Functiones et Approximatio
XXVIII (2000), 201–209

To Professor Włodzimierz Staś on the occasion of his 75th birthday

ON THE RATE OF CONVERGENCE OF THE BÉZIER TYPE OPERATORS

Paulina Pych-Taberska

Abstract: The rate of pointwise convergence of the Bézier type modification of some discrete Feller operators of locally bounded functions is estimated. In the general theorems the Chanturija modulus of variation is used. In particular, corresponding estimates for functions of bounded p-th power variation are deduced.
Keywords: Bézier type operator, rate of convergence, modulus of variation, p-th power variation

1. Preliminaries

Let $\{X_{k,x}\}_{k=1}^{\infty}$ be a family of sequences of independent and identically distributed random variables with expectation $EX_{k,x} = x$, $k \in N := \{1, 2, \dots\}$ and finite variance $\sigma^2(x)$, where x is a real parameter taking values in a bounded or unbounded interval $I \subseteq R := (-\infty, \infty)$. Suppose that the random variables $X_{1,x}, X_{2,x}, \dots$ have the common lattice distribution $F = \{p_{1,j}(x) : x \in I, j \in J_1\}$ concentrated on a set $J_1 \subseteq Z \cap I$, $Z := \{0, \pm 1, \pm 2, \dots\}$, where the ≪weights≫ $p_{1,j}$ of the ≪atoms≫ j are continuous on I. Consider the sum $S_{n,x} = X_{1,x} + X_{2,x} + \dots + X_{n,x}$ and its distribution $\{p_{n,j}(x) : x \in I, j \in J_n\}$ being the n-fold convolution of F with itself. Introduce the discrete Feller operators

$$L_n f(x) := Ef(S_{n,x}/n) = \sum_{j \in J_n} f\left(\frac{j}{n}\right) p_{n,j}(x) \quad (n \in N) \tag{1.1}$$

for real-valued functions f defined on I and satisfying $E|f(S_{n,x}/n)| < \infty$ [5, p. 218].

Suppose that J_n is of the form $\{0, 1, 2, \dots, m_n\}$ with some $m_n \in N$ or

---

1991 Mathematics Subject Classification: 41A25

$J_n = N_0 := N \cup \{0\}$ and define, as in [2], the Bézier basis functions

$$q_{n,k}(x) := \sum_{j \in J_n,\ j \geq k} p_{n,j}(x) \quad \text{for} \quad k \in J_n,$$
$$q_{n,m_n+1}(x) \equiv 0 \quad \text{if} \quad J_n = \{0, 1, 2, \dots, m_n\}.$$

Given a number $\alpha \geq 1$, write

$$Q_{n,k}^{(\alpha)}(x) := q_{n,k}^{\alpha}(x) - q_{n,k+1}^{\alpha}(x).$$

The Bézier type modification of operators (1.1) is defined by

$$L_n^{(\alpha)} f(x) := \sum_{k \in J_n} f\left(\frac{k}{n}\right) Q_{n,k}^{(\alpha)}(x). \tag{1.2}$$

Recently, several authors studied approximation properties of the special operators (1.2) in which $L_n f \equiv B_n f$ are the Bernstein polynomials of f, i.e. $p_{n,j}(x) = \binom{n}{j} x^j (1-x)^{n-j}$, $x \in I = [0,1]$, $j \in J_n = \{0, 1, \dots, n\}$. As is known [10, p.372], the Bernstein-Bézier operators $B_n^{(\alpha)} f$ are not the Feller type ones. Some results concerning the approximation by operators $B_n^{(\alpha)} f$ are given e.g. in [3], [10]. In particular, Zeng and Piriou [10] gave an estimate for the rate of convergence of $B_n^{(\alpha)} f$ for functions f belonging to the class $BV([0,1])$, i.e. for functions f of bounded variation in the Jordan sense on $[0,1]$.

In this paper we first present a general estimate for the rate of pointwise convergence of the discrete Feller-Bézier operators (1.2) in the case where f is bounded on the interval I and possesses the one-sided limits $f(x+), f(x-)$ at a fixed point x. In particular, we obtain the corresponding estimates for functions f of bounded variation in the generalized sense on I. In the special case where $f \in BV([0,1])$ and $L_n^{(\alpha)} f \equiv B_n^{(\alpha)} f$ we get the above mentioned result of Zeng and Piriou [10]. Finally, the extension of our results to unbounded functions f is presented. Note that analogous problems for the discrete Feller operators (1.1) were considered e.g. in [1], [8], [9].

