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To Professor Włodzimierz Staś on his 75th birthday

THE MULTIDIMENSIONAL DIRICHLET DIVISOR PROBLEM AND ZERO FREE REGIONS FOR THE RIEMANN ZETA FUNCTION

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Abstract: We show a connection between the multidimensional Dirichlet divisor problem and the zero free region for the Riemann zeta function.
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Let $\tau_k(n)$ denote the number of positive integer solutions of the equation $n_1 n_2 \dots n_k = n$, $k \geq 1$. Let us define the function $R_k(x)$, $x > 1$, by the equality

$$R_k(x) = \sum_{1<n\leq x} \tau_k(n) - xP_{k-1}(\log x) ,$$

where

$$xP_{k-1}(\log x) = \operatorname*{Res}_{s=1} \left(\zeta^k(s) \frac{x^s}{s} \right) .$$

and $\zeta(s)$ is the Riemann zeta - function. L. Dirichlet proved in 1848 that $R_k(x) = O(x^{1-1/k} \log^{k-2} x)$.

In [4], on the basis of the method of trigonometric sums of I. M. Vinogradov (see [13], [14]), the estimate

$$|R_k(x)| \leq x^{1-\alpha(k)} (c_1 \log x)^k , \tag{1}$$

$$\alpha_k = ck^{-\frac{2}{3}} \tag{2}$$

with absolute positive constants c and c_1 was obtained.

Let us notice that the first result here is due to H. Richert [11] (after classical works by Dirichlet-Voronoi-Hardy-Littewood-Landau), who proved the inequality:

$$|R_k(x)| \leq x^{1-\alpha(k)+\varepsilon} , \qquad x \geq x_1(\varepsilon) > 0 , \tag{3}$$

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where ε is on arbitrary small fixed positive number. Afterwards this result was repeted by the author [5]. I was informed kindly about the paper [11] of H. Richert by Professor A. Ivić. The subsequent research on this theme—in particular computing the constant c from (2)—followed the scheme of [4] and [5] (cf. [6], [1], [2], [3], [10]). The possibility of obtaining estimate the type (1) or (3) was stated also in [15] (cf. [7], pp. 127-130).

The uniform estimates of the type (1) make it possible to obtain results about a boundary for the zeros of the Riemann zeta-function. Let us note that the estimate (3) and even the Lindelöf hypothesis cannot be successfully applied in order to obtain any bound for the zeros of the Riemann zeta-function.

The aim of the paper is to establish a connection between the estimates of the type (1) and the problem to give a boundary for the zeros of the Riemann zeta-function and to estimate zeta-sums as well. Results of this type were obtained by the author in [8], p. 112, Problem 1.

In this paper the standard notation will be used; in particular:

— $s = \sigma + it$, $i^2 = -1$, where σ and t are real numbers,
— $\Gamma(s)$ is the Euler gamma-function,
— $c, c_1, c_2, \dots$ are absolute positive constants which may differ in the different statements,
— constants implied by the O-symbols are absolute,
— $P_{k-1}(x)$ denotes a polynomial of x of the degree $\leq k-1$,
— $[x]$ = integral part of x,
— $\{x\}$ = fractional part of x.

The following lemma is basic for all the paper.

Lemma. *Let $\alpha(y)$ be an arbitrary real function of the real variable y, $y \geq 2$, such that $y^{-1} \leq \alpha(y) \leq \frac{1}{2}$. Let $c \geq 2$ and k be a natural number ≥ 2. Suppose that for all $x \geq 2$ the estimate*

$$|R_k(x)| \leq x^{1-\alpha(k)}(c \log x)^k \tag{4}$$

holds. Then for all $t \geq 2$ and $\frac{3}{2} \geq \sigma > 1 - \alpha(k)$ the following inequality holds:

$$|\zeta(\sigma + it)| < 8ckt^{1/k}(\sigma + \alpha(k) - 1)^{-1-1/k}\,. \tag{5}$$

Proof. For $\operatorname{Re} s > 1$ we have

$$\zeta^k(s) = \sum_{n=1}^{\infty} \tau_k(n) n^{-s} = \lim_{N \to +\infty} \left(1 + \sum_{1<n\leq N} \tau_k(n) n^{-s}\right). \tag{6}$$

