Functiones et Approximatio
XXVIII (2000), 117–125

To Professor Włodzimierz Staś on his 75th birthday

ON SUMS OF TWO K-TH POWERS: A MEAN-SQUARE BOUND OVER SHORT INTERVALS

WERNER GEORG NOWAK

1. Introduction.

For a fixed integer $k \geq 2$, denote by $r_k(n)$ the number of representations of the positive integer n as a sum of the k-th powers of two integers taken absolutely:

$$r_k(n) = \#\{(u_1, u_2) \in \mathbb{Z}^2 \,:\, |u_1|^k + |u_2|^k = n\,\}.$$

The average order of this arithmetic function is described by the sum

$$R_k(u) = \sum_{1 \leq n \leq u^k} r_k(n)\,,$$

where u is a large real variable[1]. One is interested in precise asymptotic formulas for this summatory function $R_k(u)$.

For $k = 2$, this is the celebrated Gaussian circle problem. (An enlightening account on its history can be found in the monograph of Krätzel [10].) The sharpest published results to date[2] read

$$R_2(u) = \pi u^2 + P_2(u)\,, \tag{1.1}$$

$$P_2(u) = O\big(u^{46/73}(\log u)^{315/146}\big)\,, \tag{1.2}$$

and[3]

$$P_2(u) = \Omega_-\big(u^{1/2}(\log u)^{1/4}(\log\log u)^{\frac{1}{4}\log 2}\exp(-c\sqrt{\log\log\log u})\big) \qquad (c > 0)\,, \tag{1.3}$$

1991 Mathematics Subject Classification: 11P21, 11N37, 11L07

1 Note that, in part of the relevant literature, $t = u^2$ is used as the basic variable.

2 Actually, M. Huxley has meanwhile improved further this upper bound, essentially replacing the exponent $\frac{46}{73} = 0.6301\ldots$ by $\frac{131}{208} = 0.6298\ldots$. The author is indebted to Professor Huxley for sending him a copy of his unpublished manuscript.

3 We recall that $F_1(u) = \Omega_*(F_2(u))$ means that $\limsup\big(*F_1(u)/F_2(u)\big) > 0$ for $u \to \infty$ where $*$ is either $+$ or $-$, and $F_2(u)$ is positive for u sufficiently large.

$$P_2(u) = \Omega_+\big(u^{1/2}\exp\big(c'(\log\log u)^{1/4}(\log\log\log u)^{-3/4}\big)\big) \qquad (c' > 0)\,. \tag{1.4}$$

These are due to Huxley [4], [6], Hafner [3], and Corrádi & Kátai [1], respectively. It is a wide-standing belief that

$$\inf\{\theta \in \mathbb{R}:\ P_2(u) \ll_\theta\ u^\theta\ \} = \frac{1}{2}\,. \tag{1.5}$$

In favour of this conjecture, there is the mean-square asymptotic

$$\int_0^T \big(P_2(u)\big)^2\,\mathrm{d}u = C_2\,T^2 + O\big(T(\log T)^2\big)\,, \qquad C_2 = \frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{(r_2(n))^2}{n^{3/2}} \tag{1.6}$$

which has been established (with this precise error term) by Kátai [7].

The proofs of the results (1.3), (1.4), (1.6) were based on the fact that the generating function (Dirichlet series) of $r_2(n)$ is the Epstein zeta- function of the quadratic form $u_1^2+u_2^2$, which satisfies a well-known functional equation and thus makes available the whole toolkit of complex analysis.