Throughout the paper we use the symbol $M(I)$ [resp. $M_{\text{loc}}(I)$] for the class of all real-valued functions bounded on I [resp. bounded on every compact subinterval of I]. For positive integer k, the modulus of variation of a bounded function g on a compact interval $Y = [c, d]$ will be denoted by $v_k(g; Y)$ or $v_k(g; c, d)$ and will be defined as in [4] (see also [1], [8]). If Y is an unbounded interval, e.g. $Y = [c, \infty)$, then $v_k(g; c, \infty)$ is understood as the limit of $v_k(g; c, d)$ as $d \to \infty$. Clearly, if $g \in M(Y)$, then for every integer k, $v_k(g; Y) < \infty$. Some basic properties of the modulus of variation can be found e.g. in [4].

2. Results

Let $f \in M(I)$ or $f \in M_{\mathrm{loc}}(I)$ and let at a fixed point $x \in IntI$ the one-sided limits $f(x+), f(x-)$ exist. It is easy to verify that for all $t \in I$,

$$f(t) = 2^{-\alpha} f(x+) + (1-2^{-\alpha}) f(x-) + g_x(t) + 2^{-\alpha}(f(x+) - f(x-)) \operatorname{sgn}_x^{(\alpha)}(t) \\ + \left(f(x) - 2^{-\alpha} f(x+) - (1-2^{-\alpha}) f(x-)\right) \delta_x(t),$$

where

$$g_x(t) = \begin{cases} f(t) - f(x+) & \text{if } t > x, \\ 0 & \text{if } t = x, \\ f(t) - f(x-) & \text{if } t < x, \end{cases} \qquad \operatorname{sgn}_x^{(\alpha)}(t) = \begin{cases} 2^{\alpha} - 1 & \text{if } t > x. \\ 0 & \text{if } t = x, \\ -1 & \text{if } t < x \end{cases}$$

and $\delta_x(x) = 1$, $\delta_x(t) = 0$ if $t \neq x$ (see [10, p. 381]). Therefore

$$L_n^{(\alpha)} f(x) - 2^{-\alpha} f(x+) - (1-2^{-\alpha}) f(x-) = L_n^{(\alpha)} g_x(x) + \Delta_n^{(\alpha)}(f;x) \tag{2.1}$$

with

$$\Delta_n^{(\alpha)}(f;x) = 2^{-\alpha}\left(f(x+) - f(x-)\right) L_n^{(\alpha)} \operatorname{sgn}_x^{(\alpha)}(x) \\ + \left(f(x) - 2^{-\alpha} f(x+) - (1-2^{-\alpha}) f(x-)\right) L_n^{(\alpha)} \delta_x(x). \tag{2.2}$$

Let a, b be two arbitrary positive numbers. Write the term $L_n^{(\alpha)} g_x(x)$ in the form

$$L_n^{(\alpha)} g_x(x) = \sum_{k \in A_x(a.b)} g_x\left(\frac{k}{n}\right) Q_{n.k}^{(\alpha)}(x) + \vartheta_x(a,b) \sum_{k \in D_x(a,b)} g_x\left(\frac{k}{n}\right) Q_{n.k}^{(\alpha)}(x), \tag{2.3}$$

where $A_x(a.b) = \{k \in J_n : x - a \le k/n \le x + b\}$, $D_x(a,b) = J_n \setminus A_x(a,b)$ and $\vartheta_x(a.b) = 0$ if neither of the points $x - a, x + b$ belongs to $\mathrm{Int}I, \vartheta_x(a,b) = 1$ otherwise.

