Functiones et Approximatio
XXVIII (2000), 173–186

In honorem W. Staś annos LXXV nati

A NOTE ON LANDAU'S FORMULA

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Abstract: In this paper we obtain a weighted form of Gonek's [5] uniform version of the classical Landau formula. The results are based on Kaczorowski-Perelli's [10] technique for the Riemann-von Mangoldt explicit formula. The main feature of our results is a more flexible remainder term, which in particular gives better mean-values estimates.
Keywords: Landau's formula, explicit formulae, distribution of primes.

1. Introduction

In 1911, Landau [11] proved that for any fixed $x > 1$

$$\sum_{0<\gamma\le T} x^{\rho} = -\frac{T}{2\pi}\Lambda(x) + O(\log T) \qquad \text{as } T \to \infty, \tag{1.1}$$

where ρ runs over the non-trivial zeros of the Riemann zeta function $\zeta(s)$ and

$$\Lambda(x) = \begin{cases} \log p & x = p^m,\ p \text{ prime} \\ 0 & \text{otherwise} \end{cases}$$

is the extended von Mangoldt function. Since the use of (1.1) is limited by its lack of uniformity in x, we are interested in a version of (1.1) that is uniform in both variables. The first result in this direction was obtained by Gonek [4], [5]. He proved that for $T > 1$

$$\begin{aligned}\sum_{0<\gamma\le T} x^{\rho} = -\frac{T}{2\pi}\Lambda(x) &+ O(x\log 2xT\log\log 3x)\\ &+ O\Big(\log x \min\Big(T; \frac{x}{<x>}\Big)\Big) + O\Big(\log 2T \min\Big(T; \frac{1}{\log x}\Big)\Big)\end{aligned} \tag{1.2}$$

2000 Mathematics Subject Classification: 11M36, 11N05.
* Partially supported by KBN Grant no 2 PO3A 024 17

where $< x >$ denotes the distance between x and the nearest prime power other that x itself.

We remark that (1.1) follows from (1.2) provided that x is fixed and $T \to \infty$. We also remark that if $x = n \in \mathbb{N}$ and $T \gg n$, then the last two error terms in (1.2) are absorbed in the first one and hence (1.2) becomes

$$\sum_{0<\gamma\leq T} n^{\rho} = -\frac{T}{2\pi}\Lambda(n) + O(n \log 2nT \log\log 3n). \tag{1.3}$$

This is due to the fact that the last two error terms in (1.2) exhibit the contribution of the spikes of the zeta-zeros sum when x is a real number near to a prime power. Observe that (1.2) and (1.3) are of interest when $T = \infty(x \log x \log\log x)$ and $T = \infty(n \log n \log\log n)$, respectively. Here $f(x) = \infty(g(x))$ means $g(x) = o(f(x))$. We further remark that Fujii [2], [3] gave a stronger form of (1.2) assuming the Riemann Hypothesis.

Deeper insight into the nature of the Landau formula and its connection with the classical Riemann-von Mangoldt explicit formula can be gained by the study of the function

$$k(z) = \sum_{\gamma>0} e^{\rho z},$$

where $\mathrm{Im}\, z > 0$ and the summation is taken over non-trivial zeros with positive imaginary parts. This function was considered by Cramér [1], Guinand [6] and Kaczorowski [8], who described its basic analytic properties such as analytic continuation, functional equations and boundary values on the real axis. For simplicity let us restrict our attention to the set $\Omega = \mathbb{C}\backslash(-\infty, 0]$. Of course $k(z)$ is holomorphic on the upper half-plane and one can prove that it has meromorphic continuation to Ω. The only singularities are simple poles at logarithms of prime powers $q = p^m$, with residues

$$\operatorname*{Res}_{z=\log q} k(z) = \frac{\Lambda(q)}{2\pi i}.$$

It is now clear that the poles of $k(z)$ are responsible for the term

$$-\frac{T}{2\pi}\Lambda(x)$$

in (1.1) and for the spikes of the corresponding zeta-zeros sum.

