ON THE SUM OF A PRIME AND A k-FREE NUMBER
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Abstract: We prove a refined asymptotic formula for the number of representations of sufficiently large integer as a sum of a prime and a k-free number, k ≥ 2.
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1. Introduction

The problem of counting the number of representations of an integer as a sum of a prime and a square-free integer was first considered by Estermann [3] in 1931. He obtained an asymptotic formula that was subsequently refined by Page [11] and then by Walfisz [13] in 1936. In 1949 Mirsky [10] generalized such results to the case of the sum of a prime and a k-free number, where k ≥ 2 is a fixed integer.
He obtained, for every A > 0, that

\[ r_k(n) = \sum_{p \leq n} \mu_k(n - p) = \mathcal{G}_k(n) \text{li}(n) + O\left(\frac{n}{\log A n}\right) \quad \text{as} \quad n \to +\infty, \quad (1) \]

where \( \mu_k(n) = \sum_{\nu \mid m} \mu(\nu) \) is the characteristic function of the k-free numbers,
\( \mu(n) \) is the Möbius function, \( \text{li}(n) = \int_2^n \frac{dt}{\log t} \) and

\[ \mathcal{G}_k(n) = \prod_{p \mid n} \left(1 - \frac{1}{p^{k-1}(p-1)}\right) \quad (2) \]

is the singular series of this problem.

The aim of this paper is to prove a refinement of Walfisz-Mirsky asymptotic formula (1). This refinement depends on inserting a new term connected with the existence of the Siegel zero of Dirichlet L-functions (see Lemmas 1-2 below) and by sharpening the error term in the asymptotic formula.

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Denoting by $\Lambda(n)$ the von Mangoldt function, we define

$$H_k(n) = \sum_{m \leq n} \Lambda(m)\mu_k(n - m)$$

to be the weighted number of representations of an integer $n$ as a sum of a prime and a $k$-free number. As usual $R_k$ is easily related with $r_k$. We have the following

**Theorem.** Let $k > 2$ be a fixed integer. Then there exists a constant $c = c(k) > 0$ such that, for every sufficiently large $n \in \mathbb{N}$, we have

$$R_k(n) = \left(n - \delta_\beta \tilde{\chi}(n) \frac{n^{\beta}}{\beta}\right)G_k(n) + O_k(nG \exp(-c\sqrt{\log n})).$$

where $\beta$ is the Siegel zero, $\tilde{\chi}$ is the Siegel character, $\tilde{\tau}$ is the Siegel modulus associated with the set of Dirichlet $L$-functions with modulus $q \leq \exp(c' \sqrt{\log n})$, where $c' = c'(k) > 0$ is a suitable constant,

$$G = \begin{cases} (1 - \beta)\sqrt{\log n} & \text{if } \tilde{\beta} \text{ exists} \\ 1 & \text{if } \tilde{\beta} \text{ does not exist} \end{cases}, \quad \delta_\beta = \begin{cases} 1 & \text{if } \tilde{\beta} \text{ exists} \\ 0 & \text{if } \tilde{\beta} \text{ does not exist} \end{cases}$$

(see also Lemmas 1-2 below).

An analogous result, but with a weaker error term, can also be obtained via the circle method using some recent results on exponential sums over $k$-free numbers proved by Brüdern-Granville-Perrelli-Vaughan-Wooley [1].

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### 2. Lemmas

We recall now some analytic results on the zero-free region of Dirichlet $L$-functions.

**Lemma 1.** [Davenport [2], §13-14] Assume $T' > 0$. There exists a constant $c_1 > 0$ such that $L(\sigma + it, \chi) \neq 0$ whenever

$$\sigma \geq 1 - \frac{c_1}{\log T'}, \quad |t| \leq T'$$

for all the Dirichlet characters $\chi \mod q \leq T'$, with the possible exception of at most one primitive character $\tilde{\chi} \mod \tilde{\tau}$, $\tilde{\tau} \leq T'$. If it exists, the character $\tilde{\chi}$ is real and the exceptional zero $\tilde{\beta}$ of $L(s, \tilde{\chi})$ is unique, real, simple and there exists a constant $c_2 > 0$ such that

$$\frac{c_2}{\tilde{\tau}/2 \log^{2/3} \tilde{\tau}} \leq 1 - \tilde{\beta} \leq \frac{c_1}{\log T'}, \quad |t| \leq T'.$$
Fix now $T_1 > 0$ such that $\log T_1 \asymp \sqrt{\log \pi}$. According to Lemma 1, applied with $T'' = T_1$, we denote by $\beta$ the Siegel zero, $\tilde{\chi}$ the Siegel character and by $\tilde{\tau}$ its modulus. Let now

$$T_2 = \begin{cases} T_1 & \text{if } \tilde{\tau} \leq T_1^{1/4} \\ 1/4 & \text{otherwise.} \end{cases}$$

Now Lemma 1 remains true for $T'' = T_2$, with a suitable change in the constant $c_1$. In the following we will continue to call $c_1$ this modified constant. Hence $\tilde{\tau} \leq T_2^{1/4}$, if it exists. From now on we set $T = T_2$.