The general case, $k \geq 3$, lacks this technical advantage. Nevertheless, the problem concerning the asymptotic behaviour of $R_k(u)$, $k \geq 3$, has attracted a lot of attention, too. It has first been dealt with by Van der Corput [18] and Krätzel [9]. For a thorough account on the history of this problem and the results available until 1988, see again Krätzel's textbook [10]. It turns out that

$$R_k(u) = \frac{2\Gamma^2(1/k)}{k\Gamma(2/k)}u^2 + B_k\Phi_k(u)u^{1-1/k} + P_k(u) \tag{1.7}$$

where

$$B_k = 2^{3-1/k}\,\pi^{-1-1/k}\,k^{1/k}\,\Gamma\big(1+\frac{1}{k}\big)\,,$$

$$\Phi_k(u) = \sum_{n=1}^{\infty} n^{-1-1/k}\sin\big(2\pi nu - \frac{\pi}{2k}\big)\,,$$

and the new error term $P_k(u)$ satisfies an estimate quite analogous to (1.2), i.e.,

$$P_k(u) = O\big(u^{46/73}(\log u)^{315/146}\big)\,, \tag{1.8}$$

as was proved by Kuba [11], using Huxley's method [4], [6].

Concerning lower bounds, it was shown by the author [16] that, for any fixed $k \geq 3$,

$$P_k(u) = \Omega_-\big(u^{1/2}(\log u)^{1/4}\big)\,, \tag{1.9}$$

and by Küehleitner, Nowak, Schoißengeier & Wooley [13] that

$$P_3(u) = \Omega_+\big(u^{1/2}(\log\log u)^{1/4}\big)\,. \tag{1.10}$$

The similarity of these results to those for the case $k = 2$ suggested to extend the classic conjecture (1.5) to arbitrary $k \geq 2$. It turned out that this is again true in mean-square: In fact, the author [15] was able to show that, for T large,

$$\frac{1}{T}\int_0^T (P_k(u))^2 \, du \ll T \tag{1.11}$$

for any fixed $k \geq 3$. M. Küehleitner [12] refined this estimate, proving an asymptotic formula

$$\frac{1}{T}\int_0^T (P_k(u))^2 \, du = C_k T + O(T^{1-\omega_k+\epsilon}), \tag{1.12}$$

with explicite constants C_k and $\omega_k > 0$.

2. Statement of result

In the present note we investigate the question whether the "average moderate size" of this error term $P_k(u)$, as displayed by (1.11), can be observed only "in the long run," i.e., by averaging over an interval of order T, or if a similar estimate is possible for a "short interval mean." In fact, it turns out that it essentially suffices to average over an interval of bounded length—at the cost of a small loss of precision (extra logarithmic factor).

Theorem 2.1. *For T large and arbitrary fixed $k \geq 3$,*

$$\int_{T-\frac{1}{2}}^{T+\frac{1}{2}} (P_k(u))^2 \, du \ll T(\log T)^2,$$

with the $\ll$-constant depending on k.

Remarks. This work is inspired by a paper of Huxley [5] who investigated the corresponding problem for the lattice rest of a convex planar domain (with smooth boundary of finite nonzero curvature throughout), linearly dilated by a large factor u. He obtained the corresponding mean-square bound $O(T \log T)$, thereby including the case of a circle, i.e., that of $k = 2$ in our problem.

In geometric terms, for $k \geq 3$ we are concerned with the number of lattice points in a domain bounded by a Lamé's curve $|\xi|^k + |\eta|^k = u^k$. This has curvature 0 in its points of intersection with the coordinate axes. As a consequence, the expansion of the lattice rest into a trigonometric series, as discovered by Kendall [8] and employed by Huxley [5], is no longer available. Therefore, we use a different approach based on fractional part sums, Vaaler's transition to exponential sums, the Van der Corput transformation ("B-step"), and, in the end, Huxley's trick involving the Féjer kernel.

Catching a word of Huxley [5] (who imagined the dilation factor u as a time variable), we can say that, according to our result, these number-theoretic error terms "*have no memory,*" or, a bit more precisely, that their average small size is accomplished "*not by long-term memory, but by short-term memory.*"