In order to explain connections with the classical explicit formula, let us consider the following function defined on the upper half-plane $\mathrm{Im}\, z > 0$

$$K(z) = \int_{z+i\infty}^{z} k(s)\, ds,$$

where the path of integration is the half-line $s = z + iy$, $\infty > y \geq 0$. Of course $K(z)$ is holomorphic for $\mathrm{Im}\, z > 0$, and for such z we have

$$K(z) = \sum_{\gamma>0} \frac{e^{\rho z}}{\rho}.$$

The integral representation of $K(z)$ immediately gives analytic continuation of $K(z)$ along every curve lying inside Ω and avoiding the poles of $k(z)$. Moreover, near $\log q$, where q is a fixed prime power, we have

$$K(z) = \frac{\Lambda(q)}{2\pi i}\log(z - \log q) + G_q(z),$$

where $G_q(z)$ is holomorphic at $\log q$. Consequently, the poles of $k(z)$ become logarithmic branch points of $K(z)$.

Let us fix a real number $0 < x_0 < \log 2$. For real $x > x_0$ which is not a logarithm of a prime power, let us consider the curve consisting of the half-line from $x_0 + i\infty$ to x_0 and of the segment $[x_0, x]$ of the real axis, with small detours around poles of $k(z)$ if $x > \log 2$. One can easily observe that near $\log q$ the real part of $K(z)$ decreases by $\frac{1}{2}\Lambda(n)$ and therefore we have

$$\operatorname{Re} K(x) = -\frac{1}{2}\psi(e^x) + g(x)$$

with a continuous function $g(x)$ and

$$\psi(x) = \sum_{n \le x} \Lambda(n).$$

This computation can be made precise (see [8], Th. 4.1) and the result is as follows. For $x > 0$ let

$$F(x) = 2 \lim_{y \to 0^+} \operatorname{Re} K(x + iy).$$

Then

$$F(x) = e^x - \psi_0(e^x) - \frac{1}{2}\log(1 - e^{-2x}) - \log 2\pi,$$

where

$$\psi_0(x) = \frac{1}{2}(F(x-0) + F(x+0)).$$

This is a form of the explicit formula. One can therefore say that Landau's formula is the "derivative" of the classical explicit formula.

Our aim here is to prove a weighted form of the Landau-Gonek formula, in which the error term has a more flexible shape and a better mean-square and individual bound. We will follow the technique that Kaczorowski-Perelli [10] applied to obtain similar results for the error term of a weighted form of the classical Riemann-von Mangoldt explicit formula. Writing

$$J(N,H) = \int_{N/2}^{3N} (\psi(x+H) - \psi(x) - H)^2 dx,$$

$$\tilde{J}(N,H) = \sum_{N/2 \le n \le 3N} (\psi(n+H) - \psi(n) - H)^2,$$

$$H(x,T,n) = \frac{2}{T}\int_{T/2}^{T}\left(\int_{-\tau}^{\tau}\left(\frac{x}{n}\right)^{iu} du\right)d\tau,$$

$$w(u) = \begin{cases} 1 & 0 \le u \le \frac{1}{2} \\ 2(1-u) & \frac{1}{2} \le u \le 1 \end{cases} \quad \text{and} \quad L = \log x,$$

our first result is

Theorem 1.1. *Let $x \geq 16$, $4 \leq T \leq x/4$ and $1 \leq M \leq T/4$. Then*

$$x - \sum_{|\gamma| \leq T} w\Big(\frac{|\gamma|}{T}\Big) x^{\rho} = R(x,T), \tag{1.4}$$

where $\rho = \beta + i\gamma$ runs over the non-trivial zeros of $\zeta(s)$,

$$R(x,T) = R_1(x,T) + R_2(x,T) + R_3(x,T) \tag{1.5}$$

and

$$R_1(x,T) = \frac{1}{2\pi} \sum_{x - Mx/T < n \leq x + Mx/T} \Lambda(n) H(x,T,n), \tag{1.6}$$

$$R_2(x,T) \ll \frac{MxL}{T\log(Mx/T)} + x^{(\sigma+3)/4} L^4 + \frac{x}{T^{\alpha} L} + \frac{xL^3}{T^{1-\alpha}} N(\sigma,T) + ML + \frac{xL}{T}$$

for every $\alpha \in (0,1]$ and $\sigma \in [\frac{1}{2}, 1)$, and

$$R_3(x,T) \ll \frac{1}{T} \sum_{n \in (x/2, x - Mx/T] \cup (x + Mx/T, 2x)} \Lambda(n) \Big(\log \frac{x}{n}\Big)^{-2} \ll \frac{xL}{M\log(x/T)}.$$

