Functiones et Approximatio
XXX (2002), 83–88

ON C_p^*-SEMINORMS FOR GENERALIZED INVOLUTION

A. El Kinani

Abstract: We consider algebras endowed with a generalized involution. We show that $|\cdot|_p^{\frac{1}{p}}$ is a C^*-seminorm, for every a p-seminorm $|\cdot|_p$, $0 < p \leqslant 1$, which satisfies the C^*-property.
Keywords: Generalized involution, involutive antimorphism, C^*-seminorm, submultiplicativity.

An involutive antimorphism on a complex algebra E is a vector involution $x \longmapsto x^*$ ([1]) such that $(xy)^* = x^*y^*$ for every $x, y \in E$. A vector space involution $x \longmapsto x^*$ is said to be a generalized involution if either it is an algebra involution (i.e. $(xy)^* = y^*x^*$ for every $x, y \in E$) or an involutive antimorphism. An algebra p-norm on E is a linear p-norm $\|\cdot\|_p$, $0 < p \leqslant 1$, satisfying $\|xy\|_p \leqslant \|x\|_p \|y\|_p$ for every $x, y \in E$. A complete p-normed algebra will be called p-Banach algebra. Let $\left(E, \|\cdot\|_p\right)$, $0 < p \leqslant 1$, be a complex p-Banach algebra endowed with a generalized involution $x \longmapsto x^*$. An element a of E is said to be hermitian (resp. normal) if $a = a^*$ (resp. $aa^* = a^*a$). We designate by $H(E)$ (resp. $N(E)$) the set of hermitian (resp. normal) elements of E. We say that a p-Banach algebra $\left(E, \|\cdot\|_p\right)$ with a generalized involution is hermitian if the spectrum of every hermitian element is real. We denote Ptak's function on E by P_E that is, for every $a \in E$, $P_E(a) = \varrho_E(aa^*)^{\frac{1}{2}}$, where ϱ_E is the spectral radius i.e. $\varrho_E(a) = \sup\{|\lambda| : \lambda \in Spa\}$. Let $\left(E, \|\cdot\|_p\right)$, $0 < p \leqslant 1$, be a hermitian p-Banach algebra with an algebra involution $x \longmapsto x^*$. We show, as in the Banach case ([5]), that P_E is an algebra seminorm such that $\varrho_E \leqslant P_E$ and $P_E(a)^2 = P_E(aa^*)$ for every $a \in E$. Moreover $RadE = \{x \in E : P_E(x) = 0\}$.

Taking into account the fact that in any p-Banach algebra $\left(E, \|\cdot\|_p\right)$ we have $\varrho_E(a)^p = \lim_n \|a^n\|_p^{\frac{1}{n}}$ for every $a \in E$. One can prove, as in [5], the following result.

2000 Mathematics Subject Classification: 46K05, 46L05.

Proposition 1. *Let* $\left(E, \|\cdot\|_p\right)$, $0 < p \leqslant 1$, *be a* p*-Banach algebra with a generalized involution* $x \longmapsto x^*$. *The following assertions are equivalent:*

1) E *is hermitian.*

2) There is $c > 0$ *such that* $\varrho_E(a) \leqslant cP_E(a)$ *for every* $a \in N(E)$.

3) $\varrho_E(a) \leqslant P_E(a)$ *for every* $a \in E$.

Using Theorem 3.10 of [7] and the fact that the quotient of a p-Banach algebra by a primitive ideal is a primitive p-Banach algebra, we can extend Theorem 4.8 p.19 of Kaplansky ([4]) to the p-Banach case as follows.

Theorem 2. *Any real semi-simple* p*-Banach algebra,* $0 < p \leqslant 1$, *in which every square is quasi-invertible is necessarily commutative.*

Let E be a complex algebra with an algebra involution $x \longmapsto x^*$. A C^*-seminorm is a seminorm $|\cdot|$ on E which satisfies the C^*-property $|a^*a| = |a|^2$ for every $a \in E$. In [6] Z. Sebestyén has proved that every C^*-seminorm is automatically submultiplicative. In this paper we extend this result to the p-seminorm case as follows.