In order to estimate the terms of the right-hand side of (2.3) let us observe that $Q_{n.k}^{(\alpha)}(x) \ge 0$,

$$\sum_{k \in J_n} Q_{n.k}^{(\alpha)}(x) = \left(\sum_{j \in J_n} p_{n.j}(x)\right)^{\alpha} = 1$$

and that the variance of the avergage $S_{n.x}/n$ is equal to

$$\sum_{j \in J_n} \left(\frac{j}{n} - x\right)^2 p_{n,j}(x) = \frac{\sigma^2(x)}{n}.$$

Consequently, in view of the obvious inequality $|u^{\alpha} - v^{\alpha}| \le \alpha|u - v|$ if $\alpha \ge 1$, $0 < v \le u \le 1$, we have

$$\sum_{k \in J_n} \left(\frac{k}{n} - x\right)^2 Q_{n.k}^{(\alpha}(x) \le \alpha \sum_{k \in J_n} \left(\frac{k}{n} - x\right)^2 (q_{n.k}(x) - q_{n.k+1}(x)) = \alpha \frac{\sigma^2(x)}{n}.$$

Arguing similarly to the proof of Lemma in [1] (see also the proof of Lemma in [9]) we obtain the following fundamental estimates.

Lemma 2.1. *Let $f \in M_{\mathrm{loc}}(I)$ and let $a > 0$, $b > 0$. Then, for every $n \geq 4$,*

$$\left|\sum_{k\in A_x(a,b)} g_x\left(\frac{k}{n}\right) Q_{n,k}^{(\alpha)}(x)\right|$$
$$\leq \left(1+\frac{8\alpha\sigma^2(x)}{a^2}\right)\left(\sum_{j=1}^{m-1}\frac{1}{j^3}v_j\left(g_x; I_x(-ja/\sqrt{n})\right)+\frac{1}{m^2}v_m\left(g_x; I_x(-a)\right)\right)$$
$$+\left(1+\frac{8\alpha\sigma^2(x)}{b^2}\right)\left(\sum_{j=1}^{m-1}\frac{1}{j^3}v_j\left(g_x; I_x(jb/\sqrt{n})\right)+\frac{1}{m^2}v_m\left(g_x; I_x(b)\right)\right),$$

where $m = [\sqrt{n}]$, $I_x(h) = [x+h, x] \cap I$ if $h < 0$, $I_x(h) = [x, x+h] \cap I$ if $h > 0$. If $f \in M(I)$ and if at least one of the points $x-a, x+b$ belongs to Int$I$, *then for all $n \in N$,*

$$\left|\sum_{k\in D_x(a,b)} g_x\left(\frac{k}{n}\right) Q_{n,k}^{(\alpha)}(x)\right| \leq \frac{\alpha\sigma^2(x)}{nc^2}v_1(g_x; I),$$

where $c = \min\{a, b\}$.

Lemma 2.2. *Let $I = [0,\infty)$ or $I = (-\infty,\infty)$ and let a function f of class $M_{\mathrm{loc}}(I)$ satisfy the growth condition*

$$|f(x)| \leq \psi(x) \quad \text{for all} \quad x \in I \tag{2.4}$$

with a non-negative function $\psi \in C(I)$ such that for all $n \geq n_0$, $x \in I$,

$$\sum_{j\in J_n}\psi^2\left(\frac{j}{n}\right)p_{n,j}(x) \leq \varphi^2(x) \quad (\varphi \in C(I), \varphi(x) \geq 0 \quad \text{for all} \quad x \in I).$$

Then

$$\left|\sum_{k\in D_x(a,b)} g_x\left(\frac{k}{n}\right) Q_{n,k}^{(\alpha)}(x)\right| \leq \frac{\alpha}{c\sqrt{n}}\left(\varphi(x)\sigma(x)+\frac{1}{c}\psi(x)\sigma^2(x)\right), \tag{2.5}$$

where $c = \min\{a, b\}$, $\sigma(x) = \sqrt{\sigma^2(x)}$.

Proof. In view of (2.4) the left-hand side of (2.5) is not greater than

$$\sum_{|k/n-x|\geq c}\left(\psi\left(\frac{k}{n}\right)+\psi(x)\right)Q_{n,k}^{(\alpha)}(x)$$
$$\leq \frac{1}{c}\sum_{k\in J_n}\psi\left(\frac{k}{n}\right)\left|\frac{k}{n}-x\right|Q_{n,k}^{(\alpha)}(x)+\frac{\psi(x)}{c^2}\sum_{k\in J_n}\left(\frac{k}{n}-x\right)^2Q_{n,k}^{(\alpha)}(x)$$
$$\leq \frac{1}{c}\left(\alpha\varphi^2(x)\right)^{1/2}\left(\alpha\frac{\sigma^2(x)}{n}\right)^{1/2}+\alpha\frac{\psi(x)\sigma^2(x)}{nc^2},$$

by the Schwarz inequality. Thus, estimate (2.5) follows. ■

Inequalities given in Lemmas 2.1 and 2.2 together with identity (2.3) enable us to get the estimate of the term $L_n^{(\alpha)} g_x(x)$ in (2.1). To obtain the estimate of the term $\Delta_n^{(\alpha)}(f;x)$ in (2.1) we consider only the points $x \in I$ at which

$$\sigma^2(x) > 0 \quad \text{and} \quad \beta(x) := \sum_{j\in J_1} |j-x|^3 p_{1,j}(x) < \infty. \tag{2.6}$$

