# SOME PROPERTIES OF THE GENERALIZED FAVARD-DURRMEYER OPERATORS

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**Abstract:** The Durrmeyer modification $\widetilde{F}_n f$ of the generalized Favard operators in some weighted function spaces are considered. The rate of convergence of $\widetilde{F}_n f(x)$ at the Lebesgue points $x$ of $f$ is estimated. In particular, a corresponding estimate in the class of functions $f$ of bounded $p$-th power variation is deduced.

**Keywords:** Favard-Durrmeyer operator, rate of convergence, Lebesgue point, $p$-th power variation

## 1. Preliminaries

Let $X_\sigma(R)$ be the space of all measurable real-valued functions $f$ on the real line $R := (-\infty, \infty)$, with the norm

$$\|f\|_\sigma := \sup_{x \in R} |f(x) \exp(-\sigma x^2)| < \infty,$$

where $\sigma > 0$. For functions $f \in X_\sigma(R)$ consider first the generalized Favard operators defined by

$$F_n f(x) := \sum_{k=-\infty}^{\infty} f(k/n) p_{n,k}(x; \gamma),$$

where $x \in R, n \in N$,

$$p_{n,k}(x; \gamma) := \frac{1}{n\gamma_n \sqrt{2\pi}} \exp\left(-\frac{1}{2\gamma_n^2}\left(\frac{k}{n} - x\right)^2\right)$$

and $\gamma = (\gamma_n)_{n=1}^\infty$ is a positive sequence convergent to 0 (see [4]). In the special case where $\gamma_n^2 = \kappa/2n$ with a positive constant $\kappa$, $F_n$ become the known discrete

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Favard operators introduced in [3]. Some properties of operators $F_n f$ for continuous functions on $R$ can be found e.g. in [4] and [5]. In this paper we deal with the Durrmeyer type modification of the operators $F_n$, defined by

$$\widetilde{F}_n f(x) := n \sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma) \int_{-\infty}^{\infty} f(t) p_{n,k}(t;\gamma) dt. \tag{1.1}$$

We will examine the rate of convergence of $\widetilde{F}_n f(x)$, mainly, at those points $x \in R$ at which

$$\lim_{h \to 0} \frac{1}{h} \int_0^h (f(x+t) - f(x)) dt = 0.$$

The general estimate will be expressed in terms of the quantity

$$w_x(\delta; f) := \sup_{0 < |h| \le \delta} \left| \frac{1}{h} \int_0^h (f(x+t) - f(x)) dt \right|, \quad (\delta > 0).$$

Some analogous results for the generalized Favard-Kantorovich operators are presented in [6].

Throughout the paper, the symbols $K(\cdots), K_j(\cdots)$ $(j = 1, 2, \ldots)$ will mean some positive constants depending only on the parameters indicated in parentheses.

## 2. Main result

As is known ([2], pp. 126, 204; [4], p. 388), for all $n \in N, x \in R$,

$$\sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma) = 1 + S_n(x), \tag{2.1}$$

where

$$S_n(x) := 2 \sum_{j=1}^{\infty} \exp(-2\pi^2 j^2 n^2 \gamma_n^2) \cos(2\pi n j x) \tag{2.2}$$

and

$$|S_n(x)| \le (\pi n \gamma_n)^{-2}. \tag{2.3}$$

It is easy to see that for all $n \in N, \nu \in N, k \in Z := \{0, \pm 1, \pm 2, \ldots\}$,

$$\int_{-\infty}^{\infty} p_{n,k}(t;\gamma) dt = \frac{1}{n}, \tag{2.4}$$

$$\int_{-\infty}^{\infty} \left(\frac{k}{n} - t\right)^{2\nu} p_{n,k}(t;\gamma) dt = \frac{(2\nu - 1)!!}{n} \gamma_n^{2\nu}$$

and, by the Schwarz inequality,

$$\int_{-\infty}^{\infty} \left|\frac{k}{n} - t\right|^{\nu} p_{n,k}(t;\gamma)dt \leq \frac{\sqrt{(2\nu - 1)!!}}{n}\gamma_n^{\nu}. \tag{2.5}$$