Moreover we need also the following form of the Deuring-Heilbronn phenomenon whose proof can be found in Knapowski [9], see also §4 of Gallagher [5].

**Lemma 2.** Under the same hypotheses of Lemma 1 applied with $T'' = T$, if $\tilde{\beta}$ exists, then for all the Dirichlet characters $\chi$ modulo $q \leq T$, there exists a constant $c_3 > 0$ such that $L(\sigma + it, \chi) \neq 0$ whenever

$$\sigma \geq 1 - \frac{c_3}{\log T} \log \left( \frac{e^{c_1}}{(1 - \beta) \log T} \right), \quad |t| \leq T$$

and $\tilde{\beta}$ is still the only exception.

The next Lemma is the explicit formula for $\psi(x, \chi)$.

**Lemma 3.** [Davenport [2], §19] Let $\chi$ a Dirichlet character to the modulus $q$ and $2 \leq T \leq x$. Then

$$\sum_{m \leq x} \Lambda(m) \chi(m) = \delta_\chi x - \delta_{\chi, \overline{\chi}} \frac{x^{1/2}}{\beta} - \sum_{\rho \leq T} \frac{i x^\rho}{\rho} + O \left( \frac{x}{T} \log^2 qx + x^{1/4} \log x \right),$$

where $\delta_\chi = 1$ if $\chi$ is the principal character, $\delta_\chi = 0$ otherwise, $\delta_{\chi, \overline{\chi}} = 1$ if $\chi = \overline{\chi}$ and $\delta_{\chi, \overline{\chi}} = 0$ otherwise and $\sum'$ means that the sum runs over the non-exceptional zeros.

We will need also a zero-density result for Dirichlet's $L$-functions.

**Lemma 4.** [Huxley [7] and Ramachandra [12]] Let $\chi$ be a Dirichlet character (mod $q$) and $N(\sigma, T, \chi) = \{ \rho = \beta + i \gamma : L(\rho, \chi) = 0, \beta \geq \sigma \text{ and } |\gamma| \leq T \}$. Then, for $\sigma \in [1/2, 1]$, there exists a positive absolute constant $c_4$ such that

$$\sum_{\chi} N(\sigma, T, \chi) \ll (qT)^{12/5(1-\sigma)}(\log^3 qT)^{c_4}. \quad (3)$$
3. Proof of the theorem

Following Walfisz [13] and Mirsky [10], we have

\[ R_k(n) = \sum_{m \leq n} \Lambda(m) \sum_{d^k \mid (n-m)} \mu(d) = \sum_{m \leq n} \Lambda(m) \left[ \sum_{d \leq D} \mu(d) + \sum_{d > D} \mu(d) \right] = \]

\[ = \sum_{d \leq D} \mu(d) \sum_{m \leq n} \Lambda(m) + \sum_{d > D} \mu(d) \sum_{m \leq n} \Lambda(m) = \]

\[ = \sum_{d \leq D} \mu(d) \psi(n; d^k, n) + \sum_{d > D} \mu(d) \psi(n; d^k, n) = A + B, \]

say, where \( \psi(x; q, a) = \sum_{m \leq x, \atop m \equiv a \mod q} \Lambda(m) \) and \( 1 \leq D \leq n^{1/k} \) will be chosen later in (12).

First of all, we estimate \( B \). By Brun-Titchmarsh Theorem, see, e.g., Friedlander-Iwaniec [4], and Theorem 328 of Hardy-Wright [6], we get

\[ B \leq \sum_{d > D} \psi(n; d^k, n) \ll \sum_{d > D} \frac{n}{\varphi(d^k)} \ll_k n \sum_{d > D} \frac{\log \log d}{d^k} \ll_k n D^{1-k} \log \log D. \] (5)

Then we remark that, if \((d, n) > 1\), we have \( \psi(n; d^k, n) \ll k \log^2 (dn) \) and hence

\[ A = \sum_{d \leq D, \atop (d, n) = 1} \mu(d) \psi(n; d^k, n) + O_k(D \log^2 (Dn)). \] (6)