Theorem 3. *Let* E *be a complex algebra endowed with a generalized involution* $x \longmapsto x^*$. *If* $|\cdot|_p$ *is a linear* p*-seminorm,* $0 < p \leqslant 1$, *on* E *such that*

$$|a^*a|_p = |a|_p^2 \quad \text{for every} \quad a \in E,$$

then $|\cdot|_p^{\frac{1}{p}}$ *is an algebra seminorm and the completion of* $E/Ker\,|\cdot|_p$ *is a* C^**-algebra.*

Proof. Using the elementary algebraic identity

$$\begin{aligned} 4ab = {} & (b + a^*)^*(b + a^*) + i(b + ia^*)^*(b + ia^*) \\ & - (b - a^*)^*(b - a^*) - i(b - ia^*)^*(b - ia^*) \end{aligned}$$

valid for every $a, b \in E$, we obtain that

$$|ab|_p \leqslant 4^{1-p}\left(|a^*|_p + |b|_p\right)^2 \quad \text{for every} \quad a, b \in E.$$

So $|ab|_p \leqslant 4^{2-p}$ for every $a, b \in E$ with $|a^*|_p \leqslant 1$ and $|b|_p \leqslant 1$. This implies that

$$|ab|_p \leqslant 4^{2-p}\,|a^*|_p\,|b|_p \qquad \text{for every} \quad a, b \in E. \tag{1}$$

Hence

$$|a|_p \leqslant 4^{1-\frac{p}{2}}\,|a^*|_p \qquad \text{for every} \quad a \in E. \tag{2}$$

According to (1) and (2) we get

$$|ab|_p \leqslant 4^{3-\frac{3p}{2}}\,|a|_p\,|b|_p \qquad \text{for every} \quad a, b \in E.$$

Consider on $E/Ker\,|\cdot|_p$ the p-norm denoted by $\|\cdot\|_p$ and defined by

$$\|\pi(x)\|_p = |x|_p \quad \text{for every} \quad x \in E,$$

where π is the natural quotient map of E onto $E/Ker\,|\cdot|_p$. Denote by $\widehat{E}$ the completion of the p-normed algebra $\left(E/Ker\,|\cdot|_p\,, \|\cdot\|_p\right)$. The p-norm in $\widehat{E}$ will also be designated by $\|\cdot\|_p$. Then we have

$$\|a^*a\|_p = \|a\|_p^2 \quad \text{for every} \quad a \in \widehat{E} \tag{3}$$

and

$$\|ab\|_p \leqslant 4^{3-\frac{3p}{2}} \|a\|_p \|b\|_p \quad \text{for every} \quad a, b \in \widehat{E}. \tag{4}$$

For $a \in \widehat{E}$, put

$$|||a|||_p = \sup\{\|ab\|_p : \|b\|_p \leqslant 1\}.$$

We get an algebra p-norm on $\widehat{E}$ such that

$$4^{\frac{p}{2}-1} \|a\|_p \leqslant |||a|||_p \leqslant 4^{3-\frac{3p}{2}} \|a\|_p \quad \text{for every} \quad a \in \widehat{E}.$$

In the p-Banach algebra $\left(\widehat{E}, |||\cdot|||_p\right)$ with a generalized involution $x \longmapsto x^*$, the spectral radius $\varrho_{\widehat{E}}$ satisfies, for every $a \in N(\widehat{E})$,

$$\begin{aligned}
\varrho_{\widehat{E}}(a)^{2p} &= \lim_n \left|\left|\left| a^{2^n} \right|\right|\right|_p^{2^{-n+1}} \\
&= \lim_n \left\| a^{2^n} \right\|_p^{2^{-n+1}} \\
&= \lim_n \left\| (a^*a)^{2^n} \right\|_p^{2^{-n}} \\
&= \lim_n \left|\left|\left| (a^*a)^{2^n} \right|\right|\right|_p^{2^{-n}} \\
&= \varrho_{\widehat{E}}(a^*a)^p.
\end{aligned}$$