In view of Lemma 2.1 in [5], if $\nu \in N$ and $n\gamma_n^2 \geq c$ for all $n \in N$, with a positive absolute constant $c$, then

$$\left| \sum_{k=-\infty}^{\infty} \left(\frac{k}{n} - x\right)^{\nu} p_{n,k}(x;\gamma)\right| \leq 75 A_c(2/e)^{\nu/2}\nu!\gamma_n^{\nu},$$

where $A_c = \max\{1, (2c\pi^2)^{-1}\}$. Further, from (2.1) and (2.3) it follows that

$$\sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma) \leq 3A_c \quad \text{for all} \quad n \in N, x \in R, \tag{2.6}$$

and consequently,

$$\sum_{k=-\infty}^{\infty} \left|\frac{k}{n} - x\right|^{\nu} p_{n,k}(x;\gamma) \leq 15 A_c(2/e)^{\nu/2}\sqrt{(2\nu)!}\gamma_n^{\nu}. \tag{2.7}$$

Let $f \in X_\sigma(R)$ with some $\sigma > 0$. Then the operators (1.1) are well defined for all $x \in R$ and $n \in N$ such that $16\sigma\gamma_n^2 \leq 1$. Indeed, using the obvious inequality $(a + b)^2 \leq 2(a^2 + b^2)$ and denoting by $\sqrt{2}\gamma$ the sequence $(\sqrt{2}\gamma_n)_{n=1}^{\infty}$ we easily observe that

$$p_{n,k}(t;\gamma)\exp(\sigma t^2) \leq \frac{1}{n\gamma_n\sqrt{2\pi}}\exp(2\sigma k^2/n^2)\exp\left(-\left(\frac{1}{2\gamma_n^2} - 2\sigma\right)\left(\frac{k}{n} - t\right)^2\right)$$
$$\leq \sqrt{2}\exp(2\sigma k^2/n^2)p_{n,k}(t;\sqrt{2}\gamma)$$

and

$$p_{n,k}(t;\gamma)\exp(2\sigma k^2/n^2) \leq \sqrt{2}\exp(4\sigma x^2)p_{n,k}(x;\sqrt{2}\gamma).$$

Hence, in view of (2.1) – (2.4)

$$|\widetilde{F}_n f(x)| \leq 2n\|f\|_\sigma \exp(4\sigma x^2)\sum_{k=-\infty}^{\infty} p_{n,k}(x;\sqrt{2}\gamma)\int_{-\infty}^{\infty} p_{n,k}(t;\sqrt{2}\gamma)dt$$
$$\leq 2(1 + (\pi n\gamma_n)^{-2})\|f\|_\sigma \exp(4\sigma x^2).$$

**Theorem 2.1.** Let $f \in X_\sigma(R), \sigma > 0$ and let $\gamma = (\gamma_n)_{n=1}^{\infty}$ be a positive sequence convergent to 0 and satisfying the condition $n\gamma_n^2 \geq c$, where $c$ is a positive absolute constant. Then, given any numbers $q \in N, \varrho \geq 0$, we have

$$|\widetilde{F}_n f(x) - f(x)| \leq K(q,c)\sum_{r=0}^{\infty} \frac{w_x((r + 1)\gamma_n; f)}{(r + 1)^q \exp(\varrho r^2\gamma_n^2)} + \frac{|f(x)|}{(\pi n\gamma_n)^2}$$

for all $x \in R$ and $n \in N$ such that $16\sigma\gamma_n^2 \leq 1$, $8\varrho\gamma_n^2 \leq 1$.