We remark that the properties of the function $H(x,T,n)$ are similar to those of the function $G(x,T,n)$ in [10], see Lemmas 2.1 and 2.2 in section 2. Moreover, there is a relation between $H(x,T,n)$ and $G(x,T,n)$. In fact, following the proof of Lemma 4 of [10] we see that

$$\frac{1}{\pi}\,\mathrm{sgn}(x-n) G(x,T,n) = \frac{2}{T} \int_{T/2}^{T} \Big(\frac{1}{2\pi i} \int_{|t| \geq \tau} \Big(\frac{x}{n}\Big)^{it} \frac{dt}{t} \Big) d\tau$$

and hence, by straightforward computations, for $|n - x| \leq Mx/T$ we obtain

$$\int_0^x H(u,T,n) \frac{du}{u} = -\frac{1}{\pi}\,\mathrm{sgn}(x-n) G(x,T,n) + O(1).$$

We can write (1.4) in a form which is more similar to (1.2). Let $\{x\}$ the fractional part of x.

Corollary 1.2. *Let $x \geq 16$, $4 \leq T \leq x/4$ and $1 \leq M \leq T/4$. Then*

$$\sum_{|\gamma| \leq T} w\Big(\frac{|\gamma|}{T}\Big) x^{\rho} = L(x,T),$$

where

$$L(x,T) = L_1(x,T) + L_2(x,T) + L_3(x,T)$$

with

$$L_1(x,T) = -\frac{1}{2\pi} \sum_{x-Mx/T<n\le x+Mx/T} (\Lambda(n)-1)H(x,T,n),$$

$$\begin{aligned} L_2(x,T) &= -R_2(x,T) + O\Big(\frac{x}{M} + T + \frac{xM}{T}\log\frac{T}{M}\Big) \\ &\quad + O\Big(\{x\}\log\frac{x}{\{x\}} + (1-\{x\})\log\frac{x}{(1-\{x\})} \\ &\qquad + T\log\frac{x}{T\{x\}} + T\log\frac{x}{T(1-\{x\})}\Big) \end{aligned}$$

for $x \in \mathbb{R}\setminus\mathbb{N}$,

$$L_2(x,T) = -R_2(x,T) + O\Big(\frac{x}{M} + L + \frac{xM}{T}\log\frac{T}{M} + T\log\frac{x}{T}\Big) \quad \text{for } x\in\mathbb{N}$$

and

$$L_3(x,T) = -R_3(x,T).$$

We remark that the main term $L_1(x,T)$ absorbs the term $-T/2\pi\Lambda(x)$ and depends, in fact, on the values of $(\Lambda(n)-1)H(x,T,n)$ for n in the whole interval $(x-Mx/T, x+Mx/T]$. In other words, $L_1(x,T)$ contains the contribution of the above mentioned spikes.

We have the following explicit bounds for $R(x,T)$ and $L(x,T)$.

Corollary 1.3. *Let* $x\ge 16$ *and* $4\le T\le x/4$. *Then*

$$R(x,T) \ll \frac{xL}{\log(x/T)} \quad \text{and} \quad L(x,T) \ll \frac{xL}{\log(x/T)}. \tag{1.7}$$

Observe that Corollary 1.3 sharpens the error terms in (1.2). Our final result provides a mean-square bound for $L(x,T)$ and $R(x,T)$.

Theorem 1.4. *Let* $16\le N\le x\le 2N$, $4\le T\le N/4$ *and* $1\le M\le \min(N^{1/16}/L^4; T^{1/5}$ *Then*

$$\int_N^{2N} |L(x,T)|^2 dx \ll N^3 + (TM)^2 J\Big(N, \frac{4N}{TM}\Big) + \frac{T^2}{M^2} J\Big(N,\frac{4N}{T}\Big) \tag{1.8}$$

and

$$\begin{aligned} \sum_{N\le n\le 2N} |L(n,T)|^2 &\ll N^3 + (TM)^2 J\Big(N,\frac{4N}{TM}\Big) + \Big(\frac{T^4}{N^2}+\frac{T^2}{M^4}\Big) J\Big(N,\frac{4MN}{T}\Big) \\ &\quad + \frac{T^2}{M^2}\Big(J\Big(N,\frac{4N}{T}\Big) + \tilde{J}\Big(N,\frac{4N}{T}\Big)\Big) + NT^2M^2. \end{aligned}$$

Moreover, the same bounds hold for $R(x,T)$ and $R(n,T)$ as well.