Using partial summation we find that

$$S_N = \sum_{1<n\leq N} \tau_k(n) n^{-s} = s \int_1^N \mathbb{C}_k(u) u^{-s-1} du + \mathbb{C}_k(N) N^{-s}\,, \tag{7}$$

where

$$\mathbb{C}_k(u) = \sum_{1<n\le u} \tau_k(n) = uP_{k-1}(\log u) + R_k(u) \ . \tag{8}$$

From (7) and (8) it follows that

$$S_N = s\int_1^N u^{-s}P_{k-1}(\log u)du + s\int_1^N R_k(u)u^{-s-1}du + \mathbb{C}_k(N)N^{-s} \ . \tag{9}$$

The polynomial $P_{k-1}(\log u)$ is of the form

$$P_{k-1}(\log u) = \sum_{j=0}^{k-1} b_j(\log u)^j = \frac{1}{u}\operatorname*{Res}_{s=1}\left(\zeta^k(s)\frac{u^s}{s}\right) \ .$$

The following estimates and transformations are obvious:

$$\begin{aligned}
\int_1^N u^{-s}\log^j u du &= \int_0^{\log N} e^{-v(s-1)}v^j dv \\
&= \int_0^{\infty} e^{-v(s-1)}v^j dv + O(N^{-\sigma+1}\log^j N) \\
&= (s-1)^{-j-1}\int_0^{\infty} e^{-v}v^j dv + O(N^{-\sigma+1}\log^j N) \\
&= \Gamma(j+1)(s-1)^{-j-1} + O(N^{-\sigma+1}\log^j N) \\
&= j!(s-1)^{-j-1} + O(N^{-\sigma+1}\log^j N) \ ,
\end{aligned}$$

$$\int_1^N u^{-s}P_{k-1}(\log u)du = \sum_{j=0}^{k-1} j!b_j(s-1)^{-j-1} + O\left(N^{-\sigma+1}\sum_{j=0}^{k-1}|b_j|\log^j N\right) \ ,$$

$$\begin{aligned}
S_N &= s\sum_{j=0}^{k-1} j!b_j(s-1)^{-j-1} + s\int_1^N R_k(u)u^{-s-1}du \\
&\quad + O\left(N^{-\sigma+1}\sum_{j=0}^{k-1}|b_j|\log^j N\right) + \mathbb{C}_k(N)\cdot N^{-s} \ .
\end{aligned} \tag{10}$$

Since $\sigma > 1$ and $\mathbb{C}_k(N) = O(N\log^k N)$, we can take the limit in (10) as $N \to +\infty$ and get the new formula instead of (6):

$$\zeta^k(s) = 1 + s\sum_{j=0}^{k-1} j!b_j(s-1)^{-j-1} + s\int_1^{\infty} R_k(u)u^{-s-1}du \ . \tag{11}$$

By (4), the last improper integral converges for $\sigma = \operatorname{Re} s > 1-\alpha(k)$, i.e. (11) holds for $\operatorname{Re} s > 1-\alpha(k)$ by the principle of analytic continuation. Let us estimate the right hand side of (11) for $t \geq 2$ and $\sigma > 1 - \alpha(k)$. Estimating it and using (4) we obtain:

$$|\zeta(s)|^k \leq 1 + |s| \sum_{j=0}^{k-1} j!|b_j|t^{-j-1} + |s| \int_1^\infty u^{-\sigma-\alpha(k)}(c\log u)^k du \\ + |s| \int_1^2 |R_k(u)|u^{-\sigma-1} du \ . \tag{12}$$

Let us evaluate

$$J = \int_1^\infty u^{-\sigma-\alpha(k)}(c\log u)^k du \ .$$

Putting $\log u = v$ we successively obtain:

$$J = c^k \int_0^\infty e^{(-\sigma-\alpha(k))v+v} v^k dv = c^k(\sigma + \alpha(k) - 1)^{-k-1} \int_0^\infty e^{-w} w^k dw \\ = c^k k!(\sigma + \alpha(k) - 1)^{-k-1} \ .$$

Next, since $\mathbb{C}_k(u) = 0$ for $1 < u < 2$, we obtain for $1 < u < 2$:

$$R_k(u) = -\sum_{j=0}^{k-1} b_j(\log u)^j$$

and

$$\int_1^2 |R_k(u)|u^{-\sigma-1} du \leq \sum_{j=0}^{k-1} |b_j| \int_1^2 u^{-\sigma-1} \log^j u du \\ < \sum_{j=0}^{k-1} |b_j|(j+1)^{-1} \log 2 \ . \tag{14}$$