Hence

$$\varrho_{\widehat{E}}(a) = P_{\widehat{E}}(a) \quad \text{for every} \quad a \in N(\widehat{E}), \tag{5}$$

which implies in particular

$$\varrho_{\widehat{E}}(a)^p = \lim_n \left|\left|\left| (a^*a)^{2^n} \right|\right|\right|_p^{2^{-n-1}} = \|a^*a\|_p^{\frac{1}{2}} = \|a\|_p \quad \text{for every} \quad a \in N(\widehat{E}). \tag{6}$$

By Proposition 1 the algebra $\left(\widehat{E}, |||\cdot|||_p\right)$ is hermitian and so

$$\varrho_{\widehat{E}}(a) \leqslant P_{\widehat{E}}(a) \quad \text{for every} \quad a \in \widehat{E}. \tag{7}$$

We consider first that $x \longmapsto x^*$ is an algebra involution. In this case we get by (6) and (7)

$$\|ab\|_p^2 \leqslant \left\|bb^*(a^*a)^2bb^*\right\|_p^{\frac{1}{2}} \quad \text{for every} \quad a,b \in \widehat{E}.$$

Inductively, we obtain for every $n = 1, 2, \ldots$

$$\|ab\|_p^2 \leqslant \left\|(bb^*)^{2^{n-1}}(a^*a)^{2^n}(bb^*)^{2^{n-1}}\right\|_p^{2^{-n}} \quad \text{for every} \quad a,b \in \widehat{E}.$$

It then follows from (4) and (3) that

$$\|ab\|_p^2 \leqslant (4^{6-3p})^{2^{-n}} \|a\|_p^2 \|b\|_p^2 \quad \text{for every} \quad n = 1, 2, \ldots \quad \text{and} \quad a,b \in \widehat{E}.$$

Letting n tend to infinity, we obtain

$$\|ab\|_p \leqslant \|a\|_p \|b\|_p \quad \text{for every} \quad a,b \in \widehat{E}.$$

Therefore

$$|ab|_p \leqslant |a|_p |b|_p \quad \text{for every} \quad a,b \in E.$$

On the other hand $P_{\widehat{E}}$ is an algebra seminorm such that

$$P_{\widehat{E}}(a)^2 = P_{\widehat{E}}(a^*a) \quad \text{for every} \quad a \in \widehat{E},$$

and by (6)

$$\|a\|_p^2 = \varrho_{\widehat{E}}(aa^*)^p = P_{\widehat{E}}(a)^{2p} \quad \text{for every} \quad a \in \widehat{E}.$$

Thus

$$P_{\widehat{E}}(a) = \|a\|_p^{\frac{1}{p}} \quad \text{for every} \quad a \in \widehat{E}.$$

This implies that $\|\cdot\|_p^{\frac{1}{p}}$ is an algebra seminorm. Moreover $P_{\widehat{E}}$ is an algebra norm on $\widehat{E}$ which is equivalent to $\|\cdot\|_p$, and such that $(\widehat{E}, P_{\widehat{E}})$ is a C^*-algebra. Suppose now that $x \longmapsto x^*$ is an involutive antimorphism. We will show that in this case the algebra $\widehat{E}$ is commutative. It is sufficient to consider the real p-Banach algebra $H(\widehat{E})$. By (6) we have $Rad(H(\widehat{E})) = \{0\}$. Since $\widehat{E}$ is hermitian every square of $H(\widehat{E})$ is quasi-invertible. Hence by Theorem 2 the algebra $H(\widehat{E})$ is commutative. This completes the proof. ■

Let E be a complex algebra with a generalized involution $x \longmapsto x^*$. We define a C_p^*-seminorm as being a linear p-seminorm $|\cdot|_p$, $0 < p \leqslant 1$, on E such that $|a^*a|_p = |a|_p^2$ for every $a \in E$. If $|\cdot|_p$ is a C_p^*-seminorm, $0 < p \leqslant 1$, then by Theorem 3 $|\cdot|_p^{\frac{1}{p}}$ is a C^*-seminorm. Then we have the following result which is an extension of Theorem 4 of [6].