We now insert \( \psi(x; q, a) = \frac{1}{\varphi(q)} \sum_{\chi \mod q} \chi(a) \psi(x, \chi) \) in (6). Hence, by Lemma 3 and the previous remarks, we get

\[ A = \sum_{d \leq D, \atop (d, n) = 1} \frac{\mu(d)}{\varphi(d^k)} \left[ n - \delta \frac{\tilde{x}}{\beta} \frac{n^{\frac{\delta}{\beta}}}{\beta} - \sum_{\chi \mod d^k} \frac{\chi(n)}{\chi \neq \chi_0} \sum_{|\rho| < T} \frac{n^\rho}{\rho} + \right. \]

\[ + O \left( \varphi(d^k) \left( \frac{n}{T} \log^2 (d^k n) + n^{1/4} \log n \right) \right) + O_k(D \log^2 (Dn)) = \]

\[ = \left( n - \delta \frac{\tilde{x}}{\beta} \frac{n^{\frac{\delta}{\beta}}}{\beta} \right) \sum_{d \leq D, \atop (d, n) = 1} \frac{\mu(d)}{\varphi(d^k)} - \sum_{d \leq D} \frac{\mu(d)}{\varphi(d^k)} \sum_{\chi \mod d^k, \chi \neq \chi_0} \frac{\tilde{x}(n)}{\chi} \sum_{|\rho| < T} \frac{n^\rho}{\rho} + \right. \]

\[ + O \left( \sum_{d \leq D, \atop (d, n) = 1} \frac{n}{T} \log^2 (d^k n) + n^{1/4} \log n \right) + O_k(D \log^2 (Dn)) = \]

\[ = \Sigma_1 + \Sigma_2 + \Sigma_3, \] (7)

say.
Evaluation of $\Sigma_1$.

To evaluate the singular series we use again Theorem 328 of Hardy-Wright [6], thus obtaining

$$\sum_{\substack{d \leq D \\ (d,n) = 1}} \frac{\mu(d)}{\varphi(d^k)} = \sum_{d=1}^{+\infty} \frac{\mu(d)}{\varphi(d^k)} + O\left( \sum_{d>D} \frac{1}{\varphi(d^k)} \right) = \mathfrak{S}_k(n) + O_k(D^{1-k} \log \log D)$$

by the Euler identity and (2). Hence we easily get

$$\Sigma_1 = \left( n - \frac{\delta}{\beta} \chi(n) \tilde{\rho} \right) \mathfrak{S}_k(n) + O_k(nD^{1-k} \log \log D). \quad (8)$$

Estimation of $\Sigma_2$.

Writing $\rho = \beta + i\gamma$ we have

$$\Sigma_2 \ll \sum_{\substack{d \leq D \\ (d,n) = 1}} \frac{1}{\varphi(d^k)} \sum_{x \mod{d^k} \mid \rho \leq T \atop x \equiv \chi \bmod{n}} \frac{n^{\rho}}{|\rho|} \ll \sum_{q \leq D^k} \frac{1}{\varphi(q)} \sum_{x \mod{q} \mid |\rho| \leq T \atop x \not\equiv \chi \bmod{n}} \sum_{\rho} \frac{n^{\rho}}{|\rho|}. \quad (9)$$

Now, to estimate $\Sigma_2$, we first split the summation over $\rho$ according to $0 < |\rho| \leq 1$ and $1 < |\rho| \leq T$. Arguing as in §20 of Davenport [2] and using Lemmas 1-2, we get

$$\frac{1}{\varphi(q)} \sum_{x \mod{q} \mid |\rho| \leq T \atop x \not\equiv \chi \bmod{n}} \sum_{\rho} \frac{n^{\rho}}{|\rho|} \ll n^{1-f(T)} \log^2 n, \quad (10)$$

where $f(T) = \frac{\Theta}{\log T}$ if the Siegel zero does not exist or $f(T) = \frac{\Theta}{\log T} \log \left( \frac{-\xi_0}{\beta} \right)$ if the Siegel zero exists.

In the range $1 < |\rho| \leq T$, we follow the line of §12 of Ivić [8]. Recalling Lemmas 1-2 and 4 and Theorem 328 of Hardy-Wright [6], we have, for $D^k \leq T$, that

$$\frac{1}{\varphi(q)} \sum_{x \mod{q} \mid |\rho| \leq T \atop x \not\equiv \chi \bmod{n}} \sum_{\rho} \frac{n^{\rho}}{|\rho|} \ll (\log^{\delta_4+3} n) \max_{1/2 \leq \sigma \leq 1-f(T)} n^{\sigma} \max_{1 \leq t \leq T} (qt)^{12/\sigma(1-\sigma)-1} \quad (11)$$

where $f(T)$ is as in (10).

