof where we shall require that an implicit constant depending only upon ε be bounded above by one or other of $C_0(\varepsilon)$, $M_0(\varepsilon)$. As there are only finitely many instances of such a requirement (and as no form of self-reference is involved in any instance), we may henceforth assume that $C_0(\varepsilon)$, $M_0(\varepsilon)$ have been chosen (once and for all) sufficiently large to simultaneously satisfy all such requirements.

Given that $C_0(\varepsilon)$ was chosen sufficiently large, the 'initial result' (9.1) establishes (1.61) for those cases relevant to Theorem 1.6 in which one has $0 < Q^{1-\varepsilon} \leqslant \leqslant M$ (note that cases with $Q < 1$ or $N < 1$ are trivial).

We now fix on a choice of M satisfying (1.58) and suppose that $R > 0$ is such that (1.61) fails for $Q = R$ (with the given choice of M, and some K, y, D, $\mathbf{b}$, N and P satisfying the conditions stated in the theorem). Since (1.61) has been shown to hold if $0 < Q^{1-\varepsilon} \leqslant M$, we must have

$$R^{1-\varepsilon} > M. \tag{9.2}$$

By (1.42) and (1.43) it is clear that if (1.61) fails at $Q = R$, then it fails also at $Q = [R]$. Therefore we may take R to be the least integer for which (1.61) can fail with $Q = R$ (and with M as chosen). This means that we may assume in what follows that (1.61) holds for $0 < Q < R$ (with the fixed choice of M, and any K, y, D, $\mathbf{b}$, N and P satisfying the conditions stated in the theorem). We shall need to appeal to this last point on several occassions, and shall refer to it as our 'inductive hypothesis'.

Now suppose that K, y, D, $\mathbf{b}$, N and P are given, and satisfy the conditions stated in the theorem. We shall show that this supposition, combined with the prior assumptions concerning ε, M and R, is sufficient to deduce (1.61) with $Q = R$. Since there are no conditions upon the choice of K, y, D, $\mathbf{b}$, N and P, other than those imposed in the theorem, we shall therefore obtain a contradiction with our original assumptions about R. This will show that no R exists satisfying those original assumptions, so that there is no counterexample to the theorem with the given choice of M. This will prove the theorem, since the choice of M was restricted only by the hypothesis (1.58).

Before proceeding as just outlined, we recall our earlier comments that (1.61) is trivial for $N < 1$. Therefore $N \geqslant 1$ may be assumed in what follows. By (1.59) and (9.2), these last assumptions restrict us to cases where $R > M \geqslant N \geqslant 1$. For use in what follows, we define a new dependent variable: $\eta = \varepsilon^2/3$.

Suppose first that

$$DN \leqslant R^{2-\varepsilon}. \tag{9.3}$$

Then, by Proposition 1.2 with $Q = R$, $j = 2$ and η substituted for ε, we find that

$$\frac{S_{R,K}(\mathbf{b}, N; D, y)}{K^2} \ll_\varepsilon (DN)^\eta \left(R \|\mathbf{b}_N\|_2^2 + \int_{-\infty}^{\infty} \frac{S_{L,G}(\mathbf{b}, N; D, y+t)}{G^2 (1+|t|)^2} \,\mathrm{d}t \right),$$

for some $G \geqslant 1$ and some L satisfying

$$0 < L \leqslant C\frac{DN}{R} \ll \frac{DN}{R}. \tag{9.4}$$

By (9.3), (9.2) and (1.58),

$$\frac{DN}{R} \leqslant R^{1-\varepsilon} < \frac{R}{(M_0(\varepsilon))^{\varepsilon}} < \frac{R}{C} \tag{9.5}$$

(given that $M_0(\varepsilon)$ is sufficiently large in terms of ε). Since (9.4) and (9.5) imply $L < R$, we may therefore apply our inductive hypothesis to $S_{L,G}(\mathbf{b}, N; D, y+t)$, in order to deduce from the above application of Proposition 1.2 that

$$\begin{aligned} \frac{1}{K^2} S_{R,K}(\mathbf{b}, N; D, y) &\ll_\varepsilon \\ &\ll_\varepsilon (DN)^{\eta} \|\mathbf{b}_N\|_2^2 \times \\ &\times \left(R + C_0(\varepsilon)(LDN)^{\varepsilon}\left(L + D^{\varrho}M + (PDN)^{\varrho}\left(\min\left(L, \sqrt{DN}\right)\right)^{\zeta}\right)\right). \end{aligned} \tag{9.6}$$

Since $L < R$ we have only to note here that, by (9.4), (9.5), (9.3), (9.2) and (1.58),

$$(LDN)^{\varepsilon}/(RDN)^{\varepsilon} = (L/R)^{\varepsilon} \ll R^{-\varepsilon^2} < (M_0(\varepsilon))^{-\eta}(DN)^{-\eta}$$

and

$$(DN)^{\eta} \leqslant (DN)^{\varepsilon} = R^{-\varepsilon}(RDN)^{\varepsilon} \leqslant (M_0(\varepsilon))^{-\varepsilon}(RDN)^{\varepsilon},$$

in order to conclude that the bound (9.6) will imply (1.61) for $Q = R$ (given that $M_0(\varepsilon)$ was chosen sufficiently large in terms of ε).

The above concludes our treatment of the cases where (9.3) holds. For the remainder of the proof it may be assumed that

$$DN > R^{2-\varepsilon}. \tag{9.7}$$

As it follows from (9.7), (9.2) and (1.59) that $D > R^{2-3\varepsilon}/N > M/N \geqslant 1$, the bound (1.60) therefore implies that we may choose $D_1|D$ to satisfy

$$\frac{R^{2-3\varepsilon}}{PN} < D_1 \leqslant \frac{R^{2-3\varepsilon}}{N} \tag{9.8}$$

(note here that $D > 1$ implies $P \geqslant 2$, and that we have no reservations about choosing $D_1 = 1$ when (9.8) permits it). For later use, we note that this choice of D_1 ensures

$$\left(\frac{D}{D_1}\right)^{\varrho} R^{1-\varepsilon} < (PDN)^{\varrho} R^{1-\varepsilon-(2-3\varepsilon)\varrho} = (PDN)^{\varrho} R^{\zeta - \frac{11}{32}\varepsilon} < (PDN)^{\varrho} R^{(1-\varepsilon/2)\zeta} \tag{9.9}$$

and that, by (9.7), we have here

$$R^{1-\varepsilon/2} < \min\left(R, \sqrt{DN}\right). \tag{9.10}$$

By Proposition 1.1, we find

$$\frac{S_{R,K}(\mathbf{b},N;D,y)}{K^2} \ll_\varepsilon \frac{1}{G^2} S_{R_1,G}\left(\mathbf{b}^{\{g_0\}}, \frac{N}{g_0}; \frac{D_1}{g_1}, y\right)\left(\frac{D}{D_1}\right)^{\varrho+\eta}, \tag{9.11}$$

for some $G \geqslant 1$, some

$$R_1 \in (0, R], \tag{9.12}$$

and some g_0, g_1 and sequence $\mathbf{b}^{\{g_0\}}$, satisfying (1.51) and (1.52).

We will treat first the cases where

$$R_1^{2-\varepsilon} < D_1 N. \tag{9.13}$$

Given (9.13), it follows by (9.8) that

$$R_1 < R^{(2-3\varepsilon)/(2-\varepsilon)} = R^{1-2\varepsilon/(2-\varepsilon)} < R^{1-\varepsilon}, \tag{9.14}$$

so that $R_1 < R$ and we are therefore able to apply our inductive hypothesis on the right-hand side of (9.11). As $0 < N/g_0 \leqslant N \leqslant M$ and $\|\mathbf{b}^{\{g_0\}}_{N/g_0}\|_2^2 \leqslant \|\mathbf{b}_N\|_2^2$, while D_1/g_1 is a factor of D_1 (and therefore of D), it follows from (9.11) and the inductive hypothesis that

$$\begin{aligned} &\frac{1}{K^2} S_{R,K}(\mathbf{b},N;D,y) \ll_\varepsilon \\ &\quad \ll_\varepsilon C_0(\varepsilon)(R_1 D_1 N)^\varepsilon \times \\ &\quad \times \left(R_1 + D_1^\varrho M + (PD_1N)^\varrho\left(\min\left(R_1, \sqrt{D_1 N}\right)\right)^\zeta\right)\|\mathbf{b}_N\|_2^2 \left(\frac{D}{D_1}\right)^{\varrho+\eta} \leqslant \\ &\quad \leqslant C_0(\varepsilon)(R_1 D_1 N)^\varepsilon \times \\ &\quad \times \left(R_1\left(\frac{D}{D_1}\right)^\varrho + D^\varrho M + (PDN)^\varrho\left(\min\left(R, \sqrt{DN}\right)\right)^\zeta\right)\|\mathbf{b}_N\|_2^2 \left(\frac{D}{D_1}\right)^{\eta}. \end{aligned} \tag{9.15}$$

By (9.14), (9.9) and (9.10), we have here

$$R_1\left(\frac{D}{D_1}\right)^\varrho < (PDN)^\varrho\left(\min\left(R, \sqrt{DN}\right)\right)^\zeta$$

and

$$(R_1 D_1 N)^\varepsilon\left(\frac{D}{D_1}\right)^\eta < \left(R^{1-\varepsilon} D_1 N\right)^\varepsilon\left(\frac{D}{D_1}\right)^\varepsilon = R^{-\varepsilon^2}(RDN)^\varepsilon,$$

so that, by (9.2) and (1.58), the bounds of (9.15) imply (1.61) for $Q = R$.

It remains to consider the cases where (9.13) is false, so that

$$R_1^{2-\varepsilon} \geqslant D_1 N. \tag{9.16}$$

In such a case we apply Proposition 1.2 after (9.11)-(9.12), obtaining the bound:

$$\begin{aligned}\frac{1}{K^2} S_{R,K}(\mathbf{b}, N; D, y) &\ll_\varepsilon \\ &\ll_\varepsilon \left(\frac{D}{D_1}\right)^{\varrho+\eta} (D_1 N)^\eta \times \\ &\times \left((R_1 + N)\|\mathbf{b}_N\|_2^2 + \int_{-\infty}^{\infty} S_{L,H}\left(\mathbf{b}^{\{g_0\}}, \frac{N}{g_0}; \frac{D_1}{g_1}, y+t\right) \frac{H^{-2}\mathrm{d}t}{(1+|t|)^2} \right),\end{aligned} \tag{9.17}$$

for some $H \geqslant 1$ and some L satisfying $0 < L \leqslant C D_1 N/R_1 \ll D_1 N/R_1$ (and with R_1, g_0, g_1 and $\mathbf{b}^{\{g_0\}}$ as in (9.12), (1.51) and (1.52)). Therefore, and by (9.16) and (9.12), we have here

$$L \ll R_1^{1-\varepsilon} \leqslant R^{1-\varepsilon}, \tag{9.18}$$

which (see (9.2), (1.58)) implies $L < R$, given that $M_0(\varepsilon)$ was chosen sufficiently large in terms of ε. This enables us to apply the inductive hypothesis with (9.17), so as to obtain (given (9.12) and given that $R > M \geqslant N$):

$$\begin{aligned}\frac{1}{K^2} S_{R,K}(\mathbf{b}, N; D, y) &\ll_\varepsilon \\ &\ll_\varepsilon \left(R + C_0(\varepsilon)(LD_1N)^\varepsilon \left(L + D_1^\varrho M + (PD_1N)^\varrho \left(\min\left(L, \sqrt{D_1 N}\right)\right)^\zeta\right)\right) \times \\ &\times \left(\frac{D}{D_1}\right)^{\varrho+\eta} (D_1 N)^\eta \|\mathbf{b}_N\|_2^2 \leqslant \\ &\leqslant C_0(\varepsilon)(LDN)^\varepsilon (D_1 N)^\eta \times \\ &\times \left(L\left(\frac{D}{D_1}\right)^\varrho + D^\varrho M + (PDN)^\varrho \left(\min\left(R, \sqrt{DN}\right)\right)^\zeta\right) \|\mathbf{b}_N\|_2^2 + \\ &+ R\left(\frac{D}{D_1}\right)^\varrho (DN)^\eta \|\mathbf{b}_N\|_2^2 .\end{aligned} \tag{9.19}$$

By (9.7) we have here

$$R\left(\frac{D}{D_1}\right)^\varrho (DN)^\eta < R\left(\frac{D}{D_1}\right)^\varrho \frac{(DN)^\varepsilon}{R^\eta} = R^{-\eta}(RDN)^\varepsilon \left(\frac{D}{D_1}\right)^\varrho R^{1-\varepsilon},$$

while by (9.18) and (9.8)-(9.10),

$$(LDN)^\varepsilon (D_1 N)^\eta \ll \left(R^{1-\varepsilon} DN\right)^\varepsilon R^{2\eta} = R^{-\eta}(RDN)^\varepsilon$$

and

$$L\left(\frac{D}{D_1}\right)^\varrho \ll \left(\frac{D}{D_1}\right)^\varrho R^{1-\varepsilon} < (PDN)^\varrho \left(\min\left(R, \sqrt{DN}\right)\right)^\zeta$$

so that (1.61) for $Q = R$ follows from the bounds of (9.19) by virtue of (9.2), (1.58) and the choice of $M_0(\varepsilon)$.

Having fulfilled our promise (that we would deduce (1.61) for $Q = R$ in all relevant cases), the reasoning set out half a paragraph below (9.2) now permits us to conclude that the theorem is proved. ■

10. A special sum of Kloosterman sums

This section is concerned with the sum of Kloosterman sums $A_{Q,H,X}(\mathbf{b}, N; D, y)$, defined in (1.54)-(1.56), (1.48) and (1.69)-(1.71). We establish a chain of lemmas, with the last of them (Lemma 10.12) being a bound for $A_{Q,H,X}(\boldsymbol{\Psi}, N; D, y)$ that is valid for sequences $\boldsymbol{\Psi}$ of the same form as in (1.63). We need this bound in the next (final) section of this paper, where it enables us to prove Lemma 11.1, and so to obtain the 'initial result', (11.7), in the proof of Theorem 1.8.

We think it worth noting that the bound given by Lemma 10.12 is independent of both Q and X (so long as $X > \Delta$).

In Lemma 10.5 we arrive at sums $\mathcal{V}(D_\delta; A/c_1)$ involving terms with a factor $\gamma_n(D_\delta; A/c_1)$ that invites an analysis by the method of stationary phase (see (10.25), (10.26), and, for an application of the stationary phase method [23], Lemma 4.6). The error terms that arise in such an anlysis turn out to be significant, so we resort to a careful 'deconstruction' of the stationary phase method (see Lemmas 10.6, 10.7 and 10.8).

Our first and third lemmas link up somewhat late with the others (the former being needed for the proofs of Lemmas 10.9 and 10.10, while the latter helps to complete the proof of Lemma 10.12).