Since for $1 \le H \le N$ we have

$$J(N,H),\ \tilde{J}(N,H) \ll H^2N + HNL,$$

see Languasco [12], choosing $M = 1$ in Theorem 1.4 we obtain

$$\int_N^{2N} |L(x,T)|^2 dx \ll N^2 \max(N, TL) \qquad \text{uniformly for } T \le \frac{N}{4}$$

and

$$\sum_{N \le n \le 2N} |L(n,T)|^2 \ll N^2 \max(N, TL) \qquad \text{uniformly for } T \le \frac{N}{4},$$

and similarly for $R(x,T)$ and $R(n,T)$.

2. Proof of Theorem 1.1 and Corollaries

We start with several lemmas.

Lemma 2.1. *Let $x \ge 16$, $4 \le T \le x/4$, $|x-n| = o(x)$ and $h = o(x)$. Then*

$$H(x,T,n) \ll \min\left(T; \frac{1}{T}\left(\frac{x}{|x-n|}\right)^2\right) \tag{2.1}$$

and

$$H(x,T,n+h) - H(x,T,n) \ll \frac{T^2h}{N} \min\left(1; \frac{x}{T|x-n|}\right).$$

Proof. By straightforward computations, for $x \ne n$ we have

$$H(x,t,n) = \frac{4}{T\log(x/n)^2} \int_{T/2|\log(x/n)|}^{T|\log(x/n)|} \sin u \, du. \tag{2.2}$$

Hence the first upper bound in (2.1) follows using $\sin u \le u$ in (2.2). The second upper bound in (2.1) follows computing the integral in the right hand side of (2.2), thus obtaining

$$\begin{aligned} H(x,t,n) &= \frac{4}{T\log(x/n)^2}\left(\cos\left(\frac{T}{2}\left|\log\left(\frac{x}{n}\right)\right|\right) - \cos\left(T\left|\log\left(\frac{x}{n}\right)\right|\right)\right) \\ &\ll \frac{1}{T\log(x/n)^2}. \end{aligned} \tag{2.3}$$

Hence (2.1) follows using $\log(x/n)^{-2} \le (x/|x-n|)^2$ in (2.3) and observing that $H(n,T,n) = \frac{3}{2}T$.

The second part of Lemma 2.1 follows using Lagrange theorem, partial integration and (2.1).

Lemma 2.2. *Let* $x \geq 16$, $4 \leq T \leq x/4$ *and* $1 \leq M \leq T/4$. *Then, for* $x \in \mathbb{R} \setminus \mathbb{N}$, *we have*

$$\begin{aligned} \frac{1}{2\pi} & \sum_{x-Mx/T<n\leq x+Mx/T} H(x,T,n) \\ & = x + O\left(\frac{x}{M} + T + \frac{xM}{T}\log\frac{T}{M}\right) + O\left(\{x\}\log\frac{x}{\{x\}} + (1-\{x\})\log\frac{x}{(1-\{x\})}\right. \\ & \qquad \left. + T\log\frac{x}{T\{x\}} + T\log\frac{x}{T(1-\{x\})}\right). \end{aligned} \tag{2.4}$$

Moreover, for $x = m \in \mathbb{N}$, *we get*

$$\begin{aligned} \frac{1}{2\pi} & \sum_{m-Mm/T<n\leq m+Mm/T} H(m,T,n) \\ & \qquad = m + O\left(\frac{m}{M} + \log m + \frac{mM}{T}\log\frac{T}{M} + T\log\frac{m}{T}\right) \end{aligned} \tag{2.5}$$

and, if we further assume that $\infty(M) \leq T \leq \mathrm{o}(m)$ *and* $\infty(1) \leq M \leq T/4$ *as* $m \to \infty$,

$$\frac{1}{2\pi} \sum_{m-Mm/T<n\leq m+Mm/T} H(m,T,n) = m(1+\mathrm{o}(1)) \qquad \text{as } m \to \infty.$$