Corollary 4. *Let E be a complex algebra with a generalized involution $x \longmapsto x^*$, I a $*$-ideal in E and $|\cdot|_p$ a C_p^*-seminorm on E. The following assertions are equivalent:*

1) There exists a C^*-seminorm $|\cdot|$ on E such that $|x| = |x|_p^{\frac{1}{p}}$ for every $x \in I$.

2) For every $a \in E$

$$\sup\{|ab|_p^{\frac{1}{p}}, \quad b \in I, \quad |b|_p \leqslant 1\} < +\infty.$$

Remark 5. If $|\cdot|_p$ is linear p-seminorm, $0 < p \leqslant 1$, such that $c\,|x^*|_p\,|x|_p \leqslant |x^*x|_p$ for every $x \in E$ and some constant $c > 0$, then $|\cdot|_p$ is not necessarily submultiplicative as the following example shows:

Let $E = C([0,1])$ be the algebra of all complex-valued continuous functions on $[0,1]$ endowed with the norm

$$\|f\|_1 = \int_0^1 |f(t)|\,dt$$

and the involution $f \longmapsto f^* = \overline{f}$. It is clear that $\|f\|_1 = \|f^*\|_1$ and $\|f\|_1^2 \leqslant \|f^*f\|_1$ for every $f \in E$. But $\|\cdot\|_1$ is not submultiplicative. Actually $\|\cdot\|_1$ is a linear norm for which the product is not continuous.

Remark 6. Let $|\cdot|_p$ be a linear p-seminorm, $0 < p \leqslant 1$ such that $|x^*x|_p \leqslant c\,|x|_p\,|x^*|_p$ for every $x \in E$ and some constant $c > 0$. The same argument used in the proof of Theorem 3 shows that $|x|_p' = \max\left(|x|_p, |x^*|_p\right)$ is a linear p-seminorm, $0 < p \leqslant 1$, for which the product is continuous. It is not the case for $|\cdot|_p$ as the following example shows:

Let E denote the direct sum $C([0,1]) \oplus C([0,1])$. Define norm, product and involution in E by:

$$\|(f,g)\| = \max(\|f\|_\infty, \|g\|_1);$$
$$(f_1,g_1)(f_2,g_2) = (f_1f_2, g_1g_2),$$
$$(f,g)^* = (\overline{g}, \overline{f}),$$

where

$$\|f\|_\infty = \sup_{t\in[0,1]} |f(t)|$$

and

$$\|g\|_1 = \int_0^1 |g(t)|\,dt.$$

It is easy to verify that $\|(f,g)^*(f,g)\| \leqslant \|(f,g)\|\,\|(f,g)^*\|$ for every $f,g \in C([0,1])$. But the product is not continuous for $\|\cdot\|$.

References

[1] F.F. Bonsall and J.Duncan, *Complete normed algebras*, Ergebnisse der Mathematik, Band **80**, Springer Verlag, 1973.

[2] A. El Kinani, A. Ifzarne, M. Oudadess, *p-Banach algebras with generalized involution and C^*-algebra structure*, Turk. J. Math. **23** (2001), 275–282.

[3] A. El Kinani, M. Oudadess, *Involution généralisée et structure de C^*-algèbre*, Rev. Academia de Ciencias, Zaragoza, **52** (1997), 15–16.

[4] I. Kaplansky, *Normed algebras*, Duke Math. J. 16 (1949), 399–418.

[5] V. Ptàk, *Banach algebras with involution*, Manuscripta Math. 6 (1972), 245–290.

[6] Z. Sebestyén, *Every C^*-seminorm is automatically submultiplicative*, Periodica Mathematica Hungarica, **10** (1), (1979), 1–8.

[7] W. Żelazko, *Selected topics in topological algebras*, Lecture Notes, Serie **31** (1971).

Address: Ecole Normale Supérieure, B.P. 5118,
Takaddoum, 10105 Rabat (Morocco).
E-mail: a_elkinani@ens-rabat.ac.ma
Received: 13 November 2001