Lemma 2.3. *Under the assumptions* (2.6) *we have*

$$|\Delta_n^{(\alpha)}(f;x)| \le \frac{\alpha\tau}{\sqrt{n}\sigma^3(x)}\Big(\beta(x)|f(x+)-f(x-)| + e_n(x)|f(x)-f(x-)|(2\beta(x)+\sigma^2(x))\Big),$$

where $0<\tau\le 0.82$ *and* $e_n(x)=0$ *if* $x \ne k/n$ *for all* $k\in J_n$, $e_n(x)=1$ *if there exists a* $k' \in J_n$ *such that* $x=k'/n$.

Proof. First, let us recall that in view of the Berry-Esséen Theorem [5, p. 515],

$$\Big|\sum_{j-nx\le t\sigma(x)\sqrt{n}} p_{n,j}(x) - \mathfrak{N}(t)\Big| \le \frac{\tau\beta(x)}{\sqrt{n}\sigma^3(x)} \qquad (n\in N, t\in R)$$

where

$$\mathfrak{N}(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} \exp(-y^2/2)dy$$

and $0<\tau\le 0.82$ (see [6, p. 93]). From this it follows at once that

$$\Big|\sum_{j>nx} p_{n,j}(x) - \frac{1}{2}\Big| \le \frac{\tau\beta(x)}{\sqrt{n}\sigma^3(x)} \tag{2.7}$$

and

$$p_{n,k}(x) = \sum_{j\le k} p_{n,j}(x) - \sum_{j\le k-1} p_{n,j}(x) \le \frac{1}{\sqrt{n}}\left(\frac{2\tau\beta(x)}{\sigma^3(x)} + \frac{1}{\sqrt{2\pi}\sigma(x)}\right). \tag{2.8}$$

Further, it is easy to see (as in [10]) that

$$\begin{aligned} L_n^{(\alpha)}\mathrm{sgn}_x^{(\alpha)}(x) &= (2^\alpha-1)\sum_{k>nx} Q_{n,k}^{(\alpha)}(x) - \sum_{k<nx} Q_{n,k}^{(\alpha)}(x) \\ &= 2^\alpha \sum_{k>nx}\left(q_{n,k}^\alpha(x) - q_{n,k+1}^\alpha(x)\right) - \sum_{k>nx} Q_{n,k}^{(\alpha)}(x) - \sum_{k<nx} Q_{n,k}^{(\alpha)}(x) \\ &= 2^\alpha\Big(\sum_{j>nx} p_{n,j}(x)\Big)^\alpha - \Big(1 - e_n(x) Q_{n,k'}^{(\alpha)}(x)\Big) \end{aligned}$$

and

$$L_n^{(\alpha)}\delta_x(x) = e_n(x)Q_{n,k'}^{(\alpha)}(x).$$

Hence, in view of (2.2),

$$\begin{aligned}|\Delta_n^{(\alpha)}(f;x)| &\le |f(x+) - f(x-)| \left| \Big(\sum_{j>nx} p_{n,j}(x) \Big)^{\alpha} - 2^{-\alpha} \right| \\ &\qquad + |f(x) - f(x-)| e_n(x) Q_{n,k'}^{(\alpha)}(x) \\ &\le \alpha |f(x+) - f(x-)| \left| \sum_{j>nx} p_{n,j}(x) - \frac{1}{2} \right| \\ &\qquad + \alpha |f(x) - f(x-)| e_n(x) p_{n,k'}(x).\end{aligned}$$

Now, the result follows by (2.7) and (2.8). ∎

Combining identity (2.1) with the inequalities given in our lemmas one can formulate corresponding estimates concerning the rate of pointwise convergence of the operators $L_n^{(\alpha)} f$.