**Proof.** In view of (2.1) and (2.4)

$$\widetilde{F}_n f(x) - f(x) = n \sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma) \int_{-\infty}^{\infty} (f(t) - f(x))p_{n,k}(t;\gamma)dt + f(x)S_n(x).$$

Clearly, if $f \in X_\sigma(R)$ and $t \in R$, then

$$\left| \int_x^t (f(u) - f(x))du \right| \leq |t - x| \, \|f\|_\sigma(2\exp(\sigma x^2) + \exp(\sigma t^2))$$

and, under the assumption $16\sigma\gamma_n^2 \leq 1$,

$$\exp\left(\sigma t^2 - \frac{1}{2\gamma_n^2}\left(\frac{k}{n} - t\right)^2\right) \leq \exp\left(2\sigma\left(\frac{k}{n}\right)^2 - \frac{3}{8\gamma_n^2}\left(\frac{k}{n} - t\right)^2\right).$$

Hence, from the definition of $p_{n,k}(x;\gamma)$ it follows at once that for any $k \in Z$,

$$\lim_{t\to\pm\infty} p_{n,k}(t;\gamma) \int_x^t (f(u) - f(x))du = 0.$$

Consequently, integration by parts gives

$$\widetilde{F}_n f(x) - f(x)$$
$$= -n \sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma) \int_{-\infty}^{\infty} \left(\int_x^t (f(u) - f(x))du\right)p'_{n,k}(t;\gamma)dt + f(x)S_n(x).$$

Observing that $p'_{n,k}(t;\gamma) = p_{n,k}(t;\gamma)\left(\frac{k}{n} - t\right)/\gamma_n^2$, applying (2.3) and the definition of $w_x(\delta; f)$ we get

$$|\widetilde{F}_n f(x) - f(x)|$$
$$\leq \frac{n}{\gamma_n^2} \sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma) \int_{-\infty}^{\infty} \left|\frac{k}{n} - t\right| p_{n,k}(t;\gamma)|t - x| w_x(|t - x|)dt + \frac{|f(x)|}{(\pi n \gamma_n)^2}$$

$$= \frac{n}{\gamma_n^2} \sum_{r=0}^{\infty} Z_r(\lambda) + |f(x)|(\pi n \gamma_n)^{-2}, \qquad (2.8)$$

where $\lambda$ is an arbitrary positive number,

$$Z_r(\lambda) := \sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma) \int_{I_{r,\lambda}(x)} \left|\frac{k}{n} - t\right| p_{n,k}(t;\gamma)|t - x| w_x(|t - x|)dt,$$

$$I_{r,\lambda}(x) = \{t \in R: r\lambda \leq |t - x| < (r + 1)\lambda\} \text{ and } w_x(\delta) \equiv w_x(\delta; f).$$

Using (2.6) and (2.5) with $\nu = 1$ we obtain $Z_0(\lambda) \leq 3A_c\lambda w_x(\lambda)n^{-1}\gamma_n$.
Given any $\varrho \geq 0, q \in N$ we have for $r \geq 1$

$$Z_r(\lambda) \leq \frac{w_x((r+1)\lambda)}{r^q\lambda^q \exp(\varrho r^2\lambda^2)} \sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma)$$

$$\cdot \int_{-\infty}^{\infty} \left|\frac{k}{n} - t\right| p_{n,k}(t;\gamma)|t - x|^{q+1} \exp(\varrho(t-x)^2)dt.$$

Clearly,

$$|t - x|^{q+1} \leq 2^q \left(\left|\frac{k}{n} - x\right|^{q+1} + \left|\frac{k}{n} - t\right|^{q+1}\right)$$

and

$$\exp(\varrho(t-x)^2) \leq \exp\left(2\varrho\left(\frac{k}{n} - x\right)^2\right) \exp\left(2\varrho\left(\frac{k}{n} - t\right)^2\right).$$