Proof. Let $x \in \mathbb{R} \setminus \mathbb{N}$. By (2.2) we have

$$\begin{aligned} \frac{1}{2\pi} & \sum_{x-\frac{Mx}{T}<n\leq x+\frac{Mx}{T}} H(x,T,n) \\ & = \frac{2}{\pi T} \sum_{x-Mx/T<n\leq x+Mx/T} \log\left(\frac{x}{n}\right)^{-2} \int_{T/2|\log(x/n)|}^{T|\log(x/n)|} \sin u \, du = I, \end{aligned} \tag{2.6}$$

say. Interchanging summation and integration in (2.6) we obtain, writing $[x]$ to denote the integer part of x, that

$$\begin{aligned} I = \frac{2}{\pi T} & \left(\int_{T|\log(([x]+1)/x)|}^{T|\log(1+M/T)|} \left(\sum_{\substack{[x]+1\leq n\leq x+Mx/T \\ u/T\leq|\log(x/n)|\leq 2u/T}} \log\left(\frac{x}{n}\right)^{-2} \right) \sin u \, du \right. \\ & \left. + \int_{T|\log([x]/x)|}^{T|\log(1-M/T)|} \left(\sum_{\substack{x-Mx/T\leq n\leq [x] \\ u/T\leq|\log(x/n)|\leq 2u/T}} \log\left(\frac{x}{n}\right)^{-2} \right) \sin u \, du \right) \\ = I_1 + I_2, & \end{aligned} \tag{2.7}$$

say.

Since the two conditions in the summations are equivalent, by partial summation we get

$$
\begin{aligned}
I_1 &= \frac{2}{\pi T}\int_{T|\log(([x]+1)/x)|}^{T|\log(1+M/T)|}\frac{Tx}{2u}\sin u\, du + O\Big(\frac{x}{T}\int_{T|\log(([x]+1)/x)|}^{T|\log(1+M/T)|}\Big|\log\frac{u}{T}\Big|du\Big)\\
&\quad + O\Big(T\int_{T|\log(([x]+1)/x)|}^{1}\frac{|\sin u|}{u^2}du\Big) + O(T)\\
&= \frac{x}{\pi}\int_0^\infty\frac{\sin u}{u}du + O\Big(\frac{x}{M}+T\Big)\\
&\quad + O\Big(\frac{xM}{T}\log\frac{T}{M} + (1-\{x\})\log\frac{x}{(1-\{x\})} + T\log\frac{x}{T(1-\{x\})}\Big).
\end{aligned}
\tag{2.8}
$$

Analogously we can prove

$$
\begin{aligned}
I_2 &= \frac{x}{\pi}\int_0^\infty\frac{\sin u}{u}du\\
&\quad + O\Big(\frac{x}{M} + T + \frac{xM}{T}\log\frac{T}{M} + \{x\}\log\frac{x}{\{x\}} + T\log\frac{x}{T\{x\}}\Big).
\end{aligned}
\tag{2.9}
$$

Hence, recalling $\int_0^\infty \sin u/u du = \pi/2$, we have, by (2.7)-(2.9), that

$$
\begin{aligned}
I &= x + O\Big(\frac{x}{M} + T + \frac{xM}{T}\log\frac{T}{M}\Big)\\
&\quad + O\Big(\{x\}\log\frac{x}{\{x\}} + (1-\{x\})\log\frac{x}{(1-\{x\})} + T\log\frac{x}{T\{x\}}\\
&\qquad + T\log\frac{x}{T(1-\{x\})}\Big)
\end{aligned}
\tag{2.10}
$$

and so, by (2.6) and (2.10), we obtain (2.4).

If $x \in \mathbb{N}$, using $H(x,T,x) = \frac{3}{2}T$ we get

$$
\begin{aligned}
&\frac{1}{2\pi}\sum_{x-Mx/T<n\le x+Mx/T} H(x,T,n)\\
&\qquad = \frac{3}{4\pi}T + \sum_{n\in(x-Mx/T,x-1]\cup[x+1,x+Mx/T]} H(x,T,n).
\end{aligned}
\tag{2.11}
$$

To obtain (2.5) we evaluate the second term in (2.11) using a simplified version of the previous argument. The asymptotic formula stated in the last part of Lemma 2.2 is an immediate consequence of (2.5).