3. Proof of the Theorem 2.1

As in our earlier article [15], we start from formulae (3.57), (3.58) (and the asymptotic expansion below) in Krätzel [10], p. 148. In our notation, this reads

$$P_k(u) = -8 \sum_{\alpha u < n \le u} \psi((u^k - n^k)^{1/k}) \; + \; O(1)\,, \tag{3.1}$$

with $\psi(w) = w - [w] - \frac{1}{2}$ throughout, and $\alpha := 2^{-1/k}$. We suppose that T is sufficiently large, $u \in [T - \frac{1}{2}, T + \frac{1}{2}]$, and define q by $1/k + 1/q = 1$, i.e., $q = k/(k-1)$, and thus $1 < q \le \frac{3}{2}$. We break up the range of summation into subintervals $\mathcal{N}_j(u) =]N_j, N_{j+1}]$, where $N_j = u\,(1 + 2^{-jq})^{-1/k}$, $j = 0, 1, \ldots, J$, with J minimal such that $u - N_J < 1$ for all $u \in [T - \frac{1}{2}, T + \frac{1}{2}]$.[4] It follows that the length of any $\mathcal{N}_j(u)$ is equal to $N_{j+1} - N_j \asymp 2^{-jq}T$, and that $w \in \mathcal{N}_j(u)$ implies that $u^k - w^k \asymp 2^{-jq}T^k$. We put

$$I_j(T) := \int_{T-\frac{1}{2}}^{T+\frac{1}{2}} \Bigg(\sum_{n \in \mathcal{N}_j(u)} \psi((u^k - n^k)^{1/k}) \Bigg)^2 \mathrm{d}u$$

and infer from Cauchy's inequality, with some fixed $\epsilon > 0$ sufficiently small, that

$$\begin{aligned} &\int_{T-\frac{1}{2}}^{T+\frac{1}{2}} \Bigg(\sum_{j=0}^{J} \sum_{n \in \mathcal{N}_j(u)} \psi((u^k - n^k)^{1/k}) \Bigg)^2 \mathrm{d}u \\ &\qquad \le \sum_{j=0}^{J} 2^{-j\epsilon} \sum_{j=0}^{J} 2^{j\epsilon}\, I_j(T) \ll \sum_{j=0}^{J} 2^{j\epsilon}\, I_j(T)\,. \end{aligned} \tag{3.2}$$

We now invoke a deep result of Vaaler [17] which connects fractional parts with exponential sums. (See also Graham and Kolesnik [2], p. 116.) For every positive integer D there exists a sequence $(\alpha_{h.D})_{h=1}^{D}$ contained in the interval $[0,1]$ such that for all reals w,

$$\left| \psi(w) + \frac{1}{2\pi i} \sum_{1 \le |h| \le D} \frac{\alpha_{|h|.D}}{h}\, e(hw) \right| \le \frac{1}{2D+2} \sum_{h=-D}^{D} \Big(1 - \frac{|h|}{D+1}\Big) e(hw)\,,$$

with $e(w) = e^{2\pi i w}$ as usual. From this it is easy to see that there exists a complex-valued sequence $(\beta_{h.D})_{h=1}^{D}$ with

$$\beta_{h.D} \ll \frac{1}{h} \tag{3.3}$$

---

4 The idea of this special choice of subdivision points is that $\frac{\mathrm{d}}{\mathrm{d}w}\left((u^k - w^k)^{1/k}\right)$ assumes integer values at $w = N_j$. See the application of the Lemma below.

such that

$$I_j(T) \ll \int_{T-\frac{1}{2}}^{T+\frac{1}{2}} \left| \sum_{h=1}^{D} \beta_{h.D} \sum_{n \in \mathcal{N}_j(u)} e\left(-h(u^k - n^k)^{1/k}\right) \right|^2 du \; + \; \left(\frac{2^{-jq}T}{D}\right)^2 . \qquad (3.4)$$

We choose $D = \exp(\log 2\,[\frac{1}{2}\log T/\log 2])$, i.e., D is a power of 2 and $D \asymp \sqrt{T}$. The last term in (3.4) is thus $\ll 4^{-jq}T$.

We now transform the exponential sums under consideration by a fairly sharp form of the "Van der Corput step."