Let us estimate $|b_j|$, $j = 0, 1, \ldots, k-1$, from above. From (11) and the Cauchy residue theorem it follows that

$$j!b_j = \frac{1}{2\pi i} \int_{|s-1|=\frac{1}{2}} \zeta^k(s)(s-1)^j \frac{ds}{s} \ . \tag{15}$$

Let us use the fact that for $\mathrm{Re}\, s > 0$ we have

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + s \int_1^\infty \varrho(u) u^{-s-1} du \ ,$$

where

$$\varrho(u) = \frac{1}{2} - \{u\} \ .$$

In the formula (15) we have $s = 1 + \frac{1}{2}e^{i\varphi}$, $0 \le \varphi < 2\pi$, so $\mathrm{Re}\, s \ge \frac{1}{2}$, and $\frac{1}{2} \le |s| \le \frac{3}{2}$. Consequently,

$$|\zeta(s)| \le 2 + \frac{1}{2} + \frac{3}{4}\int_1^\infty u^{-\frac{3}{2}} du = 4$$

and

$$j!\,|b_j| \le 4^k 2^{-j} \ . \tag{16}$$

From (12)–(16), for $s = \sigma + it$, $\frac{3}{2} \ge \sigma > 1 - \alpha(k)$, $t \ge 2$ we successively obtain:

$$\begin{aligned} |\zeta(s)|^k &\le 1 + \sqrt{t^2+4}\sum_{j=0}^{k-1} 4^k \cdot 2^{-j} \cdot t^{-j-1} \\ &+ \sqrt{t^2+4} \cdot c^k \cdot k! \cdot (\sigma + \alpha(k) - 1)^{-k-1} \\ &+ \sqrt{t^2+4} \cdot \sum_{j=0}^{k-1} 4^k \cdot 2^{-j} \cdot (j!)^{-1} \cdot \log 2 \\ &< (8ck)^k \cdot t \cdot (\sigma + \alpha(k) - 1)^{-k-1} \ , \\ |\zeta(s)| &< 8ck \cdot t^{1/k} (\sigma + \alpha(k) - 1)^{-1-1/k} \ . \end{aligned}$$

The lemma is proved. ■

Thorem 1. *Let $\alpha(y)$ denote a nonincreasing function of y, $y \ge 2$. Suppose that for all $k \ge 2$ condition of the lemma are fulfilled.*
Then in the region

$$\sigma \ge 1 - 0.5\alpha(\log t) \ , \qquad t \ge e^2 \ ,$$

the following estimate holds:

$$|\zeta(\sigma + it)| \le 16 e^3 c \log^2 t \ . \tag{17}$$

Proof. Put in the Lemma $k = [\log t]$ and

$$t \ge e^2 \ , \qquad \sigma \ge 1 - 0.5\alpha(k) \ . \tag{18}$$

Then we have the inequality:

$$\sigma + \alpha(k) - 1 \ge 0.5\alpha(k) \ge 0.5k^{-1} \ge 0.5(\log t)^{-1} \ .$$

Hence, from (5) we find that

$$|\zeta(\sigma + it)| < 8c\log t \cdot e^2 (2k)^{1/k} \cdot 2\log t < 16 e^3 c \log^2 t \ .$$

Since $\alpha(y)$ in a nonincreasing function, the theorem follows from the last inequality and (18). ■

Corollary. *If (4) holds for any* $x \geq 2$ *and* $k \geq 2$, *then the function* $\alpha(y)$ *tends to zero as* $y \to +\infty$.

Proof. Let us assume the contrary. Since $\alpha(y) \geq y^{-1} > 0$ and $\alpha(y)$ in a nonincreasing function, there exists $\alpha > 0$ such that $\alpha(k) \geq \alpha > 0$, $k = 2, 3, \ldots$. Consequently, estimate (4) can be replaced by

$$|R_k(x)| \leq x^{1-\alpha}(c \log x)^k .$$

Without loss of generality we can assume that $\alpha < 0.5$. From the above theorem it follows that for $\sigma \geq 1 - 0.5\alpha$ the following estimate holds:

$$|\zeta(\sigma + it)| < 16e^3 c \log^2 t , \qquad t \geq e^2 . \tag{19}$$

On the other hand, by the known Ω-theorems, for $\frac{1}{2} < \sigma < 1$ the following relation holds:

$$\zeta(\sigma + it) = \Omega\left(\exp\left(c_1 \frac{(\log t)^{1-\sigma}}{(\log\log t)^{\sigma}}\right)\right) \tag{20}$$

(compare e.g. [9] or a weaker result in [15], p. 291 and [16]).