We need the following three lemmas, whose proofs follow by the argument in [10] and are therefore omitted.

Lemma 2.3. *Let $x \geq 16$, $4 \leq T \leq x/4$ and $1 \leq M \leq T/4$. Let further $c = 1 + 1/\log x$, $s \in \mathbb{C}$,*

$$U(s) = \sum_{x - Mx/T < n \leq x + Mx/T} \Lambda(n) n^{-s} \quad \text{and} \quad f(s) = -\frac{\zeta'}{\zeta}(s) - U(s).$$

Then, for every $\tau \in [T/2, T]$, we have

$$\frac{1}{2\pi i} \int_{c-i\tau}^{c+i\tau} f(s) x^s ds = g_1(x, \tau)$$

and

$$\frac{2}{T} \int_{T/2}^{T} g_1(x, \tau) d\tau \ll R_3(x, T) + \frac{x}{T} \log x,$$

where

$$R_3(x, T) \ll \frac{1}{T} \sum_{n \in (x/2, x - Mx/T] \cup (x + Mx/T, 2x)} \Lambda(n) \left(\log \frac{x}{n} \right)^{-2} \ll \frac{x}{M} \frac{L}{\log x/T}.$$

Moreover,

$$\int_{N}^{2N} |R_3(x, T)|^2 dx \ll \frac{T^2}{M^2} J\left(N, \frac{4N}{T}\right) + \frac{N^3}{M^2} \qquad \text{as } N \to \infty$$

and

$$\sum_{n=N}^{2N} |R_3(n, T)|^2 \ll \frac{T^2}{M^2} \tilde{J}\left(N, \frac{4N}{T}\right) + \frac{N^3}{M^2} \qquad \text{as } N \to \infty.$$

Lemma 2.4. *Let $x \geq 16$ and $4 \leq T \leq x/4$. Let further $s \in \mathbb{C}$, $\alpha \in (0, 1]$, $\sigma \in [\frac{1}{2}, 1)$ and*

$$h(x, \tau) = -\frac{1}{2\pi i} \int_{c-i\tau}^{c+i\tau} \frac{\zeta'}{\zeta}(s) x^s ds.$$

Then for every $\tau \in [T/2, T]$ we have

$$h(x, \tau) = x - \sum_{|\gamma| \leq \tau} x^{\rho} + g_2(x, \tau)$$

and

$$\frac{2}{T} \int_{T/2}^{T} g_2(x, \tau) d\tau \ll x^{(\sigma+3)/4} L^4 + \frac{x}{T^{\alpha} L} + \frac{x L^3}{T^{1-\alpha}} N(\sigma, T).$$

Lemma 2.5. *Assume the same hypotheses of Lemma 2.3. For $\tau \in [T/2, T]$, let further*

$$k(x, \tau) = \frac{1}{2\pi i} \int_{c-i\tau}^{c+i\tau} U(s) x^s ds.$$

Then

$$\frac{2}{T}\int_{T/2}^{T} k(x,\tau)d\tau$$
$$= \frac{1}{2\pi}\sum_{x-Mx/T<n\le x+Mx/T} \Lambda(n)H(x,T,n) + O\Big(\frac{MxL}{T\log(Mx/T)} + ML\Big).$$

Now we are ready for the proof of Theorem 1.1. For $\tau \in [T/2,T]$, from Lemmas 2.3, 2.4 and 2.5 we get

$$x - \sum_{|\gamma|\le\tau} x^{\rho} + g_2(x,\tau) = k(x,\tau) + g_1(x,\tau), \tag{2.12}$$

and hence Theorem 1.1 follows integrating (2.12) over $\tau \in [T/2,T]$.

Corollary 1.2 follows easily from Theorem 1.1 and Lemma 2.2.

Finally, choosing $M = 1$, $\sigma = \frac{3}{4}$ and $\alpha = \frac{1}{5}$ in Theorem 1.1, using Lemma 2.1 and Ingham-Huxley's density estimate (see chapter 11 of Ivić [7])

$$N(\sigma,T) \ll \begin{cases} T^{3(1-\sigma)/(2-\sigma)}\log^5 T & \frac{1}{2}\le\sigma\le\frac{3}{4} \\ T^{3(1-\sigma)/(3\sigma-1)}\log^{44} T & \frac{3}{4}\le\sigma\le 1, \end{cases}$$

we obtain

$$R_1(x,T) \ll x, \quad R_2(x,T) \ll x \quad \text{and} \quad R_3(x,T) \ll \frac{xL}{\log(x/T)}$$

and hence we have the first part of (1.7). The second part of (1.7) follows easily using $L(x,T) = x - R(x,T)$, and Corollary 1.3 follows.