Lemma 3.1. *Suppose that f is a real-valued function which possesses four continuous derivatives on the interval $[A, B]$. Let L and U be real parameters not less than 1 such that $B - A \asymp L$,*

$$f^{(j)}(w) \ll UL^{1-j} \qquad \text{for } w \in [A, B],\ j = 1, 2, 3, 4,$$

and, for some $C^* > 0$,

$$f''(w) \geq C^* UL^{-1} \qquad \text{for } w \in [A, B] .$$

Suppose further that $f'(A)$ and $f'(B)$ are integers, and denote by ϕ the inverse function of f'. Then it follows that

$$\sum_{A \leq k \leq B} e(f(k)) = e\left(\frac{1}{8}\right) \sum_{f'(A) \leq m \leq f'(B)}'' \frac{e\big(f(\phi(m)) - m\phi(m)\big)}{\sqrt{f''(\phi(m))}} + O\big(\log(1+U)\big) ,$$

where $\sum''$ means that the terms corresponding to $m = f'(A)$ and $m = f'(B)$ get a factor $\frac{1}{2}$. The O-constant depends on C^* and on the constants implied in the order symbols in the suppositions.

Proof. This is Lemma 2 in Kühleitner [12]. For a more general version of the same precision, as well as for comments on the history of this sort of results, see Kühleitner & Nowak [14], Lemma 2.2.

We use this formula to transform each of the sums over n in (3.4), with $[A, B] = [N_j, N_{j+1}]$, and

$$f(w) = -h\big(u^k - w^k\big)^{1/k} .$$

We readily compute the derivatives as[5]

$$f'(w) = hw^{k-1}\big(u^k - w^k\big)^{-1+1/k} \ll h\,2^j ,$$

$$f''(w) = h(k-1)u^k w^{k-2}\big(u^k - w^k\big)^{-2+1/k} \asymp hT^{-1}\,2^{j-jq} ,$$

5 Recall that $w \in \mathcal{N}_j(u)$ implies that $w \asymp T$ and $u^k - w^k \asymp 2^{-jq}T^k$.

$$f'''(w) = h(k-1)u^k w^{k-3}\big(u^k - w^k\big)^{-3+1/k}\big((k-2)u^k + (k+1)w^k\big)$$
$$\ll hT^{-2}\,2^{j-2jq}\,,$$
$$f^{(4)}(w) = h(k-1)u^k w^{k-4}\big(u^k - w^k\big)^{-4+1/k}$$
$$\times\big((k-2)(k-3)u^{2k} + (k+1)(4k-7)u^k w^k + (k+1)(k+2)w^{2k}\big)$$
$$\ll hT^{-3}\,2^{j-3jq}\,.$$

Our Lemma thus applies with $L = N_{j+1} - N_j \asymp 2^{-jq}T$, $U = h\,2^j$, and we obtain by a straightforward calculation, for $u \in [T-\frac{1}{2}, T+\frac{1}{2}]$,

$$\sum_{n\in\mathcal{N}_j(t)} e\big(-h(u^k - n^k)^{1/k}\big)$$
$$= \frac{e\left(\frac{1}{8}\right)}{\sqrt{k-1}} hu^{1/2} \sideset{}{''}\sum_{m\in\mathcal{M}_j(h)} (hm)^{-1+q/2}\,\|(h,m)\|_q^{-q+1/2}\, e\big(-u\,\|(h,m)\|_q\big)$$
$$+ O(\log T)\,, \tag{3.5}$$

with
$$\mathcal{M}_j(h) =]f'(N_j), f'(N_{j+1})] = \big]2^j h, 2^{j+1}h\big]\,,$$

and $\|\cdot\|_q$ denoting the q-norm in $\mathbb{R}^2$, i.e., $\|(u_1,u_2)\|_q = \big(|u_1|^q + |u_2|^q\big)^{1/q}$. With a look back to (3.4), we define

$$S_h(u) := \beta_{h,D}\, h \sideset{}{''}\sum_{m\in\mathcal{M}_j(h)} (hm)^{-1+q/2}\,\|(h,m)\|_q^{-q+1/2}\, e\big(-u\,\|(h,m)\|_q\big)$$

and divide the range $1 \le h \le D = 2^I$ (say) into dyadic subintervals $\mathcal{H}_i =]2^{i-1}, 2^i]$, $i = 1,\ldots,I \ll \log T$. Combining (3.4) and (3.5), we conclude by Cauchy's inequality that

$$I_j(T) \ll \int_{T-\frac{1}{2}}^{T+\frac{1}{2}} u\left|\sum_{i=1}^{I}\sum_{h\in\mathcal{H}_i} S_h(u)\right|^2 \mathrm{d}u + (\log T)^2 + 4^{-jq}T$$
$$\ll T(\log T)^2 \max_{1\le i\le I} \int_{T-\frac{1}{2}}^{T+\frac{1}{2}} \left|\sum_{h\in\mathcal{H}_i} S_h(u)\right|^2 \mathrm{d}u \;+ (\log T)^2 + 4^{-jq}T\,. \tag{3.6}$$