First, let us choose $a = b = 1$ in (2.3). Then we obtain the following

Theorem 2.4. *Let $f \in M(I)$ and let at a fixed point $x \in \mathrm{Int}I$ the one-sided limits $f(x+), f(x-)$ exist. Then, for all $n \ge 4$,*

$$\begin{aligned}&|L_n^{(\alpha)} f(x) - 2^{-\alpha} f(x+) - (1 - 2^{-\alpha}) f(x-)| \\ &\quad \le 2\left(1 + 8\alpha\sigma^2(x)\right) \left(\sum_{j=1}^{m-1} \frac{1}{j^3} v_j\left(g_x; Y_x(j/\sqrt{n}\right) + \frac{1}{m^2} v_m\left(g_x; Y_x(1)\right)\right) \\ &\qquad + \vartheta_x(1,1) \frac{\alpha\sigma^2(x)}{n} v_1(g_x; I) + |\Delta_n^{(\alpha)}(f;x)|,\end{aligned}$$

where $m = [\sqrt{n}]$, $Y_x(h) = [x - h, x + h] \cap I$, $\vartheta_x(1,1) = 0$ if neither of the points $x - 1, x + 1$ belongs to $\mathrm{Int}I$, $\vartheta_x(1,1) = 1$ otherwise, and $|\Delta_n^{(\alpha)}(f;x)|$ is estimated via Lemma 2.3.

Next, suppose that I is an unbounded interval and choose $a = b = A > 0$ in the formula (2.3). Then we get

Theorem 2.5. *Let all conditions of Lemma 2.2 be satisfied. If moreover the function f has the one-sided limits $f(x+), f(x-)$ at a fixed point $x \in IntI$ and $|x| \le A$, then for all $n \ge \max\{4, n_0\}$,*

$$\begin{aligned}&|L_n^{(\alpha)} f(x) - 2^{-\alpha} f(x+) - (1 - 2^{-\alpha}) f(x-)| \\ &\quad \le 2\Big(1 + \frac{8\alpha\sigma^2(x)}{A^2}\Big) \Big(\sum_{j=1}^{m-1} \frac{1}{j^3} v_j(g_x; Y_x(jA/\sqrt{n})) + \frac{1}{m^2} v_m(g_x; Y_x(A))\Big) \\ &\qquad + \frac{\alpha}{A\sqrt{n}} \Big(\varphi(x)\sigma(x) + \frac{1}{A}\psi(x)\sigma^2(x) \Big) + |\Delta_n^{(\alpha)}(f;x)|.\end{aligned}$$

where $m = [\sqrt{n}]$, $Y_x(h) = [x-h, x+h] \cap I$ *and* $|\Delta_n^{(\alpha)}(f;x)|$ *is estimated via Lemma 2.3.*

Remark 2.6. In view of the continuity of the function g_x at x, the right-hand sides of the inequalities given in Theorems 2.4 and 2.5 tend to 0 as $n \to \infty$ (see Remark 1 in [9]).

3. Corollaries and examples

Let $p \geq 1$. Denote by $BV_p(I)$ the class of all functions of bounded p-th power variation on the interval I and by $V_p(g;Y)$ the total p-th variation of the function g on the interval $Y \subseteq I$, defined as the upper bound of the set of all numbers

$$\left(\sum_i |g(t_i) - g(\tau_i)|^p\right)^{1/p}$$

over all finite systems of non-overlapping intervals $(t_i, \tau_i) \subset Y$. Clearly, if $V_p(g;Y) < \infty$ then for every integer j

$$v_j(g;Y) \leq j^{1-1/p} V_p(g;Y).$$

This inequality, Theorem 2.4 and some simple calculation (see e.g. [9, p. 152]) lead to

Corollary 3.1. *If* $f \in BV_p(I)$, *then for every* $x \in \operatorname{Int} I$ *and all* $n \geq 4$,

$$\begin{aligned}&|L_n^{(\alpha)} f(x) - 2^{-\alpha} f(x+) - (1-2^{-\alpha}) f(x-)|\\ &\quad \leq \frac{8(1+9\alpha\sigma^2(x))}{(\sqrt{n})^{1+1/p}} \sum_{k=0}^{n} (\sqrt{k+1})^{-1+1/p} V_p(g_x;U_k) + |\Delta_n^{(\alpha)}(f;x)|,\end{aligned}$$

where $U_0 = I, U_k = [x - 1/\sqrt{k}, x + 1/\sqrt{k}] \cap I$ *if* $k = 1, 2, \ldots, n$ *and* $|\Delta_n^{(\alpha)}(f;x)|$ *is estimated as in Lemma 2.3.*

Analogously, from Theorem 2.5 one can deduce corresponding estimate for functions f of bounded p-th power variation on every compact interval contained in I and satisfying the growth condition (2.4). We will not formulate this corollary explicite.