Lemma 10.1. *Let $\delta \in (0,1)$. Then, for $X, Y > 0$ and $A, B \in \mathbb{Z}$ with $A \neq 0$,*

$$\sum_{\substack{\delta X \leqslant c \leqslant X/\delta \\ (A,c)=1}} \min\left(Y, \left\| \frac{B\bar{c}}{A} \right\|^{-1}\right) \ll_\delta \begin{cases} XY & \text{if } A|B, \\ (X + |A|/(A,B)) \log |A| & \text{otherwise.} \end{cases}$$

Proof. The bound in the case $A|B$ is trivial, since there are no more than X/δ integers $c \in (0, X/\delta]$. Suppose now that A is not a factor of B. Then we may write $B/A = B_1/A_1$ where $(A_1, B_1) = 1$, $A_1 \in \mathbb{N} - \{1\}$ and $B_1 \in \mathbb{Z} - \{0\}$. The sum we wish to bound is

$$\sum_{\substack{\delta X \leqslant c \leqslant X/\delta \\ (A,c)=1}} \min\left(Y, \left\| \frac{B_1\bar{c}}{A_1} \right\|^{-1}\right) \leqslant \sum_{\substack{\delta X \leqslant c \leqslant X/\delta \\ (A_1,c)=1}} \left\| \frac{B_1\bar{c}}{A_1} \right\|^{-1} =$$

$$= \sum_{-A_1/2 < r \leqslant A_1/2} \left| \frac{r}{A_1} \right|^{-1} \sum_{\substack{\delta X \leqslant c \leqslant X/\delta \\ (A_1,c)=1 \\ B_1\bar{c} \equiv r \pmod{A_1}}} 1.$$

As $(A_1, B_1) = 1$ and $A_1 \geqslant 2$, we may bound the last sum above by

$$\sum_{\substack{|r|\leqslant A_1/2 \\ (A_1,r)=1}} \frac{A_1}{|r|} \sum_{\substack{\delta X\leqslant c\leqslant X/\delta \\ c\equiv B_1\bar{r} \pmod{A_1}}} 1 \ll \sum_{0<|r|\leqslant A_1/2} \frac{A_1}{|r|}\left(1+\frac{X}{\delta A_1}\right) \ll_\delta$$

$$\ll_\delta (X + A_1)\sum_{r=1}^{A_1}\frac{1}{r} \ll (X + A_1)\log|A_1|,$$

which proves the lemma, since $A_1 = |A|/(A, B)$. ■

Lemma 10.2. *Let* $\mathbf{b} = (b_n)$ *be a complex sequence. Let* $0 < \varepsilon < 1/2$, $Q, N > 0$, $D \in \mathbb{N}$, $H \geqslant 1$ *and* $X > \Delta$, *where* $\Delta \in (0, 1/2]$ *is an absolute constant. Suppose that*

$$Z = (NX)^\varepsilon X^{-1/2}. \tag{10.1}$$

Then, for some θ *satisfying*

$$1 \leqslant \theta \leqslant 1 + O(Z^2), \tag{10.2}$$

one has

$$A_{Q,H,X}(\mathbf{b}, N; D, 0) \ll O_\varepsilon\left(D\|\mathbf{b}_N\|_2^2\right) + H^3Z^3\,|\mathrm{Im}(B_{Q,X}(\mathbf{b}, N; D, \theta))|,$$

where

$$B_{Q,X}(\mathbf{b}, N; D, \theta) = \tag{10.3}$$

$$= \sum\sum_{N/2<m,n\leqslant N} \bar{b}_m b_n \times$$

$$\times \sum_{q=1}^{\infty}\sum_{\ell=1}^{\infty} \frac{\omega(q/Q)}{q\ell}\, g_X\!\left(\frac{4\pi D\sqrt{mn}}{q\ell}\right) \mathrm{e}\!\left(\frac{2\theta D\sqrt{mn}}{q\ell}\right) S(Dm, Dn; q\ell),$$

with

$$g_X(x) = x\Omega_0(x/X) \tag{10.4}$$

and with $\omega(x)$ *and* $\Omega_0(x)$ *as in* (1.49), (1.57) *and* (1.71) *(see also* (1.69)-(1.70)*).*

Proof. Let

$$B^{\perp}_{Q,X}(\mathbf{b}, N; D, \theta) = \mathrm{Im}(B_{Q,X}(\mathbf{b}, N; D, \theta)) \qquad (\theta \in \mathbb{R}). \tag{10.5}$$

As $\omega(x)$, $g_X(x)$ and $S(a, b; c)$ are real valued, it follows from (10.3) that

$$B^{\perp}_{Q,X}(\mathbf{b}, N; D, \theta) = \tag{10.6}$$

$$= \frac{1}{2i}\left(B_{Q,X}(\mathbf{b}, N; D, \theta) - \overline{B_{Q,X}(\mathbf{b}, N; D, \theta)}\right) =$$

$$= \frac{1}{2i}\left(B_{Q,X}(\mathbf{b}, N; D, \theta) - B_{Q,X}(\mathbf{b}, N; D, -\theta)\right) =$$

$$= \sum\sum_{N/2<m,n\leqslant N} \bar{b}_m b_n \times$$

$$\times \sum_{q=1}^{\infty}\sum_{\ell=1}^{\infty} \frac{\omega(q/Q)}{q\ell}\, g_X\!\left(\frac{4\pi D\sqrt{mn}}{q\ell}\right) \sin\!\left(\frac{4\pi\theta D\sqrt{mn}}{q\ell}\right) S(Dm, Dn; q\ell),$$

for $\theta \in \mathbb{R}$. By (1.54)-(1.56) and (1.48), (10.6) and (10.4),

$$A_{Q,H,X}(\mathbf{b},N;D,0) = H^3 \int_0^\infty \frac{\xi \tanh(\xi)}{e^{(H\xi)^2}} B^{\perp}_{Q,X}(\mathbf{b},N;D,\theta(\xi))\,\mathrm{d}\xi, \tag{10.7}$$

where

$$\theta(\xi) = \cosh(\xi). \tag{10.8}$$

In view of (10.6), it follows from (10.7) that we may henceforth assume that $N \geqslant 1$ (the lemma being otherwise trivial). By (10.6), (10.4), the bound $|S(Dm,Dn;q\ell)| \leqslant \leqslant q\ell$, (1.49) and (1.57), it follows that, for $\theta \in \mathbb{R}$,

$$\begin{aligned} B^{\perp}_{Q,X}(\mathbf{b},N;D,\theta) &\ll \sum\sum_{N/2<m,n\leqslant N} |b_m b_n| \sum\sum_{\substack{q\in(Q/4,2Q) \\ \frac{1}{2}<\frac{Xq\ell}{4\pi D\sqrt{mn}}<2}} \frac{DN}{q\ell} \ll \\ &\ll DN\,\|\mathbf{b}_N\|_1^2 \leqslant DN^2\,\|\mathbf{b}_N\|_2^2 . \end{aligned} \tag{10.9}$$

Therefore, given that $H \geqslant 1$, the bound $\tanh(\xi) \leqslant |\xi|$ allows us to conclude that, for $U \geqslant 1$,

$$\begin{aligned} \int_U^\infty \frac{\xi \tanh(\xi)}{e^{(H\xi)^2}} B^{\perp}_{Q,X}(\mathbf{b},N;D,\theta(\xi))\,\mathrm{d}\xi &\ll DN^2\,\|\mathbf{b}_N\|_2^2 \int_U^\infty \xi^2 e^{-\xi^2}\,\mathrm{d}\xi < \\ &< Ue^{-U^2} DN^2\,\|\mathbf{b}_N\|_2^2 . \end{aligned} \tag{10.10}$$

On the other hand, it follows from (1.54)-(1.57) and Lemma 7.4 that, for $j \in \{0,1,\ldots\}$,

$$\begin{aligned} &A_{Q,H,X}(\mathbf{b},N;D,0) = \\ &= \sum\sum_{N/2<m,n\leqslant N} \bar{b}_m b_n \sum_{q=1}^\infty\sum_{\ell=1}^\infty \frac{\omega(q/Q)}{q\ell} \times \\ &\times \Omega_0\left(\frac{4\pi D\sqrt{mn}}{Xq\ell}\right)\left(\Phi^*_H\left(\frac{4\pi D\sqrt{mn}}{q\ell}\right) + O_j(X^{1-2j\varepsilon})\right) S(Dm,Dn;q\ell), \end{aligned}$$

so that, by (7.28), (1.69)-(1.71) and (10.6), we find that, with Y as in (7.27), and $\theta(\xi)$ as in (10.8),

$$\begin{aligned} &|A_{Q,H,X}(\mathbf{b},N;D,0)| \leqslant \\ &\leqslant H^3 \int_0^Y |B^{\perp}_{Q,X}(\mathbf{b},N;D,\theta(\xi))| \frac{\xi \tanh(\xi)}{e^{(H\xi)^2}}\,\mathrm{d}\xi + O_j\left(X^{-2j\varepsilon} DN^2\,\|\mathbf{b}_N\|_2^2\right), \end{aligned} \tag{10.11}$$

(where, in bounding the error term, we essentially repeat what was done to get (10.9)).

Let

$$V = \varepsilon \log(N). \tag{10.12}$$

If

$$\varepsilon^2 \log(N) \geqslant \max(3, \log(X)), \tag{10.13}$$

then we apply (10.7) and the case $U = V$ of (10.10) (after noting that $V \geqslant 3/\varepsilon >$ > 1). Note that, given (10.12), (10.13), and given that $U = V$, we shall have, in (10.10),

$$Ue^{-U^2} = VN^{-\varepsilon^2 \log(N)} \leqslant VN^{-3} < N^{-2}.$$

If (10.13) does not hold, then

$$1 \leqslant N < \max\left(\exp\left(3\varepsilon^{-2}\right), X^{\varepsilon^{-2}}\right) \tag{10.14}$$

and we apply (10.11) with $j = [\varepsilon^{-3}] + 1$. Note that we will have, in (10.11),

$$O_j(X^{-2j\varepsilon} DN^2 \|\mathbf{b}_N\|_2^2) = O_j(X^{-2j\varepsilon} N^2) D \|\mathbf{b}_N\|_2^2,$$

and if $N < \exp(3\varepsilon^{-2})$, then

$$O_j\left(X^{-2j\varepsilon} N^2\right) = O_\varepsilon\left((1/\Delta)^{2\varepsilon^{-2}+1} \exp\left(6\varepsilon^{-2}\right)\right) = O_\varepsilon(1),$$

while if $N < X^{\varepsilon^{-2}}$, then (given that $N \geqslant 1$)

$$O_j\left(X^{-2j\varepsilon} N^2\right) = O_\varepsilon\left(X^{-2\varepsilon^{-2}} N^2\right) = O_\varepsilon(1).$$

Combining our results for the two complementary cases, (10.13) and (10.14), we conclude by (10.5) that we have shown, either by (10.11), or by (10.7) and (10.10), that it is the case that

$$|A_{Q,H,X}(\mathbf{b}, N; D, 0)| \leqslant H^3 \int_0^U |\operatorname{Im}(B_{Q,X}(\mathbf{b}, N; D, \theta(\xi)))| \frac{\xi \tanh(\xi)}{e^{(H\xi)^2}} \, d\xi + O_\varepsilon\left(D \|\mathbf{b}_N\|_2^2\right),$$

with

$$U = \begin{cases} V & \text{if (10.13) holds,} \\ Y & \text{if (10.14) holds.} \end{cases} \tag{10.15}$$

Since $U > 0$ here, since

$$\int_0^U \frac{\xi \tanh(\xi)}{e^{(H\xi)^2}} \, d\xi \leqslant \int_0^U \xi^2 d\xi = \frac{U^3}{3} \qquad (U > 0),$$

and since, by (10.8),

$$1 \leqslant \theta(\xi) = \cosh(\xi) \leqslant 1 + \xi^2 \cosh(\xi) \leqslant 1 + U^2 e^U \qquad (0 \leqslant \xi \leqslant U),$$

the lemma will follow from the above bound for $|A_{Q,H,X}(\mathbf{b}, N; D, 0)|$ (by an appeal to the first mean value theorem for integrals), provided only that we can show that, in all relevant cases,

$$U^2 e^U \ll Z^2,$$

with Z given by (10.1). To confirm this, note first that if (10.13) holds, then we certainly have $X \leqslant N^{\varepsilon/2}$ and, by (10.15) and (10.12), we also have $U = V = = \varepsilon \log(N)$, so that

$$U^2 e^U = (\varepsilon \log(N))^2 N^\varepsilon \leqslant 16 N^{3\varepsilon/2} \ll \left(N^{\varepsilon/2}/X\right)^{1-2\varepsilon} N^{3\varepsilon/2} = Z^2 N^{-\varepsilon^2} \leqslant Z^2.$$

If it is instead (10.14) that holds, then by (10.15) and (7.27), we have $U = Y = = X^{\varepsilon - 1/2}$, so that

$$U^2 e^U = X^{2\varepsilon-1} \exp\left((1/X)^{1/2-\varepsilon}\right) \leqslant N^{2\varepsilon} X^{2\varepsilon - 1} \exp\left(\sqrt{1/\Delta}\right) \ll Z^2.$$

■

Lemma 10.3. *Let* $\mathbf{b} = (b_n)$ *be a complex sequence. Let* $Q, N > 0$, $D \in \mathbb{N}$, $\theta \in \mathbb{R}$ *and* $X > \Delta$, *where* $\Delta \in (0, 1/2]$ *is an absolute constant, and suppose that* $B_{Q,X}(\mathbf{b}, N; D, \theta)$ *is given by* (10.3), (10.4) *and* (1.69)-(1.71). *Then*

$$B_{Q,X}(\mathbf{b}, N; D, \theta) \ll \log^{7/2}(1 + DN)\, \tau(D) D^{1/2} N^{3/2} X^{1/2} \|\mathbf{b}_N\|_2^2.$$

Proof. By Theorem 2.1 (Weil's bound) and the hypotheses of the lemma, we have

$$|B_{Q,X}(\mathbf{b}, N; D, \theta)| \leqslant \tag{10.16}$$
$$\leqslant \sum_{N/2<m,n\leqslant N} |b_m b_n| \sum_{\substack{Q/4<q<2Q \\ L/8<\ell<8L}} \frac{2X\tau(q\ell)}{\sqrt{q\ell}} (Dm, Dn, q\ell)^{1/2},$$

where $L = 4\pi DN/(XQ)$ (see (1.49), (1.57)). Therefore,

$$B_{Q,X}(\mathbf{b}, N; D, \theta) \ll \frac{X}{\sqrt{QL}} S_1 S_2 \asymp D^{-1/2} N^{-1/2} X^{3/2} S_1 S_2, \tag{10.17}$$

where,

$$S_1 = \sum_{Q/4<q<2Q} \sum_{L/8<\ell<8L} \tau(q\ell)(D, q\ell)^{1/2} \leqslant$$
$$\leqslant \left(\sum_{1\leqslant q<2Q} \tau^2(q) \sum_{1\leqslant \ell<8L} \tau^2(\ell)\right)^{1/2} \left(\sum_{1\leqslant q<2Q} (D,q) \sum_{1\leqslant \ell<8L} (D,\ell)\right)^{1/2} \ll$$
$$\ll \left(\log^3(1+Q)\log^3(1+L) QL\right)^{1/2} \left(\tau^2(D) QL\right)^{1/2} \ll$$
$$\ll (\log(1+Q)\log(1+L))^{3/2} \tau(D) DN/X$$

and

$$S_2 = \sum_{N/2<m,n\leqslant N}\sum |b_m b_n| (m,n)^{1/2} \leqslant$$

$$\leqslant \left(\sum_{N/2<m,n\leqslant N}\sum |b_m b_n|^2 \right)^{1/2} \left(\sum_{N/2<m,n\leqslant N}\sum (m,n) \right)^{1/2} \ll$$

$$\ll \left(\|\mathbf{b}_N\|_2^4 N^2 \log(1+N) \right)^{1/2} = \log^{1/2}(1+N) N \|\mathbf{b}_N\|_2^2 .$$

Since (10.16) shows the lemma to be trivial unless it is the case that $Q > 1/2$ and $L = 4\pi DN/(XQ) > 1/8$, it may be assumed here that $\max(Q, L) < 8QL = 32\pi DN/X$, so that one has

$$(\log(1+Q)\log(1+L))^{3/2} \log^{1/2}(1+N) \ll \log^{7/2}(1+DN),$$

(given that $X > \Delta$ and $D \in \mathbb{N}$). Therefore (10.17) and the bounds for S_1 and S_2 imply the lemma. ■

Lemma 10.4. *Let $Q, X > 0$, $\theta \in \mathbb{R}$ and $D, N, N_1 \in \mathbb{N}$ with $N/2 < N_1 \leqslant N$. Let $\Psi = (\Psi_n)$ be the sequence given by*

$$\Psi_n = \begin{cases} 1 & \text{if } N_1 \leqslant n \leqslant N, \\ 0 & \text{otherwise}, \end{cases} \qquad (n \in \mathbb{N}) \tag{10.18}$$

and suppose that $B_{Q,X}(\Psi, N; D, \theta)$ is as given by (10.3)-(10.4) (with Ψ in place of $\mathbf{b}$) and (1.69)-(1.71). Then

$$B_{Q,X}(\Psi, N; D, \theta) = \sum_{\delta | D} B_{Q,X}^{(\delta)}(\Psi, N; D, \theta),$$

where

$$B_{Q,X}^{(\delta)}(\Psi, N; D, \theta) = \sum_{\substack{QL/4<q\ell<2QL \\ (q\ell,D)=\delta}}\sum \frac{\omega(q/Q)}{q\ell} \sum_{b \bmod q\ell}^{*} \mathcal{S}_b(D/\delta, q\ell/\delta) \tag{10.19}$$

with

$$L = \frac{4\pi DN}{XQ}, \tag{10.20}$$

while

$$\mathcal{S}_b(d, c) = \sum_{m,n\in[N_1,N]}\sum g_X\left(\frac{4\pi d\sqrt{mn}}{c}\right) \mathrm{e}\left(\frac{2\theta d\sqrt{mn}}{c} + \frac{dm\bar{b} + dnb}{c}\right), \tag{10.21}$$

with $g_X(x)$ as in (10.4).