Following an idea of Huxley [5], we now use the Féjer kernel

$$\varphi(w) := \left(\frac{\sin(\pi w)}{\pi w}\right)^2\,.$$

By Jordan's inequality, $\varphi(w) \ge 4/\pi^2$ for $|w| \le \frac{1}{2}$, and the Fourier transform has the simple shape

$$\widehat{\varphi}(y) = \int_{\mathbb{R}} \varphi(w)e(yw)\,\mathrm{d}w = \max(0, 1-|y|)\,.$$

Therefore,

$$\frac{4}{\pi^2}\int_{T-\frac12}^{T+\frac12}\Big|\sum_{h\in\mathcal{H}_i} S_h(u)\Big|^2\,\mathrm{d}u \le \int_{\mathbb{R}}\varphi(u-T)\Big|\sum_{h\in\mathcal{H}_i} S_h(u)\Big|^2\,\mathrm{d}u =$$

$$= \sum_{h_1,h_2\in\mathcal{H}_i}(h_1h_2)^{q/2}\beta_{h_1,D}\overline{\beta_{h_2,D}}\sideset{}{''}\sum_{\substack{m_1\in\mathcal{M}_j(h_1),\\ m_2\in\mathcal{M}_j(h_2)}}\frac{\big(\|(h_1,m_1)\|_q\,\|(h_2,m_2)\|_q\big)^{-q+1/2}}{(m_1m_2)^{1-q/2}}$$
$$\times\, e\big(-T(\|(h_1,m_1)\|_q-\|(h_2,m_2)\|_q)\big)$$
$$\int_{\mathbb{R}}\varphi(u)e\big(-u(\|(h_1,m_1)\|_q-\|(h_2,m_2)\|_q)\big)\,\mathrm{d}u$$
$$\ll \sum_{h_1,h_2\in\mathcal{H}_i}(h_1h_2)^{-1+q/2}\sideset{}{''}\sum_{\substack{m_1\in\mathcal{M}_j(h_1),\\ m_2\in\mathcal{M}_j(h_2)}}\frac{\big(\|(h_1,m_1)\|_q\,\|(h_2,m_2)\|_q\big)^{-q+1/2}}{(m_1m_2)^{1-q/2}}$$
$$\times\max\big(0,1-\big|\|(h_1,m_1)\|_q-\|(h_2,m_2)\|_q\big|\big)\,, \tag{3.7}$$

using the bound (3.3) for the β's. We recall that $h\in\mathcal{H}_i$ implies $h\asymp 2^i$ and $m\in\mathcal{M}_j(h)$ implies that $\|(h,m)\|_q\asymp m\asymp 2^j h$. Therefore, the last expression in (3.7) is

$$\ll (2^i)^{-2+q}(2^{i+j})^{-1-q}\,\#\{(h_1,h_2,m_1,m_2)\in\mathbb{Z}^4:\ h_1,h_2\in\mathcal{H}_i,$$
$$m_1\in\mathcal{M}_j(h_1),\ m_2\in\mathcal{M}_j(h_2),\ \big|\|(h_1,m_1)\|_q-\|(h_2,m_2)\|_q\big|<1\}\,. \tag{3.8}$$

Now denote by $A_q^*(u)$ the number of lattice points $\mathbf{v}\in\mathbb{Z}^2$ with $\|\mathbf{v}\|_q\le u$, then the most elementary estimate