Note, that from Theorems 2.4 and 2.5 similar results for more general classes of functions f of bounded Φ-variation can be obtained, too (see e.g. [1], [8], [9]).

Now, we will present some simple examples.

1) Let $L_n^{(\alpha)} f \equiv B_n^{(\alpha)} f$ be the Bernstein-Bézier operators of $f \in M(I)$, defined by (1.2) in which $I = [0,1], J_n = \{0, 1, \ldots, n\}, p_{n,j}(x) = \binom{n}{j} x^j (1-x)^{n-j}$. In this case Theorem 2.4 applies with $\sigma^2(x) = x(1-x)$, $\beta(x) = x(1-x)(2x^2 - 2x + 1)$, $\vartheta_x(1,1) = 0$ and

$$|\Delta_n^{(\alpha)}(f;x)| \leq \frac{5\alpha}{2\sqrt{nx(1-x)}} \left(|f(x+) - f(x-)| + e_n(x)|f(x) - f(x-)|\right). \quad (3.1)$$

In view of the Remark 2.6,

$$\lim_{n\to\infty} B_n^{(\alpha)} f(x) = 2^{-\alpha} f(x+) + (1-2^{-\alpha}) f(x-)$$

whenever f is bounded on $[0,1]$ and $x \in (0,1)$ is its discontinuity point of the first kind. This means that our estimate can be treated as a quantitative version of Corollary 1 in [10].

Note, that we can also proceed as follows. Choosing in Lemma 2.1, $a = x$, $b = 1-x$ and using representations (2.1), (2.3), we easily get

$$\begin{aligned} &|B_n^{(\alpha)} f(x) - 2^{-\alpha} f(x+) - (1-2^{-\alpha}) f(x-)| \\ &\quad \le W_\alpha(x) \Big(\sum_{j=1}^{m-1} \frac{1}{j^3} v_j\Big(g_x; x - \frac{jx}{\sqrt{n}}, x + \frac{j(1-x)}{\sqrt{n}}\Big) + \frac{1}{m^2} v_m(g_x; 0, 1) \Big) \\ &\quad\quad + |\Delta_n^{(\alpha)}(f;x)|, \end{aligned}$$

where $m = [\sqrt{n}]$, $W_\alpha(x) = 2 + 8\alpha(1-x)/x + 8\alpha x/(1-x)$ and $|\Delta_n^{(\alpha)}(f;x)|$ is estimated in (3.1). For functions $f \in BV_p([0,1])$ the above inequality leads to the following estimate

$$\begin{aligned} &|B_n^{(\alpha)} f(x) - 2^{-\alpha} f(x+) - (1-2^{-\alpha}) f(x-)| \\ &\quad \le Z_\alpha(x) \frac{1}{(\sqrt{n})^{1+1/p}} \sum_{k=1}^{n} (\sqrt{k+1})^{-1+1/p} V_p\Big(g_x; x - \frac{x}{\sqrt{k}}, x + \frac{1-x}{\sqrt{k}}\Big) \\ &\quad\quad + |\Delta_n^{(\alpha)}(f;x)| \end{aligned}$$

for all $x \in (0,1)$, $n \ge 4$, where $Z_\alpha(x) = 10 W_\alpha(x)$. In case $p = 1$ this gives the result up to the order the same as in Theorem 1 in [10]. Clearly, in this case ($p = 1$), by the direct calculation one can get more precise value of the factor $Z_\alpha(x)$ (see [10, Th. 1]). Also, the factor $5\alpha/2\sqrt{nx(1-x)}$ in (3.1) may be slightly improved and replaced by $2\alpha/(\sqrt{nx(1-x)}+1)$ (see [10, Lemma 5]).