Proof. For $(q\ell, D) = \delta$, we may apply (2.8) to write:

$$S(Dm, Dn; q\ell) = \sum_{b \bmod q\ell}^{*} \mathrm{e}\left(Dm\frac{\overline{b}}{q\ell} + Dn\frac{b}{q\ell}\right) = \sum_{b \bmod q\ell}^{*} \mathrm{e}\left(\frac{D_1 m\overline{b} + D_1 nb}{c_1}\right),$$

where $D_1 = D/\delta$, $c_1 = q\ell/\delta$. Upon using this result with the case $\mathbf{b} = \boldsymbol{\Psi}$ of (10.3), we have only to bring the summations over m, n inside the summations over q, ℓ and b, in order to obtain the result given by the lemma.

Note that the condition $QL/4 < q\ell < 2QL$ in (10.19) is only included for later reference: it is in fact superfluous, since it follows by (10.21), (10.4) and (1.57) that $\mathcal{S}_b(D/\delta,\, q\ell/\delta) = 0$ unless $X/2 < 4\pi DN/q\ell$ and $4\pi DN_1/q\ell < 2X$. ∎

Lemma 10.5. *Let $\varepsilon, Q, X > 0$, $\theta \in \mathbb{R}$, and $D, \delta, N, N_1 \in \mathbb{N}$ with $\delta | D$, $N > 1$, $N_1 \in (N/2, N]$ and*

$$|\theta| < N/X. \tag{10.22}$$

Let the sequence $\boldsymbol{\Psi} = (\Psi_n)$ be given by (10.18). Suppose also that L is given by (10.20), that C_δ, D_δ, N_1^- and N^+ are given by

$$C_\delta = QL/\delta, \quad D_\delta = D/\delta, \tag{10.23}$$

$$N_1^- = N_1 - \tfrac{1}{2} \quad \text{and} \quad N^+ = N + \tfrac{1}{2}, \tag{10.24}$$

and that $B_{Q,X}^{(\delta)}(\boldsymbol{\Psi}, N; D, \theta)$ is as in Lemma 10.4. Then

$$B_{Q,X}^{(\delta)}(\boldsymbol{\Psi}, N; D, \theta) \ll$$

$$\ll \frac{\tau(\delta)}{\delta} DN \log\left(1 + \frac{DN}{\delta X}\right) + O_\varepsilon\left(\delta (QL)^{\varepsilon - 1} \sum_{\substack{C_\delta/4 < c_1 < 2C_\delta \\ (c_1, D_\delta) = 1}} \sum_{\substack{|A| < N^+ c_1 \\ (A, c_1) = 1}} |\mathcal{V}(D_\delta; A/c_1)|\right),$$

where

$$\mathcal{V}(D_\delta; A/c_1) = \sum_{n = N_1}^{N} \mathrm{e}\left(-\frac{D_\delta^2 \bar{A} n}{c_1}\right) \gamma_n(D_\delta; A/c_1), \tag{10.25}$$

while

$$\gamma_n(D_\delta; A/c_1) = \int_{N_1^-}^{N^+} g_X\left(\frac{4\pi D_\delta \sqrt{\mu n}}{c_1}\right) \mathrm{e}\left(\frac{2\theta D_\delta \sqrt{\mu n}}{c_1} - \frac{A}{c_1}\mu\right) \mathrm{d}\mu, \tag{10.26}$$

with $g_X(x)$ as in (10.4).

Proof. We first consider the terms $\mathcal{S}_b(D/\delta, q\ell/\delta)$ in (10.19). Each of these has the form $\mathcal{S}_b(D_\delta, c_1)$, where $(b, q\ell) = 1$, D_δ is given by (10.23) and $c_1 = q\ell/\delta \in \mathbb{N}$ satisfies

$$(c_1, D_\delta) = 1 \quad \text{and} \quad C_\delta/4 < c_1 < 2C_\delta. \tag{10.27}$$

Given any such integers, b, D_δ and c_1, suppose that A_1 is the unique integer $A_1 \in (-c_1/2, c_1/2]$ satisfying

$$D_\delta \bar{b} \equiv -A_1 \pmod{c_1}. \tag{10.28}$$

As $(D_\delta, c_1) = 1 = (b, \delta c_1)$, one then has

$$(A_1, c_1) = 1, \tag{10.29}$$

so that it follows from (10.28) that

$$b \equiv -D_\delta \overline{A_1} \pmod{c_1}. \tag{10.30}$$

By (10.28)-(10.30) and (10.21), we now have

$$\mathcal{S}_b(D/\delta, q\ell/\delta) = \mathcal{S}_b(D_\delta, c_1) = \mathcal{T}(D_\delta; A_1/c_1), \tag{10.31}$$

where, for $c, d \in \mathbb{N}$ and $a \in \mathbb{Z}$ with $(a, c) = 1$,

$$\mathcal{T}\left(d; \frac{a}{c}\right) = \sum\sum_{m,n \in [N_1, N]} g_X\left(\frac{4\pi d\sqrt{mn}}{c}\right) \mathrm{e}\left(\frac{2\theta d\sqrt{mn}}{c} - \frac{am + d^2\bar{a}n}{c}\right). \tag{10.32}$$

The notation in (10.31) emphasises that $\mathcal{S}_b(D_\delta, c_1)$ may be regarded as a function of A_1/c_1, rather than as a function of q, ℓ and $b \bmod q\ell$. Given only $D_\delta, c_1 \in \mathbb{N}$ (as above) and $A_1 \in \mathbb{Z}$, there are no more than $\tau(\delta c_1)$ choices for q and ℓ such that $q\ell/\delta = c_1$, and certainly no more than $q\ell/c_1 = \delta$ choices for $b \bmod q\ell$ that will satisfy (10.30). Therefore, and by (10.27), (10.29) and (1.69)-(1.71), we may conclude that, as a consequence of (10.19) and (10.31), one has:

$$\left|B^{(\delta)}_{Q,X}(\Psi, N; D, \theta)\right| \leqslant \sum_{\substack{C_\delta/4 < c_1 < 2C_\delta \\ (c_1, D_\delta) = 1}} \tau(\delta c_1) c_1^{-1} \sum_{\substack{A_1 \in (-c_1/2, c_1/2] \\ (A_1, c_1) = 1}} |\mathcal{T}(D_\delta; A_1/c_1)|. \tag{10.33}$$

Suppose now that c_1 and A_1 are integers satisfying the conditions of summation shown in (10.33). By (10.32) (and since $N, N_1 \in \mathbb{N}$),

$$\mathcal{T}(D_\delta; A_1/c_1) = \sum_{n=N_1}^{N} \mathrm{e}\left(-\frac{D_\delta^2 \overline{A_1} n}{c_1}\right) \mathcal{U}_n(D_\delta; A_1/c_1), \tag{10.34}$$

where

$$\mathcal{U}_n(D_\delta; A_1/c_1) = \sum_{N_1^- \leqslant m \leqslant N^+} g(m) \mathrm{e}(f(m)),$$

with N_1^-, N^+ as in (10.24), and, for $\mu > 0$:

$$g(\mu) = g_X\left(\frac{4\pi D_\delta \sqrt{\mu n}}{c_1}\right), \tag{10.35}$$

$$f(\mu) = \frac{2\theta D_\delta \sqrt{\mu n}}{c_1} - \frac{A_1}{c_1}\mu. \tag{10.36}$$

Supposing that $n \in [N_1, N] \cap \mathbb{Z}$, we have now

$$f'(\mu) = \frac{\theta D_\delta \sqrt{n/\mu}}{c_1} - \frac{A_1}{c_1} \qquad (\mu > 0),$$

which is a decreasing function, and satisfies:

$$|f'(\mu)| \leqslant \frac{|\theta| D_\delta \sqrt{N/N_1^-}}{C_\delta/4} + \frac{1}{2} \leqslant \frac{\sqrt{2}X}{\pi N}|\theta| + \frac{1}{2} < 1,$$

for $N_1^{-\cdot} \leqslant \mu \leqslant N^+$ (see (10.23), (10.24), (10.20), (10.22) and the hypotheses relating to N and N_1). Therefore, and since $\|N_1^-\| = \|N^+\| = 1/2$, the case $\kappa = 0$ of [25], Lemma 5.1, applies with $W_1 = N \geqslant 2$, showing that, for $n = N_1, \ldots, N$, we have, in (10.34),

$$\mathcal{U}_n(D_\delta; A_1/c_1) = O\left(\frac{V_g + V_{g'}}{N}\right) + \sum_{-N<w<N} c_w, \qquad (10.37)$$

where V_h denotes the total variance plus maximum modulus of $h(\mu)$ on the interval $[N_1^-, N^+]$, and

$$c_w = \int_{N_1^-}^{N^+} g(\mu) \mathrm{e}(f(\mu) - w\mu) \mathrm{d}\mu.$$

By (10.36),

$$f(\mu) - w\mu = \frac{2\theta D_\delta \sqrt{\mu n}}{c_1} - \frac{(A_1 + c_1 w)}{c_1}\mu, \qquad (\mu > 0)$$

so that in (10.37) we have (see (10.26) and (10.35)):

$$c_w = \gamma_n \left(D_\delta; (A_1 + c_1 w)/c_1\right). \qquad (10.38)$$

We also have that

$$V_g + V_{g'} \ll X + XN^{-1} \ll X, \qquad (10.39)$$

since (10.35), (10.4), (1.69)-(1.71) and (1.57) imply that, for $j = 0, 1, 2$ and $N_1^- \leqslant \mu \leqslant N^+$, one has $g^{(j)}(\mu) \ll_j XN^{-j}$.

By (10.37), (10.38) and (10.39), we may conclude that, for integers c_1, A_1 satisfying the conditions of summation in (10.33),

$$\mathcal{U}_n(D_\delta; A_1/c_1) = O(X/N) + \sum_{-N<w<N} \gamma_n \left(D_\delta; (A_1 + c_1 w)/c_1\right) \quad (n = N_1, \ldots, N).$$

From this it follows, by (10.34), (10.24) and (10.25), that

$$|\mathcal{T}(D_\delta; A_1/c_1)| = O(X) + \left| \sum_{-N<w<N} \sum_{n=N_1}^{N} \mathrm{e}\left(-\frac{D_\delta^2 \overline{A_1} n}{c_1}\right) \gamma_n\left(D_\delta; \frac{A_1 + c_1 w}{c_1}\right)\right| \leqslant$$
$$\leqslant O(X) + \sum_{\substack{|A|<N^+ c_1 \\ A \equiv A_1 (\bmod c_1)}} |\mathcal{V}(D_\delta; A/c_1)|,$$

for c_1, A_1 as in the summations of (10.33).

To complete the proof we need only use the above bound for $|\mathcal{T}(D_\delta; A_1/c_1)|$ in (10.33). The lemma then follows by (10.20) and (10.23), since

$$\sum_{C_\delta/4<c_1<2C_\delta} \frac{\tau(\delta c_1)}{c_1} \sum_{|A_1|\leqslant c_1/2} O(X) \ll \tau(\delta)X \sum_{C_\delta/4<c_1<2C_\delta} \tau(c_1) \ll \tau(\delta)XC_\delta \log(1+C_\delta)$$

and $\tau(\delta c_1)c_1^{-1} = \tau(\delta c_1)(\delta c_1)^{-1}\delta \ll_\varepsilon (\delta C_\delta)^{\varepsilon-1}\delta$, for $c_1 \in (C_\delta/4, 2C_\delta) \cap \mathbb{Z}$. ■

Lemma 10.6. *Let $Q, X, \theta > 0$ and $D, \delta, N, N_1 \in \mathbb{N}$ with $\delta | D$, $N > 1$ and $N_1 \in (N/2, N]$. Suppose also that L, C_δ, D_δ, N_1^- and N^+ are as in Lemma 10.5. Let c_1, A be a pair of integers satisfying*

$$c_1 \in (C_\delta/4, 2C_\delta), \quad (A, c_1) = 1 \quad \text{and} \quad A \neq 0. \tag{10.40}$$

Then, for some ν and N_2 satisfying

$$\nu \in [N/64, N^+] \quad \text{and} \quad N_2 \in \{N_1, \ldots, N\}, \tag{10.41}$$

one has

$$\mathcal{V}(D_\delta; A/c_1) \ll X (Nc_1/|A|)^{1/2} \left| \sum_{n=N_1}^{N_2} F_\nu(n) e(\phi n) \right|,$$

where $\mathcal{V}(D_\delta; A/c_1)$ is given by (10.25)-(10.26) of Lemma 10.5, while

$$F_\nu(n) = \int_{\alpha_\nu^- + \beta_n}^{\alpha^+ + \beta_n} e(\sigma t^2)\, dt \qquad (\nu > 0 \text{ and } n \in \mathbb{N}), \tag{10.42}$$

with $\sigma = -A/|A|$,

$$\alpha_\nu^- = \sqrt{\max(N_1^-, \nu)\,|A|/c_1}, \qquad \alpha^+ = \sqrt{N^+|A|/c_1}, \tag{10.43}$$

$$\beta_n = \sigma\theta D_\delta \sqrt{n/|A|c_1}, \tag{10.44}$$

and

$$\phi = \frac{D_\delta^2\theta^2}{c_1 A} - \frac{D_\delta^2 \overline{A}}{c_1}. \tag{10.45}$$

Proof. We begin by considering the factor $g_X(4\pi D_\delta\sqrt{\mu n}/c_1)$, which occurs in (10.26). By (10.4) and (1.57), we are able to write

$$x g_X(x) = x^2 \Omega_0(x/X) = \int_0^x \left(\frac{d}{dy}\, y^2 \Omega_0(y/X) \right) dy \qquad (x > 0),$$

so that, for $x > 0$,

$$g_X(x) = \frac{1}{x}\int_0^x G_X(y)\mathrm{d}y, \tag{10.46}$$

where $G_X(y) = 2y\Omega_0(y/X) + y^2X^{-1}\Omega_0'(y/X)$. Note that, by (1.69)-(1.71) and (1.57), we have here:

$$G_X(y) = 0 \qquad (y \notin (-X/2, 2X)) \tag{10.47}$$

and

$$G_X^{(j)}(x) \ll_j X^{1-j} \qquad (x > 0 \text{ and } j = 0, 1, \ldots). \tag{10.48}$$