For $\sigma = 1 - 0.5\alpha$ the estimates (19) and (20) contradict each other. Therefore our assumption that $\alpha(y) \not\to 0$ as $y \to +\infty$ is not true. The corollary is proved. ■

In what follows we assume that $\alpha(y) \longrightarrow 0$ monotorically as $y \to +\infty$.

Theorem 2. *Suppose that the assumptions of Theorem 1 are fulfilled. Then* $\zeta(s) \neq 0$ *in the region:*

$$\sigma \geq 1 - c_2 \frac{\alpha(\log|t|)}{\log\log|t|} , \qquad t \geq e^2 .$$

Proof. Assume that $t \geq e^2$. We use the following proposition (cf. [12], p. 57):

Let

$$\zeta(s) = O(e^{\varphi(t)})$$

as $t \to +\infty$ in the region

$$1 - \Theta(t) \leq \sigma \leq 2 , \qquad t \geq e^2 ,$$

where $\varphi(t)$ and $\Theta^{-1}(t)$ positive nondecreasing functions such that $\Theta(t) \leq 1$, $\varphi(t) \to +\infty$, and

$$\frac{\varphi(t)}{\Theta(t)} = o(e^{\varphi(t)}) .$$

Then $\zeta(s) \neq 0$ in the region

$$\sigma \geq 1 - c_1 \frac{\Theta(2t+1)}{\varphi(2t+1)} , \qquad t \geq e^2 .$$

Put here $\Theta(t) = \alpha(\log t)$, $\varphi(t) = 2 \log\log t$. Since $\alpha(y) \geq y^{-1}$, it follows that

$$\frac{\varphi(t)}{\Theta(t)} \leq (2\log\log t)\log t = o(e^{\varphi(t)}) = o(\log^2 t) .$$

It is clear that $\varphi(t)$ and $\Theta^{-1}(t)$ are nondecreasing positive functions and $\Theta(t) < 1$. Therefore $\zeta(s) \neq 0$ in the region

$$\sigma \geq 1 - c_1 \frac{\alpha(\log(2t+1))}{2\log\log(2t+1)} , \qquad t \geq e^2 .$$

From this the theorem follows. ∎

Examples. Let us consider some examples of concrete functions $\alpha(k)$ in Theorem 2.

1. Let $\alpha(k) = k^{-\alpha}$, $0 < \alpha < 1$. Then $\zeta(s) \neq 0$ in the region

$$\sigma \geq 1 - \frac{c_2}{\log^{\alpha} |t| \log\log |t|} , \qquad |t| \geq e^2 .$$

In particular, putting $\alpha = \frac{2}{3}$ we obtain the result of I. M. Vinogradov [13].

2. Let $\alpha(k) = (\log k)^{-\alpha}$, $\alpha > 0$. Then $\zeta(s) \neq 0$ in the region

$$\sigma \geq 1 - \frac{c_2}{(\log\log |t|)^{\alpha+1}} , \qquad |t| \geq e^2 .$$

3. Let $\alpha(k) = (\log\log k)^{-\alpha}$, $\alpha > 0$. Then $\zeta(s) \neq 0$ in the region

$$\sigma \geq 1 - \frac{c_2}{(\log\log |t|)(\log\log\log |t|)^{\alpha}} , \qquad |t| \geq e^{e^e} .$$

From Theorem 1 estimates for short zeta-sum can be derived. For $t \geq e^2$ the following trigonometric sum

$$S(a) = \sum_{n \leq a} n^{it} , \quad 0 < a \leq t$$

is called a zeta - sum. The number a is called the length of $S(a)$. We say that the sum $S(b)$ is shorter than the sum $S(a)$ if $b < a$. The upper estimates for $|S(a)|$ are closely related to the estimates for $|\zeta(s)|$ (compare e.g. [8], [15]).