"Number of lattice points = area + O(length of boundary)"

implies, for any fixed (h_1,m_1), $h_1\in\mathcal{H}_i$, $m_1\in\mathcal{M}_j(h_1)$, that

$$A_q^*(\|(h_1,m_1)\|_q+1)-A_q^*(\|(h_1,m_1)\|_q-1)\ll\|(h_1,m_1)\|_q\ll m_1\,.$$

Thus, combining (3.7) and (3.8), it follows that

$$\int_{T-\frac12}^{T+\frac12}\Big|\sum_{h\in\mathcal{H}_i} S_h(u)\Big|^2\,\mathrm{d}u\ll(2^i)^{-2+q}(2^{i+j})^{-1-q}\sum_{h_1\in\mathcal{H}_i,}\ \sum_{m_1\in\mathcal{M}_j(h_1)} m_1\ll(2^j)^{1-q}\,,$$

uniformly in $i=1,\dots,I$. Using this in (3.6), we get

$$I_j(T)\ll 2^{-j(q-1)}\,T(\log T)^2+(\log T)^2\,.$$

Recalling (3.1), (3.2), and the fact that $q = k/(k-1) > 1$, we complete the proof of the Theorem. ■

References

[1] K. Corrádi and I. Kátai, *A comment on K. S. Gangadharan's paper "Two classical lattice point problems"* (Hungarian), Magyar Tud. Akad. mat. fiz. Oszt. Közl. **17** (1967), 89–97.

[2] S.W. Graham and G. Kolesnik, *Van der Corput's method of exponential sums*, Cambridge University Press, Cambridge, 1991.

[3] J.L. Hafner, *New omega theorems for two classical lattice point problems,* Invent. Math. **63** (1981), 181–186.

[4] M.N. Huxley, *Exponential sums and lattice points II,* Proc. London Math. Soc. **66** (1993), 279–301.

[5] M.N. Huxley, *The mean lattice point discrepancy*, Proc. Edinburgh Math. Soc. **38** (1995), 523–531.

[6] M.N. Huxley, *Area, lattice points, and exponential sums*, LMS Monographs, New Ser. **13**, Oxford 1996.

[7] I. Kátai, *The number of lattice points in a circle*, Ann. Univ. Sci. Budap. Rolando Eötvös, Sect. Math. **8** (1965), 39–60.

[8] D.G. Kendall, *On the number of lattice points inside a random oval*, Quart. J. Math. (Oxford) **19** (1948), 1–26.

[9] E. Krätzel, *Bemerkungen zu einem Gitterpunktproblem*, Math. Ann. **179** (1969), 90–96.

[10] E. Krätzel, *Lattice Points,* Dt. Verl. d. Wiss., Berlin, 1988.

[11] G. Kuba, *On sums of two k-th powers of numbers in residue classes II*, Abh. Math. Sem. Hamburg **63** (1993), 87–95.

[12] M. Kühleitner, *On sums of two kth powers: an asymptotic formula for the mean square of the error term*, Acta Arithm. **92** (2000), 263–276.

[13] M. Kühleitner, W. G. Nowak, J. Schoißengeier and T. Wooley, *On sums of two cubes: An Ω_+-estimate for the error term*, Acta Arithm. **85** (1998), 179–195.

[14] M. Kühleitner and W. G. Nowak, *The asymptotic behaviour of the mean-square of fractional part sums*, Proc. Edinburgh Math. Soc., **43** (2000), 309–323.

[15] W. G. Nowak, *On sums of two k-th powers: a mean-square bound for the error term*, Analysis **16** (1996), 297–304.

[16] W. G. Nowak, *Sums of two k-th powers: an Omega estimate for the error term*, Arch. Math. **68** (1997), 27–35.

[17] J. D. Vaaler, *Some extremal problems in Fourier analysis*, Bull. Amer. Math. Soc. (2) **12** (1985), 183–216.

[18] J.G. Van der Corput, *Over roosterpunkten in het plate vlak*, Thesis, Groningen, 1919.

Address: Institut für Mathematik u. Ang.Stat., Universität für Bodenkultur, A-1180 Wien, Austria.
E-mail: nowak@mail.boku.ac.at, http://www.boku.ac.at/math/nth.html
Received: 14 May 2000