2) Next, let us consider the Szász-Mirakyan operators $S_n f \equiv L_n f$ given by (1.1) in which $p_{n,j}(x) = e^{-nx}(nx)^j/j!$ for $x \in I = [0,\infty)$, $j \in J_n = N_0$. Denote by $S_n^{(\alpha)} f$ their modification of the form (1.2). If f is bounded on $[0,\infty)$ then one can apply Theorem 2.4 with $\sigma^2(x) = x$, $\beta(x) \le x\sqrt{1+3x}$, $\vartheta_x(1,1) = 1$ and

$$|\Delta_n^{(\alpha)}(f;x)| \le \frac{5\alpha\sqrt{1+3x}}{2\sqrt{nx}} \left(|f(x+) - f(x-)| + e_n(x) |f(x) - f(x-)| \right).$$

Suppose, further, that f is unbounded on I and that it satisfies the growth condition (2.4) with $\psi(x) = (1+x)^q$ where q is a positive integer. It is easy to see that

$$\sum_{j=0}^{\infty} \psi^2\left(\frac{j}{n}\right) p_{n,j}(x) \le 2^{2q-1} \left((1+x)^{2q} + \sum_{j=0}^{\infty} \left(\frac{j}{n} - x\right)^{2q} p_{n,j}(x) \right) \le c(q)(1+x)^{2q}$$

for all $n \in N$, $x \in I$, where $c(q)$ is some positive constant depending only on q (cf. e.g. [7, Lemma 3.7]). Hence, one can apply Theorem 2.5 with the above values of $\sigma^2(x), \beta(x), |\Delta_n^{(\alpha)}(f;x)|, \psi(x)$, $n_0 = 1$, $\varphi(x) = \sqrt{c(q)}(1+x)^q$ and arbitrary positive number A (in paricular, A may be choosen x). Also, analogous results can be obtained for functions $f \in M_{\mathrm{loc}}(I)$ satisfying condition (2.4) with some exponential function ψ, e.g. $\psi(x) = \exp(\rho x)$, $\rho > 0$. In this case we have

$$\sum_{j=0}^{\infty} \psi^2\left(\frac{j}{n}\right) p_{n,j}(x) = \exp\left(nx(e^{2\rho/n}-1)\right) \leq \exp\left(\frac{2\rho x}{1-2\rho/n}\right),$$

i.e. Theorem 2.5 holds true with $\varphi(x) = \exp(2\rho x)$ and $n \geq n_0$ where $n_0 = [4\rho]+1$.

Finally, let us mention that corresponding results can be obtained for the Bézier type modification (1.2) of the discrete Baskakov operators (defined e.g. as in [7]).

References

[1] G. Anioł, *On the rate of convergence of some discrete operators*, Demonstratio Math. **27** (1994), 367–377.

[2] P. Bézier, *Numerical Control-Mathematics and Applications*, Wiley, London 1972.

[3] G. Chang, *Generalized Bernstein-Bézier polynomial*, J. Comput. Math. **1**, No. 4 (1983), 322–327.

[4] Z.A. Chanturija, *Modulus of variation of function and its application in the theory of Fourier series*, Dokl. Akad. Nauk SSSR **214** (1974), 63–66 (in Russian).

[5] W. Feller, *An Introduction to Probability Theory and its Applications*, Vol. II, Wiley, New York 1966.

[6] S. Guo and M.K. Khan, *On the rate of convergence of some operators on functions of bounded variation*, J. Approx. Theory **58** (1989), 90–101.

[7] M. Heilmann, *Direct and converse results for operators of Baskakov-Durrmeyer type*, Approx. Theory Appl. **5**, No. 1 (1989), 105–127.

[8] P. Pych-Taberska, *On the rate of pointwise convergence of Bernstein and Kantorovich polynomials*, Functiones et Approximatio, Comment. Math. **16** (1988), 63–76.

[9] P. Pych-Taberska, *On the rate of convergence of the Feller operators*, Ann. Soc. Math. Pol., Commentat. Math. **31** (1991), 147-156.

[10] X.M. Zeng and A. Piriou, *On the rate of convergence of two Bernstein-Bézier type operators for bounded variation functions*, J. Approx. Theory **95** (1998), 369–387.

Address: Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Matejki 48/49, 60-769 Poznań, Poland

E-mail: ppych@amu.edu.pl

Received: 6 Apr 2000