Choosing now

$$T = D^{2k} \quad \text{and} \quad T = \exp(C \sqrt{\log n}), \quad (12)$$
where \( C > 0 \) is an absolute constant, we split the interval over \( \sigma \) in two parts: the first one is for \( \sigma \in [1/2, 7/12] \) and the second one is for \( \sigma \in [7/12, 1 - f(T)] \). In the first case the maxima are attained at \( t = T \) and \( \sigma = 7/12 \) and in the second case they are attained at \( t = 1 \) and \( \sigma = 1 - f(T) \). The total contribution of (11) is then
\[
\ll (n^{7/12} + n^{1 - f(T)}) T^{1/2} \log^E n \ll T^{1/2} n^{1 - f(T)} \log^E n,
\]
(13)
where \( E > 0 \) is a suitable constant, not necessarily the same at each occurrence. An analogous argument for (10) gives the same estimate. Hence, by (10) and (12)–(13), we obtain
\[
\Sigma_2 \ll T^{1/2} n^{1 - f(T)} \log^E n.
\]
(14)
If the Siegel zero does not exist than we have
\[
\Sigma_2 \ll_k T^{1/2} n \exp(-c_1 \frac{\log n}{\log T}) \log^E n,
\]
(15)
while, if the Siegel zero exists, we get
\[
\Sigma_2 \ll_k T^{1/2} n \exp \left( -c_3 \frac{\log n}{\log T} \log \left( \frac{e c_1}{(1 - \tilde{\beta}) \log T} \right) \right) \log^E n \ll
\ll T^{1/2} n \left( (1 - \tilde{\beta}) \frac{\log n}{\log T} \right) \exp(-c_3 \frac{\log n}{\log T}) \log^E n,
\]
(16)
and hence, combining (15)–(16) we finally have
\[
\Sigma_2 \ll T^{1/2} n G \exp(-c_5 \frac{\log n}{\log T}) \log^E n,
\]
(17)
where \( c_5 = \min(c_1, c_3) \) and
\[
G = \begin{cases} 
(1 - \tilde{\beta}) \sqrt{\log n} & \text{if } \tilde{\beta} \text{ exists} \\
1 & \text{if } \tilde{\beta} \text{ does not exist.}
\end{cases}
\]

**Estimation of \( \Sigma_3 \) and the final argument.**

Recalling \( T = D^{2k} \) and \( T = \exp(C \sqrt{\log n}) \), we get from (17) that
\[
\Sigma_2 \ll_k n G \exp(-c_6 \sqrt{\log n}),
\]
(18)
with
\[
C = \sqrt{c_5} \quad \text{and} \quad c_6 = \sqrt{c_5}/3.
\]
(19)
From (8) we obtain
\[
\Sigma_1 = \left( n - \delta \tilde{\chi}(n) \frac{\gamma}{\tilde{\beta}} \right) \mathcal{S}_k(n) + O_k(n \exp(-C \frac{k - 1}{3k} \sqrt{\log n})).
\]
(20)
Moreover, the error terms collected in $\Sigma_3$ can be estimated as follows:

$$\Sigma_3 \ll nD \frac{nD}{T} \log^2(D^k n) + n^{1/4} D \log n + D \log^2(Dn) \ll$$

$$\ll n \exp\left(-C \frac{2k - 1}{3k} \sqrt{\log n}\right). \quad (21)$$

Hence, if the Siegel zero does not exist, inserting (18)–(21) into (4)–(5) and (7) we have the Theorem with $c = C \frac{k - 1}{3k}$ provided that $C < \frac{3k}{k - 1} c_0$ (which holds by (19)).

If the Siegel zero exists, we remark that

$$\approx \frac{n^\beta}{\beta} \gg \exp\left(-C \frac{2k - 1}{3k} \sqrt{\log n}\right),$$

and, by Lemma 1, that

$$G \gg \frac{\sqrt{\log n}}{\sqrt{T} \log^2 \frac{\sqrt{T}}{\sqrt{\log n}}} \gg \exp\left(-C \frac{2k - 1}{3k} \sqrt{\log n}\right),$$

since $\sqrt{T} \leq T^{1/4} = \exp((C/4) \sqrt{\log n})$.

Provided that $C < \frac{3k}{k - 1} c_0$ (which holds by (19)), the Theorem follows also in this case with $c = C \frac{k - 1}{3k}$ by inserting (18)–(21) into (4)–(5) and (7).

References


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