3. Proof of Theorem 1.4

By (1.4) we get

$$\int_N^{2N}\Big|\sum_{|\gamma|\le T} w\Big(\frac{|\gamma|}{T}\Big)x^{\rho}\Big|^2 dx = \int_N^{2N} |x - R(x,T)|^2\, dx.$$

Choosing $\sigma = \frac{3}{4}$ and $\alpha = \frac{1}{5}$ in Theorem 1.1 and using Ingham-Huxley's density estimate we get

$$R_2(x,T) \ll \frac{N}{M}. \tag{3.1}$$

Hence, by (1.5), (3.1), Lemma 2.3 and the inequality $(a+b)^2 \le 2a^2+2b^2$ we have

$$\int_N^{2N} |x - R(x,T)|^2 dx \ll \int_N^{2N} |R_1(x,T)|^2 dx + N^3 + \frac{T^2}{M^2}J\Big(N,\frac{4N}{T}\Big). \tag{3.2}$$

To estimate the contribution of $R_1(x,T)$ we follow the argument in Kaczorowski-Perelli [9]. We subdivide the interval $(x-Mx/T, x+Mx/T]$ into $P \ll M^2$ subintervals of the form

$$I_j(x) = (n_j, n_j + K(x)], \quad K(x) = \frac{x}{TM}, \quad n_j = x \pm jK(x), \quad j = 1, \dots, P.$$

Hence

$$R_1(x,T) \ll \sum\nolimits_1 + \sum\nolimits_2, \tag{3.3}$$

where

$$\sum\nolimits_1 = \sum_{j=1}^{P} |H(x,T,n_j)| |\sum_{n \in I_j(x)} \Lambda(n)|$$

and

$$\sum\nolimits_2 = \sum_{j=1}^{P} \sum_{n \in I_j(x)} \Lambda(n) |H(x,T,n) - H(x,T,n_j)|. \tag{3.4}$$

By Lemma 2.1 we get

$$H(x,T,n_j) \ll \begin{cases} T & 1 \le j \le M \\ T(M/j)^2 & M \le j \le P, \end{cases}$$

$$H(x,T,n) - H(x,T,n_j) \ll \begin{cases} T/M & 1 \le j \le M \\ T/j & M \le j \le P, \end{cases} \tag{3.5}$$

for $n \in I_j(x)$.

Now we proceed to estimate the mean-square of $\sum_2$. By (3.4) and (3.5) we get

$$\int_N^{2N} |\sum\nolimits_2|^2 dx \ll J_1 + J_2, \tag{3.6}$$

where

$$J_1 = \frac{T^2}{M^2} \int_N^{2N} \left| \sum_{j=1}^{M} \sum_{n \in I_j(2N)} \Lambda(n) \right|^2 dx$$

and

$$J_2 = T^2 \int_N^{2N} \left| \sum_{j=M}^{P} j^{-1} \sum_{n \in I_j(2N)} \Lambda(n) \right|^2 dx.$$

Using the Cauchy-Schwarz inequality, the substitution $y = x \pm jK(2N)$ and the fact that $[N \pm jK(2N), 2N \pm jK(2N)] \subset [N/2, 3N]$, we have

$$J_1 \ll T^2 \int_{N/2}^{3N} \left| \sum_{n=y}^{y+K(2N)} \Lambda(n) \right|^2 dy$$

and

$$J_2 \ll T^2 \log^2(M+1) \int_{N/2}^{3N} \Big| \sum_{n=y}^{y+K(2N)} \Lambda(n) \Big|^2 dy.$$