By applying (10.46) for $x = 4\pi D_\delta\sqrt{\mu n}/c_1$, and then changing the variable of integration to ν, where $y = 4\pi D_\delta\sqrt{\nu n}/c_1$, we obtain:

$$g_X\left(\frac{4\pi D_\delta\sqrt{\mu n}}{c_1}\right) = \frac{1}{2\sqrt{\mu\nu}}\int_0^\mu G_X\left(\frac{4\pi D_\delta\sqrt{\nu n}}{c_1}\right)\mathrm{d}\nu,$$

for $n \in \mathbb{N}$ and $\mu > 0$. Therefore, for $n \in \mathbb{N}$, we may rewite the definition (10.26) as:

$$\begin{aligned}\gamma_n(D_\delta; A/c_1) &= \frac{1}{2}\int_{N_1^-}^{N^+}\int_0^\mu G_X\left(\frac{4\pi D_\delta\sqrt{\nu n}}{c_1}\right)\frac{\mathrm{d}\nu}{\sqrt{\nu}}\,\mathrm{e}\left(\frac{2\theta D_\delta\sqrt{\mu n}}{c_1} - \frac{A}{c_1}\mu\right)\frac{\mathrm{d}\mu}{\sqrt{\mu}} = \\ &= \int_0^{N^+} G_X\left(\frac{4\pi D_\delta\sqrt{\nu n}}{c_1}\right)E_\nu(n)\frac{\mathrm{d}\nu}{\sqrt{\nu}},\end{aligned} \tag{10.49}$$

where

$$E_\nu(n) = \frac{1}{2}\int_{N^*(\nu)}^{N^+}\mathrm{e}\left(\frac{2\theta D_\delta\sqrt{\mu n}}{c_1} - \frac{A}{c_1}\mu\right)\frac{\mathrm{d}\mu}{\sqrt{\mu}}$$

with

$$N^*(\nu) = \max\left(N_1^-, \nu\right). \tag{10.50}$$

Suppose now that $\nu > 0$ and $n \in \mathbb{N}$. By making the change of variable $\mu = \theta^2D_\delta^2A^{-2}ny^2$ (where the new variable, y, is constrained to lie in $(0, \infty)$), we obtain:

$$E_\nu(n) = \frac{\theta D_\delta\sqrt{n}}{|A|}\int_{\lambda_1}^{\lambda}\mathrm{e}\left(-\frac{\theta^2D_\delta^2n}{Ac_1}\left(y^2 + 2\sigma y\right)\right)\mathrm{d}y,$$

where $\sigma = -A/|A|$,

$$\lambda_1 = \frac{|A|}{\theta D_\delta}\sqrt{\frac{N^*(\nu)}{n}} \qquad \text{and} \qquad \lambda = \frac{|A|}{\theta D_\delta}\sqrt{\frac{N^+}{n}}.$$

Now,

$$-\frac{\theta^2 D_\delta^2 n}{c_1 A}\left(y^2 + 2\sigma y\right) = \sigma\frac{\theta^2 D_\delta^2 n}{c_1|A|}(y+\sigma)^2 + \frac{\theta^2 D_\delta^2 n}{c_1 A},$$

so that, by making the linear substitution,

$$\frac{\theta D_\delta \sqrt{n}}{\sqrt{c_1|A|}}(y+\sigma) = t,$$

we are able to conclude from the above (and (10.50)) that

$$E_\nu(n) = \sqrt{\frac{c_1}{|A|}}\,\mathrm{e}\!\left(\frac{\theta^2 D_\delta^2 n}{c_1 A}\right) F_\nu(n) \qquad (\nu > 0 \text{ and } n \in \mathbb{N}),$$

where $F_\nu(n)$ is as in (10.42)-(10.44).

Using the last result with (10.25) and (10.49), we obtain:

$$\mathcal{V}(D_\delta; A/c_1) = \sum_{n=N_1}^{N} \mathrm{e}\!\left(-\frac{D_\delta^2 \bar{A} n}{c_1}\right)\int_0^{N^+} G_X\!\left(\frac{4\pi D_\delta\sqrt{\nu n}}{c_1}\right) E_\nu(n)\frac{\mathrm{d}\nu}{\sqrt{\nu}} =$$

$$= \sqrt{\frac{c_1}{|A|}}\int_0^{N^+}\left(\sum_{n=N_1}^{N} h_\nu(n)\mathrm{e}(\phi n)F_\nu(n)\right)\frac{\mathrm{d}\nu}{\sqrt{\nu}},$$

with

$$h_\nu(x) = G_X\!\left(\frac{4\pi D_\delta\sqrt{\nu x}}{c_1}\right), \tag{10.51}$$

and with ϕ given by (10.45). Note that the last integrand is trivially zero for $\nu \leqslant N/64$, since it follows by (10.51), (10.47), (10.40), (10.23) and (10.20) that $h_\nu(n)$ is zero if $\sqrt{\nu n} \leqslant N/8$. Therefore, and by an appeal to the bound

$$\int_{N/64}^{N^+} \frac{\mathrm{d}\nu}{\sqrt{\nu}} \ll \sqrt{N},$$

we may conclude that, for some ν satisfying (10.41), one has:

$$\mathcal{V}(D_\delta; A/c_1) \ll (Nc_1/|A|)^{1/2}\left|\sum_{n=N_1}^{N} h_\nu(n)\mathrm{e}(\phi n)F_\nu(n)\right|.$$

By (10.51), (10.47) and (10.48), we have (given $\nu > 0$):

$$h_\nu(x) \ll X \quad \text{and} \quad h'_\nu(x) \ll X/N \qquad (N/2 \leqslant x \leqslant N).$$

The lemma therefore follows by partial summation from the last bound for $\mathcal{V}(D_\delta; A/c_1)$. ∎

Lemma 10.7. *Let $Q, X > 0$, $\theta \geqslant 1$ and $D, \delta, N, N_1 \in \mathbb{N}$ with $\delta | D$, $N > 1$ and $N_1 \in (N/2, N]$. Suppose also that L, C_δ, D_δ, N_1^-, N^+ and $\mathcal{V}(D_\delta; A/c_1)$ are as in Lemma 10.5, with c_1, A some pair of integers satisfying (10.40), and such that*

$$|A| \leqslant D_\delta/4. \tag{10.52}$$

Then

$$\mathcal{V}(D_\delta; A/c_1) \ll \frac{N}{\theta} \min\left(N,\, (1+\theta X) \left\| \frac{D_\delta^2 \overline{A}}{c_1} \right\|^{-1}\right) + $$
$$+ \frac{N}{\theta} \min\left(1 + \frac{|A|N}{D_\delta \theta},\, (1 + (\theta - 1)X) \left\| \frac{D_\delta^2 \overline{c_1}}{A} \right\|^{-1}\right).$$

Proof. In view of Lemma 10.6, it will suffice to obtain suitable bounds for a sum

$$\mathcal{W} = \sum_{n=N_1}^{N_2} F_\nu(n) \mathrm{e}(\phi n), \tag{10.53}$$

where $F_\nu(n)$ and ϕ are as in (10.42)-(10.45), with $\sigma = -A/|A|$, and where the pair ν, N_2 (henceforth taken as given) satisfy (10.41). Note that, by (10.40), (10.41), (10.43), (10.44) and (10.24), we have

$$\sqrt{\frac{N|A|}{2c_1}} \leqslant \alpha_\nu^- < \alpha^+ \leqslant \sqrt{\left(N + \frac{1}{2}\right)\frac{|A|}{c_1}} \tag{10.54}$$

and, for $N/2 \leqslant n \leqslant N$,

$$\theta D_\delta \sqrt{\frac{N/2}{|A|c_1}} \leqslant \sigma \beta_n \leqslant \theta D_\delta \sqrt{\frac{N}{|A|c_1}}. \tag{10.55}$$

It follows by (10.52) that

$$\frac{\alpha^+}{|\beta_n|} < \frac{2|A|}{\theta D_\delta} \leqslant \frac{1}{2\theta} \leqslant \frac{1}{2} \qquad (n \in [N/2, N]). \tag{10.56}$$

Integrating by parts in (10.42), one finds that, for $n = N_1, \ldots, N_2$,

$$F_\nu(n) = \frac{\sigma}{4\pi i}\left(\int_{\alpha_\nu^- + \beta_n}^{\alpha^+ + \beta_n} t^{-2} \mathrm{e}(\sigma t^2)\, \mathrm{d}t + \frac{\mathrm{e}\big(\sigma(\alpha^+ + \beta_n)^2\big)}{\alpha^+ + \beta_n} - \frac{\mathrm{e}\big(\sigma(\alpha_\nu^- + \beta_n)^2\big)}{\alpha_\nu^- + \beta_n}\right).$$

Using this to rewrite (10.53), we obtain:

$$\mathcal{W} = \frac{\sigma}{4\pi i}\left(\mathcal{W}^* + \mathcal{W}^+ - \mathcal{W}^-\right), \tag{10.57}$$

where

$$\mathcal{W}^* = \sum_{n=N_1}^{N_2} \int_{\beta_n+\alpha_\nu^-}^{\beta_n+\alpha^+} t^{-2}\mathrm{e}(\sigma t^2)\,\mathrm{d}t\,\mathrm{e}(\phi n), \tag{10.58}$$

$$\mathcal{W}^+ = \sum_{n=N_1}^{N_2} \left(\beta_n + \alpha^+\right)^{-1} \mathrm{e}\Big(\sigma\left(\beta_n + \alpha^+\right)^2 + \phi n\Big) \tag{10.59}$$

and

$$\mathcal{W}^- = \sum_{n=N_1}^{N_2} \left(\beta_n + \alpha_\nu^-\right)^{-1} \mathrm{e}\Big(\sigma\left(\beta_n + \alpha_\nu^-\right)^2 + \phi n\Big) \tag{10.60}$$

We shall complete our proof by establishing suitable bounds for $\mathcal{W}^*$, $\mathcal{W}^+$ and $\mathcal{W}^-$.

Interchanging summation and integration in (10.58), we obtain:

$$\mathcal{W}^* = \int_{\beta^-+\alpha_\nu^-}^{\beta^++\alpha^+} t^{-2}\mathrm{e}(\sigma t^2)\left(\sum_{\substack{n=N_1 \\ t-\alpha^+\leqslant\beta_n\leqslant t-\alpha_\nu^-}}^{N_2} \mathrm{e}(\phi n)\right)\mathrm{d}t,$$

where

$$\beta^+ = \max_{N_1\leqslant n\leqslant N_2} \beta_n \qquad \text{and} \qquad \beta^- = \min_{N_1\leqslant n\leqslant N_2} \beta_n.$$

It follows trivially from this and (10.54)-(10.56) that

$$\mathcal{W}^* \ll \frac{1}{\theta D_\delta}\sqrt{\frac{|A|c_1}{N}}\max_{t\in\mathbb{R}}\left|\sum_{\substack{n=N_1 \\ t-\alpha^+\leqslant\beta_n\leqslant t-\alpha_\nu^-}}^{N_2} \mathrm{e}(\phi n)\right|. \tag{10.61}$$

By (10.43) and (10.44),

$$\frac{\partial}{\partial n}\alpha_\nu^- = \frac{\partial}{\partial n}\alpha^+ = 0 \qquad \text{and} \qquad \frac{\partial}{\partial n}\beta_n = \frac{\beta_n}{2n} \qquad (n>0), \tag{10.62}$$

so it follows by (10.54)-(10.56) that the sum over n in (10.61) is either empty, or takes the form:

$$\sum_{n=N_3}^{N_3+H} \mathrm{e}(\phi n) = \mathrm{e}\left(\phi N_3\right)\sum_{n=0}^{H}\mathrm{e}(\phi n) = \mathrm{e}\left(\phi N_3\right)\mathcal{Z} \qquad \text{(say)},$$

where N_3 and H are some integers, with

$$0 \leqslant H \ll \frac{|A|N}{D_\delta \theta}. \tag{10.63}$$

For a non-trivial bound on $|\mathcal{Z}|$, we first note that, by (10.45),

$$\phi = \frac{D_\delta^2 \overline{c_1}}{A} + \frac{(\theta^2 - 1)\, D_\delta^2}{A c_1}, \tag{10.64}$$

where $\overline{c_1}$ and $\overline{A}$ (in (10.45)) may be any pair of integer solutions for the equation

$$A\overline{A} + c_1 \overline{c_1} = 1. \tag{10.65}$$

Then, by (10.64), (10.63) and partial summation we obtain:

$$|\mathcal{Z}| \ll \left(1 + H \frac{(\theta^2 - 1)\, D_\delta^2}{A c_1}\right) \max_{K \in \mathbb{N}} \left| \sum_{n=0}^{K-1} \mathrm{e}\left(\frac{D_\delta^2 \overline{c_1} n}{A}\right) \right| \ll \left(1 + \frac{(\theta - 1) N D_\delta}{c_1}\right) \left\| \frac{D_\delta^2 \overline{c_1}}{A} \right\|^{-1}$$

(ineffective when $A | D_\delta^2$). Using this together with the bound $|\mathcal{Z}| \leqslant 1 + H$, and the upper bound of (10.63), we conclude through (10.61) that

$$\mathcal{W}^* \ll \sqrt{\frac{|A| c_1}{\theta^2 D_\delta^2 N}} \min\left(1 + \frac{|A|N}{D_\delta \theta},\ \left(1 + \frac{(\theta - 1) N D_\delta}{c_1}\right) \left\| \frac{D_\delta^2 \overline{c_1}}{A} \right\|^{-1}\right). \tag{10.66}$$

We have yet to provide bounds for the terms $\mathcal{W}^+$ and $\mathcal{W}^-$ in (10.57). It suffices to give only the treatment of $\mathcal{W}^-$, since $\mathcal{W}^+$ may be dealt with in the same way. The first step is to observe that in (10.60) one has

$$\sigma \left(\beta_n + \alpha_\nu^-\right)^2 = \sigma \beta_n^2 + 2\sigma \beta_n \alpha_\nu^- + \sigma \left(\alpha_\nu^-\right)^2,$$

where, by (10.43), α_ν^- is independent of n (as is the α^+ in (10.59)). Then, by (10.43) and (10.44), we obtain

$$\sigma \beta_n^2 = \sigma \theta^2 D_\delta^2 n / |A| c_1 = -\theta^2 D_\delta^2 n / A c_1$$

and

$$\sigma \beta_n \alpha_\nu^- = \theta D_\delta \sqrt{\max\left(N_1^-, \nu\right) n / c_1^2}.$$

It therefore follows by (10.45) and (10.40) that, for $n = N_1, \ldots, N_2$,

$$\sigma \left(\beta_n + \alpha_\nu^-\right)^2 + \phi n = \sigma \left(\alpha_\nu^-\right)^2 + \frac{2\theta D_\delta \sqrt{\max\left(N_1^-, \nu\right) n}}{c_1} - \frac{D_\delta^2 \overline{A} n}{c_1}.$$

Using this in (10.60), we find that

$$|\mathcal{W}^-| = \left|\sum_{n=N_1}^{N_2} k(n) \mathrm{e}\left(-\frac{D_\delta^2 \overline{A} n}{c_1}\right)\right|, \tag{10.67}$$

where

$$k(n) = \frac{1}{\left(\beta_n + \alpha_{\bar{\nu}}\right)} \mathrm{e}\left(\frac{2\theta D_\delta \sqrt{\max\left(N_1^-, \nu\right) n}}{c_1}\right).$$