Theorem 3. *Let the assumptions of Theorem 1 are fulfilled. Then the following estimate for $|S(a)|$ holds:*

$$|S(a)| \leq c_1 a^{1-0.5\alpha(\log t)} (\log t)^3 . \tag{21}$$

Proof. Using the inversion formula (see e.g. [8], p. 75, [15], p. 347) we obtain

$$S(a) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} \zeta(w+it) \frac{a^w}{w} dw + O\left(\frac{a^b}{T(b-1)}\right) + O\left(\frac{a \log a}{T}\right) ,$$

where $2 \geq b > 1$, $T \geq 1$ and the constants implied by the O-symbols are absolute. Set here

$$b = 1 + (\log a)^{-1} , \quad a \geq e^2 , \quad T = 0.5t .$$

We obtain

$$S(a) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} \zeta(w+it) \frac{a^w}{w} dw + O\left(\frac{a \log a}{T}\right) .$$

Consider the rectangular Γ with the vertices $b \pm iT$, $u \pm iT$, where

$$u = 1 - 0.5\alpha(\log t) .$$

Using the Cauchy residue theorem we find that

$$\frac{1}{2\pi i} \int_\Gamma \zeta(w+it) \frac{a^w}{w} dw = 0 .$$

Consequently,

$$\left| \frac{1}{2\pi i} \int_{b-iT}^{b+iT} \zeta(w+it) \frac{a^w}{w} dw \right| \leq J_1 + J_2 + J_3 . \tag{22}$$

where

$$J_1 = \frac{1}{2\pi} \left| \int_{-T}^{T} \zeta(u + i(v+t)) \frac{a^{u+iv}}{u+iv} dv \right| ,$$

$$J_2 = \frac{1}{2\pi} \left| \int_{u}^{b} \zeta(\sigma + i(T+t)) \frac{a^{\sigma+iT}}{\sigma+iT} d\sigma \right| ,$$

$$J_3 = \frac{1}{2\pi} \left| \int_{u}^{b} \zeta(\sigma + i(-T+t)) \frac{a^{\sigma-iT}}{\sigma-iT} d\sigma \right| .$$

Let us estimate J_1, J_2 and J_3 from above. Applying (17) to $|\zeta(s)|$ we obtain:

$$J_1 = O\left((\log^2 t)\int_0^T \frac{a^u dv}{\sqrt{u^2+v^2}}\right) = O(a^u \log^3 t) .$$

$$J_2 = O\left((\log^2 t)\int_0^b \frac{a^\sigma d\sigma}{T}\right) = O\left(\frac{a}{T}\log^2 t\right) .$$

$$J_3 = O\left(\frac{a}{T}\log^2 T\right) .$$

From (22) we find that

$$S(a) = O(a^u \log^3 t) + O\left(\frac{a}{T}\log^2 t\right) = O(a^u \log^3 t) .$$

The theorem is proved. ■

Remarks. 1. The estimate (21) is non-trivial if

$$a > \exp\left(\frac{6\log\log t + 2\log c_1}{\alpha(\log t)}\right) .$$

From this it follows that the estimates for $S(a)$ obtained in this way are of any value only if

$$\alpha(k) \geq \frac{c_2 \log k}{k} .$$

Let us note that in the classical Dirichlet theorem we have $\alpha(k) = 1/k$ (compare e.g. [12]: pp. 313–314).

2. Let $\alpha(k) = k^{-\alpha}$, $0 < \alpha < 1$. Then (21) is of the form:

$$|S(a)| \leq c_1 a^{1-0.5(\log t)^{-\alpha}} \log^3 t = a\Delta .$$

$$\Delta = c_1 a^{-0.5(\log t)^{-\alpha}} \log^3 t .$$

Putting $\alpha = \frac{2}{3}$ we obtain

$$\Delta = c_1 \exp\left(-0.5\frac{\log a}{(\log t)^{2/3}}\right)\log^3 t . \tag{23}$$

The known estimate of I. M. Vinogradov is of the form:

$$|S(a)| \leq a\Delta_1 .$$

$$\Delta_1 = c_1 \exp\left(-c_2\frac{\log^3 a}{\log^2 t}\right) . \tag{24}$$

Comparing the estimates (23) and (24) we can easily see that for all a the estimate (24) is the better one.

Let us finally note that the estimate (23) is nontrival for

$$a \geq \exp(c_3(\log^{2/3} t)(\log\log t)) ,$$

and the estimate (24) is nontrival for

$$a \geq \exp(c_1(\log t)^{2/3}) .$$

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