Hence, using the inequality $(a+b)^2 \le 2a^2 + 2b^2$, we obtain

$$J_1 \ll T^2 J\Big(N, \frac{4N}{TM}\Big) + \frac{N^3}{M^2},$$
$$J_2 \ll \Big(T^2 J\Big(N, \frac{4N}{TM}\Big) + \frac{N^3}{M^2}\Big) \log^2(M+1),$$

and, by (3.6), we finally get

$$\int_N^{2N} \Big| \sum\nolimits_2 \Big|^2 dx \ll \Big(T^2 J\Big(N, \frac{4N}{TM}\Big) + \frac{N^3}{M^2}\Big) \log^2(M+1). \tag{3.7}$$

The estimation of the mean-square of $\sum_1$ can be performed analogously, thus obtaining

$$\int_N^{2N} \Big| \sum\nolimits_1 \Big|^2 dx \ll T^2 M^2 J\Big(N, \frac{4N}{TM}\Big) + N^3. \tag{3.8}$$

Hence (1.8) follows from (3.2), (3.3), (3.7) and (3.8).

Now we prove the second part of Theorem 1.4. We have, by (1.6) and (3.1), that

$$\begin{aligned}
&\sum_{|\gamma| \le T} w\Big(\frac{|\gamma|}{T}\Big)[x]^\rho - \sum_{|\gamma| \le T} w\Big(\frac{|\gamma|}{T}\Big)x^\rho \\
&\qquad = R(x,T) - R([x],T) + O(1) \\
&\qquad = \frac{1}{2\pi} \sum_{0 < |n-x| \le M[x]/T} \Lambda(n)(H(x,T,n) - H([x],T,n)) \\
&\qquad\quad + \frac{1}{2\pi} \sum_{M[x]/T < |n-x| \le Mx/T} \Lambda(n) H(x,T,n) \\
&\qquad\quad + R_3(x,T) - R_3([x],T) + O\Big(\frac{N}{M}\Big).
\end{aligned} \tag{3.9}$$

Clearly

$$\begin{aligned}
&\sum_{0 < |n-x| \le M[x]/T} \Lambda(n)(H(x,T,n) - H([x],T,n)) \\
&\qquad \ll \frac{1}{T} \sum_{|n-x| \le 2MN/T} \Lambda(n) \Big| \int_{T/2}^{T} \int_{-\tau}^{\tau} \Big(\frac{[x]}{n}\Big)^{iu} \Big[\Big(\frac{x}{[x]}\Big)^{iu} - 1\Big] du\, d\tau \Big| \\
&\qquad \ll \frac{T^2}{N} \sum_{|n-x| \le 2MN/T} \Lambda(n).
\end{aligned} \tag{3.10}$$

Moreover, by (2.1), we also obtain

$$\sum_{M[x]/T<|n-x|\le Mx/T} \Lambda(n)H(x,T,n) \ll \frac{T}{M^2} \sum_{|n-x|\le 2MN/T} \Lambda(n). \tag{3.11}$$

Hence, by (3.9)-(3.11), we have

$$\begin{aligned}\sum_{|\gamma|\le T} w\left(\frac{|\gamma|}{T}\right)[x]^\rho - \sum_{|\gamma|\le T} w\left(\frac{|\gamma|}{T}\right)x^\rho \\ \ll \left(\frac{T^2}{N}+\frac{T}{M^2}\right) \sum_{|n-x|\le 2MN/T} \Lambda(n) + R_3(x,T) + R_3([x],T) + \frac{N}{M}.\end{aligned} \tag{3.12}$$

So, for any $n \in [N, 2N]$, we obtain, by (3.12), that

$$\begin{aligned}\left|\sum_{|\gamma|\le T} w\left(\frac{|\gamma|}{T}\right)n^\rho\right|^2 \ll \int_{n-1}^{n} \left|\sum_{|\gamma|\le T} w\left(\frac{|\gamma|}{T}\right)x^\rho\right|^2 dx \\ + \left(\frac{T^4}{N^2}+\frac{T^2}{M^4}\right)\int_{n-1}^{n}\left|\sum_{|m-x|\le 2MN/T}\Lambda(m)\right|^2 dx \\ + \int_{n-1}^{n} |R_3(x,T)|^2 dx + |R_3(n,T)|^2 + \frac{N^2}{M^2}.\end{aligned}$$

The second part of Theorem 1.4 now follows summing over n, using $(a+b)^2 \le 2a^2+2b^2$, Lemma 2.3 and (1.8).

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Received: 14 May 2000