By (10.54)-(10.56), (10.43), (10.62) and (10.41), we have, for $N/2 \leqslant n \leqslant N$, the two bounds:

$$k(n) \ll \frac{1}{|\beta_n|} \ll \frac{1}{\theta D_\delta}\sqrt{\frac{|A| c_1}{N}}, \tag{10.68}$$

$$k'(n) \ll \frac{1}{|\beta_n|}\left(\frac{1}{N} + \frac{\theta D_\delta}{c_1}\right) \ll \frac{1}{\theta D_\delta}\sqrt{\frac{|A| c_1}{N^3}}\left(1 + \frac{\theta N D_\delta}{c_1}\right).$$

Therefore it follows by partial summation from (10.67) that one has

$$\mathcal{W}^- \ll \frac{1}{\theta D_\delta}\sqrt{\frac{|A| c_1}{N}}\left(1 + \frac{\theta N D_\delta}{c_1}\right)\left|\sum_{N_1}^{N_3} \mathrm{e}\left(-\frac{D_\delta^2 \overline{A} n}{c_1}\right)\right|,$$

for some $N_3 \in \{N_1, \ldots, N_2\}$. As (10.67) and (10.68) also imply the trivial bound,

$$\mathcal{W}^- \ll \frac{\sqrt{|A| c_1 N}}{\theta D_\delta},$$

we may now conclude that

$$\mathcal{W}^- \ll \sqrt{\frac{|A| c_1}{\theta^2 D_\delta^2 N}} \min\left(N, \left(1 + \frac{\theta N D_\delta}{c_1}\right)\left\|\frac{D_\delta^2 \overline{A}}{c_1}\right\|^{-1}\right).$$

With reference to the last bound, to the bound (10.66), and to Lemma 10.6, we now observe that

$$X\sqrt{\frac{N c_1}{|A|}}\sqrt{\frac{|A| c_1}{\theta^2 D_\delta^2 N}} = \frac{X c_1}{\theta D_\delta},$$

and that $c_1 \asymp D_\delta N / X$, by (10.40), (10.23) and (10.20). The lemma therefore follows directly from Lemma 10.6, (10.53), (10.57), the bound (10.66), the above bound for $\mathcal{W}^-$ and a similar bound for the sum $\mathcal{W}^+$ of (10.59). ■

Lemma 10.8. *Let $Q, X > 0$, $\theta \geqslant 1$ and $D, \delta, N, N_1 \in \mathbb{N}$ with $\delta | D$, $N > 1$ and $N_1 \in (N/2, N]$. Suppose also that L, C_δ, D_δ, N_1^-, N^+ and $\mathcal{V}(D_\delta; A/c_1)$ are as in Lemma 10.5, with c_1, A some pair of integers satisfying* (10.40), *and such that*

$$|A| > D_\delta / 4. \tag{10.69}$$

Then

$$\mathcal{V}(D_\delta; A/c_1) \ll$$
$$\ll (1+\theta X)\,|A|^{-1} D_\delta N \times$$
$$\times \min\left(N, \left(1+\frac{\theta^2 X D_\delta}{|A|}\right) \left\| \frac{D_\delta^2 \overline{A}}{c_1} \right\|^{-1}, \left(1+\frac{(\theta-1)\,\theta X D_\delta}{|A|}\right) \left\| \frac{D_\delta^2 \overline{c_1}}{A} \right\|^{-1} \right).$$

Proof. As was the case for our proof of Lemma 10.7, it suffices (by virtue of Lemma 10.6) that we obtain sufficiently strong bounds for the sum $\mathcal{W}$ of (10.53), where $F_\nu(n)$ and ϕ are as in (10.42)-(10.45), with $\sigma = -A/|A|$ there, and with ν, N_2 (assumed given) satisfying (10.41). By partial summation

$$\mathcal{W} = F_\nu(N_2) \sum_{n=N_1}^{N_2} \mathrm{e}(\phi n) - \int_{N_1}^{N_2} F_\nu'(\xi) \left(\sum_{N_1 \leqslant n \leqslant \xi} \mathrm{e}(\phi n) \right) \mathrm{d}\xi, \tag{10.70}$$

where it follows by (10.42)-(10.44), that one has, for $\xi \in [N_1, N_2]$,

$$F_\nu'(\xi) = \left(\mathrm{e}\left(\sigma\left(\alpha^+ + \beta_\xi\right)^2\right) - \mathrm{e}\left(\sigma\left(\alpha_\nu^- + \beta_\xi\right)^2\right)\right) \frac{\partial}{\partial \xi}\beta_\xi \ll \tag{10.71}$$
$$\ll \left| \frac{\partial}{\partial \xi}\beta_\xi \right| = \left| \frac{\beta_\xi}{2\xi} \right| \ll \frac{\theta D_\delta}{\sqrt{N|A|c_1}}.$$

Suppose now that, in place of (10.69), we have the stronger:

$$|A| \geqslant 4\theta D_\delta. \tag{10.72}$$

Just as in the proof of Lemma 10.7, we have at our disposal the bounds (10.54) and (10.55). We do not, however, have (10.56): on the contrary, it follows by (10.54), (10.55) and (10.72) that

$$\frac{|\beta_n|}{\alpha_\nu^-} < \frac{2\theta D_\delta}{|A|} \leqslant \frac{1}{2} \qquad (n \in [N/2, N]).$$

Therefore the first derivative test [23], Lemma 4.2, applies to $F_\nu(N_2)$ (given by (10.42)), showing that

$$F_\nu(N_2) \ll \frac{1}{\alpha_\nu^-} \ll \sqrt{\frac{c_1}{N|A|}} \qquad \text{if} \quad |A| \geqslant 4\theta D_\delta \tag{10.73}$$

(see (10.43), (10.41)).

If (10.72) does not hold, then we appeal to the second derivative test [23], Lemma 4.4, in order to establish that

$$F_\nu(N_2) \ll 1 < \sqrt{\frac{4\theta D_\delta}{|A|}} \quad \text{if} \quad |A| < 4\theta D_\delta$$

(see (10.42)). By this and (10.73) we have the bound

$$F_\nu(N_2) \ll \left(\sqrt{\frac{c_1}{ND_\delta}} + \sqrt{\theta}\right)\sqrt{\frac{D_\delta}{|A|}},$$

valid whether or not (10.72) holds.

By (10.70), (10.71), our bound for $F_\nu(N_2)$, and (10.41), we have

$$\mathcal{W} \ll \left(\left(\sqrt{\frac{c_1}{ND_\delta}} + \sqrt{\theta}\right)\sqrt{\frac{D_\delta}{|A|}} + \frac{N\theta D_\delta}{\sqrt{N|A|c_1}}\right)\left|\sum_{n=N_1}^{N_3} \mathrm{e}(\phi n)\right|,$$

for some $N_3 \in \{N_1, \ldots, N_2\}$. Since (10.40), (10.23) and (10.20) imply

$$c_1 \asymp \frac{D_\delta N}{X}, \tag{10.74}$$

we may simplify the above bound for $\mathcal{W}$ as follows:

$$\mathcal{W} \ll \sqrt{\left(\frac{1}{X} + \theta + \theta^2 X\right)\frac{D_\delta}{|A|}}\,|\mathcal{Z}| < (1 + \theta X)\sqrt{\frac{D_\delta}{X|A|}}\,|\mathcal{Z}|,$$

where

$$\mathcal{Z} = \mathrm{e}(\phi N_1) + \ \ldots\ + \mathrm{e}(\phi N_3).$$

Now recall from (10.45) and (10.64)-(10.65) that

$$\phi = -\frac{D_\delta^2 \overline{A}}{c_1} + \frac{D_\delta^2 \theta^2}{Ac_1} = \frac{D_\delta^2 \overline{c_1}}{A} + \frac{(\theta^2 - 1)\, D_\delta^2}{Ac_1},$$

where, by (10.74), we have

$$\frac{D_\delta^2 \theta^2}{Ac_1} \ll \frac{\theta^2 X D_\delta}{N|A|} \qquad \text{and} \qquad \frac{(\theta^2 - 1)\, D_\delta^2}{Ac_1} \ll \frac{(\theta - 1)\theta X D_\delta}{N|A|}$$

(given that $\theta \geqslant 1$). Therefore, either by bounding the sum $\mathcal{Z}$ trivially, or by bounding it through partial summation and evaluation of a geometric series, we are now able to conclude that

$$\mathcal{W} \ll T \min\left(N,\ \left(1 + \frac{\theta^2 X D_\delta}{|A|}\right)\left\|\frac{D_\delta^2 \overline{A}}{c_1}\right\|^{-1},\ \left(1 + \frac{(\theta - 1)\theta X D_\delta}{|A|}\right)\left\|\frac{D_\delta^2 \overline{c_1}}{A}\right\|^{-1}\right),$$

where $T = (1 + \theta X) D_\delta^{1/2} (X|A|)^{-1/2}$. The lemma follows directly from this last bound, since Lemma 10.6, (10.53) and (10.74) imply $\mathcal{V}(D_\delta; A/c_1) \ll N\sqrt{XD_\delta/|A|}\,|\mathcal{W}|$. ∎

Lemma 10.9. *Let* $\varepsilon \in (0, 1/2)$, $Q > 0$, $\theta \geqslant 1$ *and* $D, \delta, N, N_1 \in \mathbb{N}$, *with* $\delta | D$, $N > 1$, $N_1 \in (N/2, N]$, *and*

$$\Delta < X < N, \tag{10.75}$$

where $\Delta \in (0,1/2]$ *is an absolute constant. Suppose also that* L, C_δ, D_δ, N_1^- *and* N^+ *are as in Lemma* 10.5. *Let*

$$\mathcal{B}_1 = \sum_{\substack{C_\delta/4<c_1<2C_\delta \\ (c_1,D_\delta)=1}} \sum_{\substack{|A|\leqslant D_\delta/4 \\ (A,c_1)=1}} |\mathcal{V}(D_\delta; A/c_1)|, \tag{10.76}$$

where $\mathcal{V}(D_\delta; A/c_1)$ *is as in Lemma* 10.5. *Then*

$$\mathcal{B}_1 \ll \frac{QLND}{\delta^2} U \log(D) + O_\varepsilon\left(\frac{X(QL)^{1+\varepsilon}ND^{1+\varepsilon}}{\delta^2} + \frac{QLN^2D^{2\varepsilon}}{\delta}\right),$$

where

$$U = 1 + (\theta - 1)X. \tag{10.77}$$

Proof. Note first that (10.75), (10.23) and (10.20) imply

$$C_\delta = 4\pi D_\delta N/X > 4\pi D_\delta > 4, \tag{10.78}$$

so that, in the summations shown in (10.76), the variable of summation c_1 is constrained to run over a subset of $\{2,3,4\ldots\}$. As those summations require $(A,c_1)=1$, it follows that the variable of summation A is implicitly constrained not to equal 0. In view of this, and of the other constraints explicit in the summations of (10.76), we may conclude that c_1 and A there are summed only over integer values for which both (10.40) and (10.52) hold. For such c_1 and A, Lemma 10.7 applies (given our other hypotheses), yielding the bound:

$$\mathcal{V}(D_\delta; A/c_1) \ll N\min\left(N,\, X\left\|\frac{D_\delta^2\overline{A}}{c_1}\right\|^{-1}\right) + N\min\left(N,\, U\left\|\frac{D_\delta^2\overline{c_1}}{A}\right\|^{-1}\right).$$

Using this bound in (10.76), we find that

$$\mathcal{B}_1 \ll (X\mathcal{E} + U\mathcal{E}^*)\,N, \tag{10.79}$$

where

$$\mathcal{E} = \sum_{\substack{C_\delta/4<c_1<2C_\delta \\ (c_1,D_\delta)=1}} \sum_{\substack{|A|\leqslant D_\delta/4 \\ (A,c_1)=1}} \min\left(\frac{N}{X}, \left\|\frac{D_\delta^2\overline{A}}{c_1}\right\|^{-1}\right), \tag{10.80}$$

$$\mathcal{E}^* = \sum_{0\neq|A|\leqslant D_\delta/4} \sum_{\substack{C_\delta/4<c_1<2C_\delta \\ (c_1,A)=1}} \min\left(\frac{N}{U}, \left\|\frac{D_\delta^2\overline{c_1}}{A}\right\|^{-1}\right). \tag{10.81}$$

We shall bound the sum $\mathcal{E}$ of (10.80) in terms of sums

$$\mathcal{N}(H) = \sum_{\substack{C_\delta/4<c_1<2C_\delta \\ (c_1,D_\delta)=1}} \sum_{\substack{|A|\leqslant D_\delta/4 \\ (A,c_1)=1}} \Lambda_H\left(\frac{D_\delta^2\overline{A}}{c_1}\right), \tag{10.82}$$

where

$$\Lambda_H(x) = \begin{cases} 1 & \text{if } \|x\| \leqslant H, \\ 0 & \text{otherwise,} \end{cases} \tag{10.83}$$

and $H > 0$ is given.

If $c_1 \in \mathbb{N}$ and $A \in \mathbb{Z}$ with $(A, c_1) = 1$, then

$$\left\| \frac{D_\delta^2 \overline{A}}{c_1} \right\| = \frac{|B|}{c_1},$$

where B is the solution of $B \equiv D_\delta^2 \overline{A} \pmod{c_1}$ with $-1/2 < B/c_1 \leqslant 1/2$. Therefore it follows from (10.82)-(10.83) that, for $H > 0$,

$$\begin{aligned} \mathcal{N}(H) = & \sum_{\substack{C_\delta/4 < c_1 < 2C_\delta \\ (c_1, D_\delta) = 1}} \sum_{\substack{B \in (-c_1/2, c_1/2] \\ |B| \leqslant Hc_1}} \sum_{\substack{|A| \leqslant D_\delta/4 \\ (A, c_1) = 1 \\ D_\delta^2 \overline{A} \equiv B \pmod{c_1}}} 1 \leqslant \\ \leqslant & \sum_{\substack{C_\delta/4 < c_1 < 2C_\delta \\ (c_1, D_\delta) = 1}} \sum_{|B| \leqslant Hc_1} \sum_{\substack{|A| \leqslant D_\delta \\ AB \equiv D_\delta^2 \pmod{c_1}}} 1. \end{aligned}$$

Since (10.78) implies $c_1 > 1$ in the last summation, and since the condition $(D_\delta, c_1) = 1$ implies $D_\delta^2 \not\equiv 0 \pmod{c_1}$ if $c_1 \in \{2, 3, 4, \ldots\}$, it is therefore an implicit condition of the last summations above that AB not be equal to 0. This observation allows us to conclude that, for $H > 0$,

$$\mathcal{N}(H) \leqslant \sum_{C_\delta/4 < c_1 < 2C_\delta} \sum_{\substack{0 < m \leqslant Hc_1 D_\delta \\ m \equiv \pm D_\delta^2 \pmod{c_1}}} 4\tau(m) \ll_\varepsilon (HC_\delta D_\delta)^\varepsilon (1 + HD_\delta) C_\delta. \tag{10.84}$$

By (10.82), (10.83) and (10.75), the sum $\mathcal{E}$ of (10.80) satisfies

$$\mathcal{E} \leqslant \sum_{\substack{j=0 \\ H_j < 1}}^{\infty} \frac{\mathcal{N}(H_j)}{H_{j-1}},$$

where

$$H_j = 2^j X/N \qquad (j \in \mathbb{Z}).$$

Applying (10.84) to the terms of the last sum, we obtain:

$$\begin{aligned} \mathcal{E} \ll_\varepsilon & \sum_{\substack{j=0 \\ H_j < 1}}^{\infty} (H_j C_\delta D_\delta)^\varepsilon \left(\frac{1}{H_j} + D_\delta \right) C_\delta \ll_\varepsilon \\ & \ll_\varepsilon (H_0 C_\delta D_\delta)^\varepsilon H_0^{-1} C_\delta + (C_\delta D_\delta)^{1+\varepsilon} = \left(4\pi D_\delta^2\right)^\varepsilon X^{-1} N C_\delta + (C_\delta D_\delta)^{1+\varepsilon} \end{aligned}$$

(see (10.23) and (10.20)).

As for the sum $\mathcal{E}^*$ in (10.79) and (10.81), it follows by Lemma 10.1 and (10.78) that one has

$$\mathcal{E}^* \ll \sum_{\substack{0<|A|\leqslant D_\delta \\ A|D_\delta^2}} C_\delta N/U + \sum_{0<|A|\leqslant D_\delta} C_\delta \log|A| \ll \tau(D_\delta^2)\, U^{-1} C_\delta N + C_\delta D_\delta \log(D_\delta)\,.$$

To finish the proof we note that, by (10.79) and the above bounds for $\mathcal{E}$ and $\mathcal{E}^*$, we have

$$\mathcal{B}_1 \ll O_\varepsilon\Big(D^{2\varepsilon} N^2 C_\delta + XN\,(C_\delta D_\delta)^{1+\varepsilon}\Big) + \tau(D^2)\, C_\delta N^2 + UNC_\delta D_\delta \log(D).$$

Since $\tau(D^2) \ll_\varepsilon D^{2\varepsilon}$ here, one need only recall the definitions of C_δ and D_δ in (10.23) in order to verify that the above bound for $\mathcal{B}_1$ implies the one given by the lemma. ∎

Lemma 10.10. *Let $Q > 0$, $\theta \geqslant 1$ and $D, \delta, N, N_1 \in \mathbb{N}$, with $\delta|D$, $N > 1$ and $N_1 \in (N/2, N]$. Let $X \in \mathbb{R}$ satisfy (10.75), with Δ there being an absolute constant lying in the interval $(0, 1/2]$, and suppose also that $U \in \mathbb{R}$ is given by (10.77), while L, C_δ, D_δ, N_1^- and N^+ are as in Lemma 10.5. Let*

$$\mathcal{B}_2 = \sum_{\substack{C_\delta/4<c_1<2C_\delta \\ (c_1,D_\delta)=1}} \sum_{\substack{D_\delta/4<|A|<N^+c_1 \\ (A,c_1)=1}} |\mathcal{V}(D_\delta; A/c_1)|\,, \tag{10.85}$$

with $\mathcal{V}(D_\delta; A/c_1)$ as in Lemma 10.5. Then

$$\mathcal{B}_2 \ll \theta^3 U \log(N) \log(DN) \frac{XQLND}{\delta^2}\left(1 + \frac{X}{\sqrt{N}}\right) + \theta\tau(D^2)\, \frac{XQLN^2}{\delta}.$$

Proof. Let M be an arbitrary integer with

$$M \in \{1, \ldots, N-1\} \tag{10.86}$$

and define

$$M^- = M - \frac{1}{2}. \tag{10.87}$$

Then, for $c_1 > C_\delta/4$, we have

$$D_\delta < C_\delta/8 < c_1/2 \leqslant M^- c_1 \qquad \text{and} \qquad M^- < M < N < N^+, \tag{10.88}$$

since (10.75), (10.23) and (10.20) imply the inequalities (10.78) (noted in the proof of Lemma 10.9), and since $N^+ = N + 1/2$. By (10.88) and (10.85),

$$\mathcal{B}_2 \leqslant \mathcal{B}_{2,1} + \mathcal{B}_{2,2}, \tag{10.89}$$

where

$$\mathcal{B}_{2,1} = \sum_{D_\delta/4<|A|\leqslant 2MC_\delta} \sum_{\substack{C_\delta/4<c_1<2C_\delta \\ (c_1,A)=1}} |\mathcal{V}(D_\delta; A/c_1)|, \tag{10.90}$$

$$\mathcal{B}_{2,2} = \sum_{\substack{C_\delta/4<c_1<2C_\delta \\ (c_1,D_\delta)=1}} \sum_{\substack{M^-c_1<|A|<N^+c_1 \\ (A,c_1)=1}} |\mathcal{V}(D_\delta; A/c_1)|. \tag{10.91}$$

Given (10.77), and given that $\theta X \gg 1$, Lemma 10.8 yields the bound

$$\mathcal{V}(D_\delta; A/c_1) \ll \theta^2 UXND_\delta |A|^{-1} \min\left(\frac{N}{\theta U}, \left\| \frac{D_\delta^2 \overline{c_1}}{A} \right\|^{-1}\right),$$

for $c_1, A \in \mathbb{Z}$ satisfying (10.40) and (10.69). Using this bound with (10.90), and then applying Lemma 10.1, we find by (10.23), (10.86), (10.78) and (10.75) that

$$\begin{aligned} \mathcal{B}_{2,1} &\ll \theta^2 UXND_\delta \left(\sum_{\substack{A \mid D_\delta^2 \\ |A|>D_\delta/4}} \frac{C_\delta N}{|A|\theta U} + \sum_{D_\delta/4<|A|<2MC_\delta} \frac{(C_\delta + |A|)}{|A|} \log|A| \right) \ll \\ &\ll \theta^2 UXND_\delta \left(\tau(D_\delta^2) \frac{C_\delta N}{D_\delta \theta U} + \left(\log\left(\frac{C_\delta}{D_\delta}\right) + M\right) C_\delta \log(MC_\delta) \right) \ll \\ &\ll \theta\tau(D^2) XN^2 C_\delta + \theta^2 U \log(N) \log(DN) XNC_\delta D_\delta M. \end{aligned} \tag{10.92}$$

By (10.88) (and given that $\theta X \gg 1$), Lemma 10.8 also implies that, for $c_1, A \in \mathbb{Z}$ satisfying (10.40) with $(c_1, D_\delta) = 1$ and $|A| > M^- c_1$, we have

$$\mathcal{V}(D_\delta; A/c_1) \ll \theta \frac{XND_\delta}{|A|} \left(1 + \frac{\theta^2 XD_\delta}{|A|}\right) \left\| \frac{D_\delta^2 \overline{A}}{c_1} \right\|^{-1}$$

(note that (10.88) implies $c_1 > 2D_\delta$, so that c_1, being coprime to D_δ, cannot divide D_δ^2 here). Using this bound with (10.91), we find that

$$\mathcal{B}_{2,2} \ll \theta XND_\delta \mathcal{E}_{2,2}^{(1)} + \theta^3 X^2 ND_\delta^2 \mathcal{E}_{2,2}^{(2)},$$

where

$$\mathcal{E}_{2,2}^{(r)} = \sum_{\substack{C_\delta/4<c_1<2C_\delta \\ (c_1,D_\delta)=1}} \sum_{\substack{M^-c_1<|A|<N^+c_1 \\ (A,c_1)=1}} |A|^{-r} \left\| \frac{D_\delta^2 \overline{A}}{c_1} \right\|^{-1} \qquad (r = 1, 2),$$

so that, by (10.24) and (10.86)-(10.88), one has

$$\mathcal{E}_{2,2}^{(r)} < \sum_{\substack{C_\delta/4<c_1<2C_\delta \\ (c_1,D_\delta)=1}} \sum_{n=M}^{N} \left(\left(n-\frac{1}{2}\right)c_1\right)^{-r} \sum_{\substack{-c_1/2<|A|-nc_1\leqslant c_1/2 \\ (A,c_1)=1}} \left\|\frac{D_\delta^2\overline{A}}{c_1}\right\|^{-1} \ll$$

$$\ll \sum_{C_\delta/4<c_1<2C_\delta} c_1^{-r} \sum_{n=M}^{N} n^{-r} \sum_{B \bmod c_1}^{*} \|B/c_1\|^{-1} \ll$$

$$\ll \left(\sum_{C_\delta/4<c_1<2C_\delta} c_1^{1-r}\log(c_1)\right)\left(\sum_{n=M}^{N} n^{-r}\right) \ll$$

$$\ll \begin{cases} C_\delta^{2-r}\log(C_\delta)\log(N) & \text{if } r=1, \\ C_\delta^{2-r}\log(C_\delta)\,M^{-1} & \text{if } r=2. \end{cases}$$

Therefore, and by (10.23), (10.78) and (10.75), we conclude that

$$\mathcal{B}_{2,2} \ll \theta X N D_\delta C_\delta \log(DN)\log(N) + \theta^3 X^2 N D_\delta^2 \log(DN)\,M^{-1}.$$

By the bound (10.92) for $\mathcal{B}_{2,1}$, by the above bound for $\mathcal{B}_{2,2}$, and by (10.89), (10.86), (10.77) and our hypotheses concerning N, X and θ, we have now shown that

$$\mathcal{B}_2 \ll \theta\tau(D^2)XN^2C_\delta + \theta^3 U\log(N)\log(DN)\left(XNC_\delta D_\delta M + \frac{X^2ND_\delta^2}{M}\right),$$

which (see (10.23) and (10.78)) implies:

$$\mathcal{B}_2 \ll \theta\tau(D^2)\,\frac{XQLN^2}{\delta} + \theta^3 U\log(N)\log(DN)\frac{XQLND}{\delta^2}\left(M + \frac{X^2}{MN}\right).$$

The lemma therefore follows on choosing $M = \max(1, [X/\sqrt{N}])$ to optimise the above bound. Note that this choice of M does satisfy (10.86), given that (10.75) holds, and that $N \geqslant 2$. ∎

Lemma 10.11. *Let $\varepsilon \in (0,1/2)$, $Q>0$ and $D,N,N_1 \in \mathbb{N}$ with $N>1$ and $N_1 \in (N/2,N]$. Let $X>\Delta$, where $\Delta \in (0,1/2]$ is an absolute constant. Suppose that $\theta \geqslant 1$ and that θ satisfies (10.22). Let the parameter U and sequence $\Psi = (\Psi_n)$, be given by (10.77) and (10.18), respectively. Then the sum $B_{Q,X}(\Psi,N;D,\theta)$ given by (10.3), (10.4) and (1.69)-(1.71) satisfies*

$$B_{Q,X}(\Psi,N;D,\theta) \ll_\varepsilon (DN)^\varepsilon\left(\theta^3 U X^{3/2} N D + \theta X N^2\right).$$

Proof. Note first that (10.22) and our other hypotheses concerning X and θ together imply that (10.75) holds. Now suppose that $\delta \in \mathbb{N}$ with $\delta|D$. Then, by Lemma 10.5 and (10.75),

$$B_{Q,X}^{(\delta)}(\Psi,N;D,\theta) \ll_\varepsilon (DN)^\varepsilon\left(\frac{DN}{\delta} + \frac{(\mathcal{B}_1+\mathcal{B}_2)\,\delta}{QL}\right),$$

where $\mathcal{B}_1$, $\mathcal{B}_2$ and $B^{(\delta)}_{Q,X}(\Psi, N; D, \theta)$ and L are as given by (10.76), (10.85), (10.19)-(10.21), (10.4) and (1.69)-(1.71). By Lemmas 10.9 and 10.10, and by (10.75) and (10.77), we have here:

$$\mathcal{B}_1 + \mathcal{B}_2 \ll_\varepsilon \frac{\theta^3 U X^{3/2} Q L (DN)^{1+2\varepsilon}}{\delta^2} + \frac{\theta X Q L N^2 D^{2\varepsilon}}{\delta},$$

so that

$$B^{(\delta)}_{Q,X}(\Psi, N; D, \theta) \ll_\varepsilon (DN)^\varepsilon \left(\frac{DN}{\delta} + \frac{\theta^3 U X^{3/2} (DN)^{1+2\varepsilon}}{\delta} + \theta X N^2 D^{2\varepsilon} \right) \ll$$
$$\ll (DN)^{3\varepsilon} \left(\theta^3 U X^{3/2} N D + \theta X N^2 \right).$$

Since Lemma 10.4 implies that

$$B_{Q,X}(\Psi, N; D, \theta) \ll_\varepsilon D^\varepsilon \max_{\delta | D} \left| B^{(\delta)}_{Q,X}(\Psi, N; D, \theta) \right|,$$

and since $\varepsilon/4$ may be substituted for ε without affecting our hypotheses, we may therefore conclude that the result given by the lemma is a consequence of the bound obtained for $B^{(\delta)}_{Q,X}(\Psi, N; D, \theta)$. ∎

Lemma 10.12. *Let $0 < \varepsilon < 1/4$, $Q > 0$ and $D, N, N_1 \in \mathbb{N}$ with $N_1 \in (N/2, N]$. Suppose that $H \geqslant 1$, and that $X > \Delta$, where $\Delta \in (0, 1/2]$ is an absolute constant. Let the sequence $\Psi = (\Psi_n)$ be given by (10.18). Then*

$$A_{Q,H,X}(\Psi, N; D, 0) \ll_\varepsilon (DN)^\varepsilon (D + N) N H^3.$$

Proof. By Lemma 10.2 and (10.18), there exists some θ satisfying (10.2) such that

$$A_{Q,H,X}(\Psi, N; D, 0) \ll O_\varepsilon(DN) + H^3 Z^3 \left| B_{Q,X}(\Psi, N; D, \theta) \right|, \tag{10.93}$$

where Z and $B_{Q,X}(\Psi, N; D, \theta)$ are as given by (10.1), (10.3)-(10.4) and (1.69) -(1.71). Taking this θ as given, we shall complete the proof by obtaining a sufficiently strong upper bound for $|B_{Q,X}(\Psi, N; D, \theta)|$.

Suppose first that $N > 1$ and that N, X and θ satisfy (10.22) (in addition to (10.1)-(10.2)). Then (as observed while proving Lemma 10.11) the condition (10.75) must hold, so that we have $Z \leqslant N^{2\varepsilon} X^{-1/2}$ and, consequently,

$$1 \leqslant \theta \leqslant 1 + O\left(N^{4\varepsilon} X^{-1} \right)$$

(see (10.1) and (10.2)). Therefore, and since $X > \Delta$, $\varepsilon > 0$ and $D, N \geqslant 1$, it follows by Lemma 10.11, (10.77) and (10.75) that

$$B_{Q,X}(\Psi, N; D, \theta) \ll_\varepsilon (DN)^{13\varepsilon} (1 + (\theta - 1) X)(D + N) N X^{3/2} \ll \tag{10.94}$$
$$\ll (DN)^{17\varepsilon} (D + N) N X^{3/2} <$$
$$< (DN)^{20\varepsilon} (D + N) N X^{3/2 - 3\varepsilon}.$$

The last is the bound we have been seeking. However, in order to reach it, we had need of our assumptions that $N > 1$ and that (10.22) holds. We shall next show that (10.94) holds if one (or both) of those assumptions are false. It will in fact suffice to show that the falsity of either assumption implies

$$N \ll X, \tag{10.95}$$

since, if $N \ll X$, then Lemma 10.3 and (10.18) imply

$$B_{Q,X}(\Psi, N; D, \theta) \ll_\varepsilon (DN)^\varepsilon D^{1/2} N^{5/2} X^{1/2} \ll$$
$$\ll (DN)^\varepsilon D^{1/2} N^{3/2+3\varepsilon} X^{3/2-3\varepsilon} \leqslant (DN)^{4\varepsilon} D^{1/2} N^{3/2} X^{3/2-3\varepsilon}$$

(given that $\varepsilon \in (0, 1/4)$), which, by virtue of the arithmetic-geometric mean inequality, is at least as strong as (10.94).

Taking first the cases where one does not have $N > 1$, we may note that in such cases $N = 1$ (since $n \in \mathbb{N}$), so that the bound (10.95) follows trivially as a consequence of the hypothesis that $X > \Delta$, where Δ is positive and absolute.

In cases where (10.22) is false we have $|\theta| \geqslant N/X$, so that (10.1) and (10.2) imply

$$N/X \leqslant 1 + O\left(N^{2\varepsilon} X^{2\varepsilon - 1}\right),$$

which, in turn, implies that either $N/X \leqslant 2$, or $2 < N/X \ll N^{2\varepsilon} X^{2\varepsilon-1}$. As the former alternative would mean that $N \leqslant 2X$, while the latter would imply that $N \ll (NX)^{2\varepsilon} < \sqrt{NX/\Delta} \ll \sqrt{NX}$ (given the lemma's hypotheses concerning ε, N, X and Δ), we may therefore conclude that (10.95) does hold in all the cases where (10.22) is false.

Since we showed earlier that (10.94) holds in cases where $N > 1$ and (10.22) is true, and since it was also found that the conclusions reached in last two paragraphs would imply that (10.94) must also hold in all the remaining cases, it therefore now follows that we are free to use (10.94) with (10.93), and so to obtain:

$$A_{Q,H,X}(\Psi, N; D, 0) \ll_\varepsilon DN + H^3 Z^3 (DN)^{20\varepsilon} (D + N) N X^{3/2-3\varepsilon} \ll$$
$$\ll (DN)^{23\varepsilon} (D + N) N H^3$$

(see (10.1) and the lemma's hypotheses concerning ε, N, D and H). The lemma follows on substitution of $\varepsilon/23$ for $\varepsilon \in (0, 1/4)$. ■

11. Proving Theorem 1.8

From our hard won results in the last section we obtain the next lemma, helping us to get started on the proof of Theorem 1.8 that follows it. Lemma 11.1 is not likely to be of interest for any applications we have in mind, since it is weak when D significantly exceeds both Q and N in magnitude. Nevertheless, it is worthwhile pointing out that this lemma gives a result as good as (or better than) Theorem 1.8 when D has a prime divisor P such that $D \leqslant P^{\varrho/(1-\varrho)} N$ (where $\varrho/(1-\varrho) = 7/25$).

Lemma 11.1. *Let $0 < \varepsilon < 1/2$, $Q > 0$, $K \geqslant 1$ and $D, N, N_1 \in \mathbb{N}$ with $N_1 \in (N/2, N]$, and let the sequence $\boldsymbol{\Psi} = (\Psi_n)$ be given by* (10.18). *Then, for $y \in \mathbb{R}$,*

$$S_{Q,K}(\boldsymbol{\Psi}, N; D, y) \ll_\varepsilon \left(1 + y^2\right)(DN)^\varepsilon (Q + D + N) N K^2.$$

Proof. We seek first to establish the above bound in cases where $y = 0$. This bound is trivial for $Q \in (0,1)$ (see (1.43)), so we assume henceforth that $Q \geqslant 1$.

By Lemmas 8.1 and 10.12, and by (10.18),

$$S_{Q,K}(\boldsymbol{\Psi}, N; D, 0) \ll$$

$$\ll O_{\varepsilon,j}\left(D^{1/j}\left(QK^2 + N^{1+\varepsilon}\right) N\right) + O_\varepsilon\left(\sum_{\substack{h=0 \\ U^h \leqslant K}}^{\infty} U^{-h} \sum_{\substack{r \in \mathbb{Z} \\ \Delta < 2^r \leqslant Y}} (DN)^{\varepsilon/3}(D+N) N U_h^3\right),$$

where j is an arbitrary element of $\mathbb{N}$ and $\Delta \in (0, 1/2]$ is an absolute constant, while $U = U_0$, Y and the sequence (U_n) are given by (8.1) and (8.2) (with $\vartheta = 7/64$ there). Therefore, on taking $j = [1/\varepsilon] + 1$, we find that

$$S_{Q,K}(\boldsymbol{\Psi}, N; D, 0) \ll_\varepsilon D^\varepsilon\left(QK^2 + N^{1+\varepsilon}\right) N + (DN)^{\varepsilon/3}(D+N) N T,$$

where

$$T = \left(\sum_{\substack{h=0 \\ U^h \leqslant K}}^{\infty} U^{2h+3}\right)\left(\sum_{\substack{r \in \mathbb{Z} \\ \Delta < 2^r \leqslant Y}} 1\right) \ll$$

$$\ll K^2 U^3 \log\left(1 + \frac{Y}{\Delta}\right) = 8K^2 D^{6\vartheta/j} \log\left(1 + 16\pi \Delta^{-1} Q^{-1} DN\right) \ll_\varepsilon (DN)^{2\varepsilon/3} K^2,$$

since $6\vartheta = 21/32 < 2/3$, $1/j < \varepsilon$, $1/\Delta \ll 1$ and $1/Q \leqslant 1$. Given that $\varepsilon > 0$ and that $K, N \geqslant 1$, the bounds just found imply the case $y = 0$ of the lemma.

In cases where $y \neq 0$, we take (see [7], page 277) the road travelled in the proof of [7], Theorem 7. Given a complex sequence $\mathbf{c} = (c_n)$, it follows by (10.18) and partial summation that

$$\sum_{N/2 < n \leqslant N} \Psi_n n^{iy} c_n = \sum_{n=N_1}^{N} c_n n^{iy} = N_1^{iy} \sum_{n=N_1}^{N} c_n + iy \int_{N_1}^{N} x^{iy}\left(\sum_{x \leqslant n \leqslant N} c_n\right) \frac{\mathrm{d}x}{x}.$$

so that, by the Cauchy-Schwarz inequality, (10.18), and the hypotheses concerning N and N_1, one has

$$\left|\sum_{N/2 < n \leqslant N} \Psi_n n^{iy} c_n\right|^2 \leqslant 2\left|\sum_{n=N_1}^{N} c_n\right|^2 + 2\log(2) y^2 \int_{N_1}^{N} \left|\sum_{x \leqslant n \leqslant N} c_n\right|^2 \frac{\mathrm{d}x}{x}.$$

As this bound can be applied with either $\mathbf{c} = \mathbf{f}$, or $\mathbf{c} = \mathbf{g}^{(r)}$, where

$$f_n = \rho_{j\infty}(Dn) \quad \text{and} \quad g_n^{(r)} = n^{ir}\varphi_{\mathfrak{c}\infty}\left(Dn, \tfrac{1}{2} + ir\right) \quad (n \in \mathbb{N})$$

(with j, $\mathfrak{c}$ and r denoting the indices of summation and variable of integration in (1.42)), we are therefore able to use it with (1.42) and (1.43), so as to obtain:

$$S_{Q,K}(\Psi, N; D, y) \ll \left(1 + y^2\right) S_{Q,K}(\Psi', N; D, 0), \tag{11.1}$$

with, for some $x \in [N_1, N]$,

$$\Psi'_n = \begin{cases} 1 & \text{if } x \leqslant n \leqslant N, \\ 0 & \text{otherwise,} \end{cases} \qquad (n \in \mathbb{N}).$$

On comparing the definition of Ψ' (in which $x \geqslant N_1 > N/2$) with the definition of Ψ in (10.18), it is evident that the hypotheses of the lemma will continue to hold following the substitution of Ψ' for the sequence Ψ. Therefore, by the case of the lemma already established,

$$S_{Q,K}(\Psi', N; D, 0) \ll_\varepsilon (DN)^\varepsilon (Q + D + N) N K^2.$$

Using this with (11.1) we obtain the desired result for all $y \in \mathbb{R}$, so completing the proof. ■

Proof of Theorem 1.8. As was the case in our proof of Theorem 1.6, we may here assume that $\varepsilon \in (0, 1/2)$. See also the remarks concerning $C_0(\varepsilon)$ and $M_0(\varepsilon)$ in the first paragraph of Theorem 1.6's proof: similar considerations apply here in respect of $C_1(\varepsilon)$ and $M_1(\varepsilon)$. The constants ε, $C_1(\varepsilon)$ and $M_1(\varepsilon)$ are indeed constant, from the beginning to the end of our proof, and this should be understood at points where we refer to (1.64) or (1.66). This will not prevent us from sometimes applying Proposition 1.1 or 1.2 with the constant parameter ε of the proposition replaced by a new constant, η. As $C_1(\varepsilon)$ and $M_1(\varepsilon)$ must be unvarying for our arguments to work, we have to keep track of both in any expressions, so as to avoid overlooking any point in the argument that might require a change in either constant. Therefore we do not allow implicit constants which might depend on $C_1(\varepsilon)$ or $M_1(\varepsilon)$ (forbidding ourselves, in particular, from simplifying a bound such as $X \ll_\varepsilon C_1(\varepsilon) Y$ to just $X \ll_\varepsilon Y$). Before plunging in to the details of the proof, we mention that, like the proof of Theorem 1.6, it consists of an 'initial result' followed by an 'inductive step' (with some intervening explanation).

For our proof of the 'initial result' we shall suppose we are given $M > 1$ and y, K, Q, D, N, P, N_1 and Ψ as indicated after (1.64). It then follows from (1.59), (1.65) and (1.63) that Ψ will satisfy (10.18) if N and N_1 there are replaced by the integers $N' = [N] \geqslant 0$ and $N_1' = [N_1] + 1 \geqslant 1$, respectively. By (1.65) we have $N_1' > N'/2$. Moreover, it follows from (1.63) that, if $N_1' > N'$, then $\Psi_n = 0$ for all $n \in \mathbb{N}$. Therefore, and in view of (1.42)-(1.43), one obtains

$$S_{Q,K}(\Psi, N; D, y) = S_{Q,K}(\Psi, N'; D, y) \ll_\varepsilon \tag{11.2}$$
$$\ll_\varepsilon \left(1 + y^2\right) (DN)^\varepsilon (Q + D + N) N K^2,$$

either trivially, or as a consequence of Lemma 11.1. Given that $C_1(\varepsilon)$ was chosen sufficiently large, it follows from (11.2) and (1.59) that (1.66) holds if $D \leqslant Q+M$ (note that $(DN)^{\varepsilon} \leqslant (QDN)^{\varepsilon}$ if $Q \geqslant 1$, and that (1.66) is trivial if $0 < Q < 1$).

Suppose now that
$$D > Q + M. \tag{11.3}$$
Since $Q + M > 1$, it follows from (11.3) and (1.60) that $P \geqslant 2$, and that we can find $D_1 | D$ with
$$(Q+M)/P < D_1 \leqslant Q + M. \tag{11.4}$$
By Proposition 1.1 and (1.63), we have
$$S_{Q,K}(\Psi, N; D, y) \ll_{\varepsilon} \frac{K^2}{G^2} S_{Q_1,G}\left(\Psi^{\{g_0\}}, \frac{N}{g_0}; \frac{D_1}{g_1}, y\right)\left(\frac{D}{D_1}\right)^{\varrho+\varepsilon}, \tag{11.5}$$
for some $G \geqslant 1$, some $Q_1 \in (0, Q]$, some $g_0, g_1 \in \mathbb{N}$ with $g_1 | D_1$, and the sequence $\Psi^{\{g_0\}}$ given by
$$\Psi_n^{\{g_0\}} = \begin{cases} 1 & \text{if } N_1 < g_0 n \leqslant N, \\ 0 & \text{otherwise,} \end{cases} \qquad (n \in \mathbb{N}). \tag{11.6}$$
As (1.63) holds with $\Psi^{\{g_0\}}$, N/g_0 and N_1/g_0 substituted for Ψ, N and N_1 (respectively), it follows that we may apply the bound (11.2) (with appropriate substitutions), so as to obtain:
$$S_{Q_1,G}\left(\Psi^{\{g_0\}}, \frac{N}{g_0}; \frac{D_1}{g_1}, y\right) \ll_{\varepsilon} \left(1+y^2\right)\left(D_1 N\right)^{\varepsilon}\left(Q + D_1 + N\right) N G^2.$$
By (1.59), (11.4) and (11.5), this enables us to conclude that
$$S_{Q,K}(\Psi, N; D, y) \ll_{\varepsilon} \left(1+y^2\right)(DN)^{\varepsilon}\left(\frac{D}{D_1}\right)^{\varrho}(Q+M) N K^2. \tag{11.7}$$
Moreover, as (11.4) implies
$$\left(\frac{D}{D_1}\right)^{\varrho}(Q+M) < \frac{D^{\varrho}(Q+M)}{((Q+M)/P)^{\varrho}} = (PD)^{\varrho}(Q+M)^{1-\varrho} = \\ = (PDM)^{\varrho} M^{\varsigma}\left(1 + \frac{Q}{M}\right)^{1-\varrho},$$
and since (1.66) holds trivially if $0 < Q < 1$, it therefore follows (given the hypotheses regarding $C_1(\varepsilon)$) that a sufficient condition for (1.66) to hold is that $Q/M \leqslant Q^{\varepsilon}$. Recalling our conclusion immediately prior to the assumption of (11.3), we may now further conclude that, regardless of whether or not (11.3) is satisfied, the bound (1.66) will hold if $Q^{1-\varepsilon} \leqslant M$. This 'initial result' supplies what is needed for the 'inductive step' that comes next.

For the remainder of the proof M is supposed to satisfy (1.64), and remains fixed. We suppose that $R > 0$ is such that (1.66) fails for $Q = R$ (with the given choice of M, and some y, K, D, N, P, N_1 and Ψ satisfying the conditions stated in the theorem). By the 'initial result', obtained in the previous paragraph, we must have the inequality

$$R^{1-\varepsilon} > M. \tag{11.8}$$

By reasoning as in the proof of Theorem 1.6 (see the paragraph containing (9.2)), we may additionally suppose R to be a natural number satisfying the 'inductive hypothesis' that, with the given choice of M, the bound (1.66) holds for $0 < Q < R$ and all y, K, D, N, P, N_1 and Ψ satisfying the conditions stated in the theorem.

Our aim now is to complete a proof by induction, although we present it as a proof by contradiction (the idea being to show that our original assumption concerning R can be ruled out as a possibility). By adapting the logic of the argument set out in the proof of Theorem 1.6 (half a paragraph below (9.2)), we find here that what suffices for completion of the current proof is to establish that, given our assumptions concerning ε, M and R, it must follow that (1.66) holds for $Q = R$ and y, K, D, N, P, N_1 and Ψ satisfying the conditions stated in the theorem (but otherwise arbitrary). We therefore now suppose such a choice of y, K, D, N, P, N_1 and Ψ to have been given. By (1.42) and (1.43), the bound (1.66) is trivial for $0 < N < 1$, so that $N \geqslant 1$ may also be assumed henceforth. For our later convenience we define two new constants:

$$\delta = \frac{1-4\varrho}{\zeta} = \frac{2}{9} \qquad \text{and} \qquad \eta = \frac{\delta\varepsilon^2}{3}. \tag{11.9}$$

We begin the inductive step by considering cases where

$$DN \leqslant R^{2-\delta\varepsilon}. \tag{11.10}$$

By Proposition 1.2, and by (1.63), (11.8)-(11.10) and (1.59), we find that

$$\frac{S_{R,K}(\Psi, N; D, y)}{K^2} \ll_\varepsilon (DN)^\eta \left(RN + \int_{-\infty}^{\infty} \frac{S_{L,G}(\Psi, N; D, y+t)}{(1+t^4)G^2} \,\mathrm{d}t \right), \tag{11.11}$$

for some $G \geqslant 1$ and some L satisfying

$$0 < L \ll DN/R \leqslant R^{-\delta\varepsilon/2}\sqrt{DN} \leqslant R^{1-\delta\varepsilon}. \tag{11.12}$$

By (11.9), (11.8) and (1.64), the last bounds may be assumed to imply

$$L < \sqrt{DN} < R. \tag{11.13}$$

Therefore our 'inductive hypothesis' applies to show that, for $t \in \mathbb{R}$,

$$\begin{aligned} &S_{L,G}(\Psi, N; D, y+t) \leqslant \\ &\quad \leqslant \left(1 + (y+t)^2\right) C_1(\varepsilon)(LDN)^\varepsilon \big(L + M + (PDM)^\varrho (L+M)^\zeta\big) NG^2 \ll \\ &\quad \ll \left(1+t^2\right)\left(1+y^2\right) C_1(\varepsilon)(LDN)^\varepsilon \big(R + (PDM)^\varrho (L+M)^\zeta\big) NG^2 \end{aligned}$$

(the final bound following by (11.8)). By this and (11.11) it follows that

$$\frac{S_{R,K}\left(\Psi,N;D,y\right)}{(1+y^2)\,K^2} \ll_\varepsilon (DN)^\eta C_1(\varepsilon)(1+L)^\varepsilon (DN)^\varepsilon \left(R+(PDM)^\varrho (L+M)^\varsigma\right) N,$$

where, by (11.10), (11.12), (11.9), (11.8) and (1.64),

$$(DN)^\eta (1+L)^\varepsilon \ll R^{2\eta}\left(R^{1-\delta\varepsilon}\right)^\varepsilon = R^{\varepsilon-\eta} < \left(M_1(\varepsilon)\right)^{-\eta} R^\varepsilon .$$

Therefore, given (11.9) and (11.13), and given that $M_1(\varepsilon)$ is supposed sufficiently large in terms of ε, we are now entitled to conclude that in cases where (11.10) holds we do obtain the bound (1.66) for $Q=R$.

As our consideration of the cases satisfying (11.10) has reached a satisfactory conclusion, we shall assume henceforth that

$$DN > R^{2-\delta\varepsilon}. \tag{11.14}$$

Given this assumption we have (see (11.9)) the inequality $D > R^{2-2\varepsilon}/N$, while it follows by (11.8) and (1.59) that $R^{2-2\varepsilon}/N \geqslant R/N > M/N \geqslant 1$. Therefore, and by (1.60), we can find $D_1|D$ such that

$$\frac{R^{2-2\varepsilon}}{PN} < D_1 \leqslant \frac{R^{2-2\varepsilon}}{N}. \tag{11.15}$$

Then, by Proposition 1.1 and (11.9), we obtain

$$\frac{S_{R,K}(\Psi,N;D,y)}{K^2} \ll_\varepsilon \frac{1}{G^2}\, S_{R_1,G}\left(\Psi^{\{g_0\}},\frac{N}{g_0};\frac{D_1}{g_1},y\right)\left(\frac{D}{D_1}\right)^{\varrho+\eta}, \tag{11.16}$$

for some $G \geqslant 1$, some $R_1 \in (0,R]$, some $g_0, g_1 \in \mathbb{N}$ satisfying $g_1|D_1$, and the sequence $\Psi^{\{g_0\}}$ given by (11.6).

The size of the parameter R_1 in (11.16) determines how we can best proceed. We consider first the cases in which

$$R_1^{2-\varepsilon} < D_1 N. \tag{11.17}$$

In such cases it follows by (11.15) that

$$R_1 < R^{(2-2\varepsilon)/(2-\varepsilon)} = R^{1-\varepsilon/(2-\varepsilon)} \leqslant R^{1-\varepsilon/2}. \tag{11.18}$$

Therefore $R_1 < R$, so that the 'inductive hypothesis' applies (given that (11.6) implies (1.63) with N_1/g_0 and $N/g_0 \in (0,M]$ substituted in place of N_1 and N). Consequently we have

$$\frac{1}{(1+y^2)\,G^2}\, S_{R_1,G}\left(\Psi^{\{g_0\}},\frac{N}{g_0};\frac{D_1}{g_1},y\right) \leqslant$$
$$\leqslant C_1(\varepsilon)\left(R_1 D_1 N\right)^\varepsilon \left(R_1 + M + (PD_1M)^\varrho \left(R_1+M\right)^\varsigma\right) N.$$

By this, (11.16), and the inequality $\eta \leqslant \varepsilon$ (see (11.9)), we obtain

$$\frac{1}{(1+y^2)\,K^2}\,S_{R,K}(\Psi,N;D,y) \ll_\varepsilon \qquad (11.19)$$
$$\ll_\varepsilon C_1(\varepsilon)\,(R_1 DN)^\varepsilon\left(\left(\frac{D}{D_1}\right)^\varrho (R_1+M) + (PDM)^\varrho (R_1+M)^\zeta\right) N.$$

Here we can note that, by (11.18), (11.8), (11.9) and (11.14),

$$R_1 + M \ll R^{1-\varepsilon/2} < R^{1-\delta\varepsilon/2} < \min\left(R, \sqrt{DN}\right),$$

with it following by (11.15) and (1.59) that we have

$$\left(\frac{D}{D_1}\right)^\varrho (R_1+M) \ll \frac{D^\varrho R^{1-\varepsilon/2}}{(R^{2-2\varepsilon}/PN)^\varrho} \leqslant (PDM)^\varrho R^{\zeta-(1-4\varrho)\varepsilon/2}.$$

As $1-4\varrho = \delta\zeta > 0$ (by (11.9)), and

$$R^{\zeta-\delta\zeta\varepsilon/2} = R^{(1-\delta\varepsilon/2)\zeta} < \left(\min\left(R,\sqrt{DN}\right)\right)^\zeta$$

(by (11.14)), we may therefore conclude that (11.19) shows

$$\frac{S_{R,K}(\Psi,N;D,y)}{(1+y^2)\,K^2 N} \ll_\varepsilon C_1(\varepsilon)\,(R_1 DN)^\varepsilon\,(PDM)^\varrho\left(\min\left(R,\sqrt{DN}\right)\right)^\zeta,$$

which, by (11.18), (11.8) and (1.64), is sufficient to guarantee that we obtain (1.66) with $Q = R$.

As the above succesfully concludes our treatment of the cases in which (11.16) holds with R_1 satisfying (11.17), we shall henceforth be assuming that, in addition to (11.14)-(11.16), it holds that

$$R_1^{2-\varepsilon} \geqslant D_1 N. \qquad (11.20)$$

Note this implies that $R_1 \geqslant 1$, and so also that $R_1^2 \geqslant D_1 N$. As the 'inductive hypothesis' may not be available to us in some of the cases we are now considering, we appeal instead to Proposition 1.2, which, by virtue of (1.63), (1.59), (11.8), (11.9) and the inequalities $D_1 N/R_1 \leqslant R_1 \leqslant R$, provides us with the bound:

$$\frac{1}{G^2}\,S_{R_1,G}\left(\Psi^{\{g_0\}},\frac{N}{g_0};\frac{D_1}{g_1},y\right) \ll_\varepsilon \qquad (11.21)$$
$$\ll_\varepsilon (D_1 N)^\eta \left(RN + \int_{-\infty}^{\infty} \frac{1}{H^2}\,S_{L_1,H}\left(\Psi^{\{g_0\}},\frac{N}{g_0};\frac{D_1}{g_1},y+t\right)\frac{dt}{(1+t^4)}\right),$$

for some $H \geqslant 1$ and some L_1 satisfying $0 < L_1 \ll D_1 N/R_1$. By (11.20), the upper bound for L_1, and the inequality $R_1 \leqslant R$, we have here

$$L_1 \ll R_1^{1-\varepsilon} \leqslant R^{1-\varepsilon}, \tag{11.22}$$

so that (given (11.8) and (1.64)) it may be assumed that $0 < L_1 < R$. Therefore the 'inductive hypothesis' applies to show that, for $t \in \mathbb{R}$,

$$S_{L_1,H}\left(\Psi^{\{g_0\}}, \frac{N}{g_0}; \frac{D_1}{g_1}, y+t\right) \leqslant$$
$$\leqslant \left(1+(y+t)^2\right) C_1(\varepsilon)\,(L_1 D_1 N)^{\varepsilon}\left(L_1 + M + (PD_1M)^{\varrho}\,(L_1+M)^{\zeta}\right) NH^2.$$

By (11.16), (11.21) and the last bound, we conclude (using $\eta \leqslant \varepsilon$) that

$$\frac{1}{(1+y^2)\,K^2 N}\, S_{R,K}(\Psi, N; D, y) \ll_{\varepsilon} \tag{11.23}$$
$$\ll_{\varepsilon} (D_1N)^{\eta}\, C_1(\varepsilon)\,(L_1DN)^{\varepsilon}\left(\left(\frac{D}{D_1}\right)^{\varrho}(L_1+M) + (PDM)^{\varrho}\,(L_1+M)^{\zeta}\right) +$$
$$+ (DN)^{\eta}\left(\frac{D}{D_1}\right)^{\varrho} R.$$

By (11.15) and (1.59), we have here

$$\left(\frac{D}{D_1}\right)^{\varrho} R < \frac{D^{\varrho}R}{(R^{2-2\varepsilon}/PN)^{\varrho}} \leqslant (PDM)^{\varrho}R^{\zeta+2\varrho\varepsilon} = R^{\varepsilon}(PDM)^{\varrho}R^{(1-\varepsilon)\zeta}. \tag{11.24}$$

Moreover, by (11.22), (11.8), (11.14) and (11.9),

$$L_1 + M \ll R^{1-\varepsilon} \ll \min\left(R, \sqrt{DN}\right), \tag{11.25}$$

so that, by (11.24),

$$\left(\frac{D}{D_1}\right)^{\varrho}(L_1+M) \ll \left(\frac{D}{D_1}\right)^{\varrho} R^{1-\varepsilon} < (PDM)^{\varrho}R^{(1-\varepsilon)\zeta}. \tag{11.26}$$

Since (11.14), (11.9), (11.15) and (11.22) show

$$(DN)^{\eta}R^{\varepsilon} \leqslant (DN)^{-\varepsilon/2}(RDN)^{\varepsilon} < R^{-\varepsilon/2}(RDN)^{\varepsilon}$$

and

$$(D_1N)^{\eta}\,(L_1DN)^{\varepsilon} \ll R^{2\eta}\left(R^{1-\varepsilon}DN\right)^{\varepsilon} \leqslant R^{-\varepsilon^2/2}(RDN)^{\varepsilon},$$

it follows from (11.23) and (11.24)-(11.26) that

$$\frac{1}{(1+y^2)\,K^2}\, S_{R,K}\,(\Psi, N; D, y) \ll_{\varepsilon} R^{-\varepsilon^2/2}C_1(\varepsilon)(RDN)^{\varepsilon}(PDM)^{\varrho}R^{(1-\varepsilon)\zeta}N,$$

which, by (11.25), (11.8) and (1.64), may be assumed to imply (1.66) for $Q = R$.

As the cases we have considered cover all the possible choices of y, K, D, N, P, N_1 and Ψ satisfying the conditions stated in the theorem, and since we found that, with $Q = R$, the bound (1.66) followed in all these cases (given the assumptions concerning ε, M and R), we have therefore now met the conditions earlier deemed sufficient for completion of the proof (see the paragraph ending with (11.9)). In fact we have obtained a contradiction with our original assumption that (1.66) failed for $Q = R$ (with some choice of y, K, D, N, P, N_1 and Ψ satisfying the conditions stated in the theorem). Therefore (1.66) must hold, for the given fixed choice of M, whenever y, K, D, N, P, N_1 and Ψ are as stated in the theorem. This completes the proof, since, aside from the condition (1.64), our choice of M was arbitrary. ■

References

[1] W.R. Alford, A. Granville and C. Pomerance, There are infinitely many Carmichael numbers, *Annals of Math.*, **140** (1994), 703–722.

[2] T.M. Apostol, *Mathematical Analysis* (second edition), World Student Series, **0284**, Addison Wesley, Reading Mass. 1979.

[3] V. Blomer, G. Harcos and P. Michel (with a second appendix by Z. Mao), A Burgess-like subconvexity bound for twisted L-functions, *Forum Math.*, to appear.

[4] D. Bump, *Automorphic Forms and Representations*, Cambridge Studies in Advanced Math., **55**, Cambridge Univ. Press, 1998.

[5] D. Bump, W. Duke, J. Hoffstein and H. Iwaniec, An estimate for the Hecke eigenvalues of Maass forms, *Duke Math. J. Research Notices*, **4** (1992), 75–81.

[6] P. Deligne, Formes modulaires et représentations l-adiques, in: *Séminaire Bourbaki, 1968/69 exposés*, Lect. Notes in Math., **179**, Springer, Berlin-Heidelberg-New York (1971), pp. 347–363.

[7] J.-M. Deshouillers and H. Iwaniec, Kloosterman sums and Fourier coefficients of cusp forms, *Invent. Math.*, **70** (1982), 219–288.

[8] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher Transcendental Functions II*, McGraw-Hill, New York-Toronto-London 1953.

[9] G. Harman, On the number of Carmichael numbers up to x, *Bull. London Math. Soc.*, **37** (2005), 641–650.

[10] G. Harman, N. Watt and K. Wong, A new mean-value result for Dirichlet L-functions and polynomials, *Quart. J. Math.*, **55** (2004), 307–324.

[11] D.A. Hejhal, *The Selberg Trace Formula for* $PSL(2,\mathbb{R})$. *II.*, Lect. Notes in Math., **1001**, Springer, Berlin-Heidelberg-New York 1983.

[12] M.N. Huxley, Introduction to Kloostermania in: *Elementary and Analytic Theory of Numbers*, Banach Center Publ., **17**, Warsaw (1985), pp. 217–306.

[13] H. Iwaniec, *Spectral Theory of Automorphic Functions*, unpublished lecture notes, Rutgers Univ. 1987.

[14] H. Iwaniec, *Topics in Classical Automorphic Forms*, Graduate Studies in Math., **17**, Amer. Math. Soc., Providence R.I. 1997.

[15] H. Iwaniec, *Spectral Methods of Automorphic Forms* (second edition), Graduate Studies in Math. **53**, Amer. Math. Soc. (with Revista Matemática Iberoamericana), Providence R.I. 2002.

[16] H.H. Kim (with appendices by D. Ramakrishnan and by H.H. Kim and P. Sarnak), Functoriality for the exterior square of GL_4 and the symmetric fourth of GL_2, *J. Amer. Math. Soc.,* **16** (2003), 139–183.

[17] H. Kim and F. Shahidi, Cuspidality of symmetric powers with applications, *Duke Math. J.,* **112** (2002), 177–197.

[18] W. Luo, Z. Rudnick and P. Sarnak, On Selberg's eigenvalue conjecture, *Geom. Funct. Anal.,* **5** (1995), 384–401.

[19] Y. Motohashi, The Riemann Zeta-function and the Hecke congruence subgroups, in: *RIMS Kokyuroku,* **958** (1996), pp. 166–177.

[20] M.R. Murty, On the estimation of eigenvalues of Hecke operators, *Rocky Mountain J. Math.,* **15** (1985), 521–533.

[21] I. Satake, Spherical functions and Ramanujan conjecture, in: A. Borel and G. D. Mostow (Eds.), *Proc. Sympos. Pure Math. Vol. 9 (Algebraic Groups and Discontinuous Subgroups),* Amer. Math. Soc., Providence R.I. (1966), pp. 258–264.

[22] A. Selberg, On the estimation of Fourier coefficients of modular forms, in: *Proc. Sympos. Pure Math. Vol. 8,* Amer. Math. Soc., Providence R.I. (1965), pp. 1–15.

[23] E.C. Titchmarsh (revised by D.R. Heath-Brown), *The Theory of the Riemann Zeta-function,* Clarendon Press, Oxford 1986.

[24] N. Watt, Kloosterman sums and a mean value for Dirichlet polynomials, *J. Number Theory,* **53** (1995), 179–210.

[25] N. Watt, On the mean squared modulus of a Dirichlet L-function over a short segment of the critical line, *Acta Arith.,* **111** (2004), 307–403.

[26] N. Watt, Bounds for a mean value of character sums, preprint.

[27] E.T. Whittaker and G.N. Watson, *A Course Of Modern Analysis,* Cambridge Univ. Press, 1973.

Address: Department of Mathematics, Royal Holloway, University of London, Egham Hill, Egham, Surrey TW20 0EX, United Kingdom

E-mail: nigel.watt@rhul.ac.uk

Received: 11 April 2005