Moreover, if $8\varrho\gamma_n^2 \leq 1$, then

$$p_{n,k}(x;\gamma)\exp\left(2\varrho\left(\frac{k}{n} - x\right)^2\right) = \frac{1}{n\gamma_n\sqrt{2\pi}} \exp\left(-\left(\frac{1}{2\gamma_n^2} - 2\varrho\right)\left(\frac{k}{n} - x\right)^2\right)$$

$$\leq \frac{1}{n\gamma_n\sqrt{2\pi}} \exp\left(-\frac{1}{4\gamma_n^2}\left(\frac{k}{n} - x\right)^2\right)$$

$$= \sqrt{2}p_{n,k}(x;\sqrt{2}\gamma).$$

From the above inequalities and the estimates (2.5), (2.6), (2.7) it follows that

$$\sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma) \int_{-\infty}^{\infty} \left|\frac{k}{n} - t\right| p_{n,k}(t;\gamma)|t - x|^{q+1} \exp(\varrho(t-x)^2)dt$$

$$\leq \frac{2^{q+1}}{n} \sum_{k=-\infty}^{\infty} p_{n,k}(x;\sqrt{2}\gamma)\left(\sqrt{(2q+3)!!}(\sqrt{2}\gamma_n)^{q+2} + \left|\frac{k}{n} - x\right|^{q+1}\sqrt{2}\gamma_n\right)$$

$$\leq \frac{2^{q+1}}{n}\gamma_n^{q+2}(3A_c\sqrt{(2q+3)!!}(\sqrt{2})^{q+2} + 15A_c\sqrt{2}\sqrt{(2q+2)!}(2/e)^{(q+1)/2}).$$

Hence

$$Z_r(\lambda) \leq K_1(q,c)\frac{w_x((r+1)\lambda)}{r^q\lambda^q \exp(\varrho r^2\lambda^2)}\gamma_n^{q+2}n^{-1} \quad \text{for } r > 0.$$

Choosing $\lambda = \gamma_n$ and using (2.8) we get the desired result immediately.  ■

It is easy to see that under the assumptions $f \in X_\sigma(R)$ and $\delta > 0$ we have

$$w_x(\delta; f) \leq (\exp(\sigma x^2) + \exp(2\sigma x^2)) \exp(2\sigma\delta^2)\|f\|_\sigma$$

(see e.g. [6]). Consequently,

$$w_x((r+1)\gamma_n; f) \leq \exp(2\sigma x^2)(1 + \exp(4\sigma r^2\gamma_n^2)) \exp(4\sigma\gamma_n^2)\|f\|_\sigma.$$

From this inequality it follows at once that the right-hand side of the estimate given in Theorem 2.1 (with $q \geq 2, \varrho \geq 4\sigma$) converges to 0 as $n \to \infty$, at every Lebesgue point $x$ of $f$.

## 3. Corollaries

Let $f \in X_\sigma(R)$ be continuous on $R$ and let $\Omega(\delta; f)_\sigma$ be its weighted modulus of continuity defined by

$$\Omega(\delta; f)_\sigma := \sup_{|h| \leq \delta} \|f(\cdot + h) - f(\cdot)\|_\sigma \quad (\delta > 0).$$

Then, given any $x \in R$ and $r \in N$ we have

$$w_x(r\delta; f) \leq r \exp(2\sigma x^2 + 2\sigma(r - 1)^2 \delta^2) \Omega(\delta; f)_\sigma$$

(see [6], p. 149). This inequality and Theorem 2.1 with $q = 3, \varrho = 2\sigma$ lead to

**Corollary 3.1.** If the sequence $\gamma = (\gamma_n)_{n=1}^\infty$ satisfies the conditions of Theorem 2.1 and if $f \in X_\sigma(R)$ is continuous on $R$, then

$$\|\widetilde{F}_n f - f\|_{2\sigma} \leq K(c) \left( \Omega(\gamma_n; f)_\sigma + \frac{1}{n} \|f\|_{2\sigma} \right)$$

for all $n \in N$ such that $16\sigma\gamma_n^2 \leq 1$.

For some $m \in N_0$ let $C_m(R)$ be the space of all continuous functions $f$ on $R$ such that

$$\|f\|_m^\circ := \sup_{x \in R} |f(x)(1 + x^{2m})^{-1}| < \infty.$$

Clearly, $C_m(R) \subset X_\sigma(R)$ for arbitrary $m \in N_0$, $\sigma > 0$. Moreover, for any $x \in R$ and $r \in N_0$ there holds the inequality

$$w_x((r + 1)\delta; f) \leq (1 + (2x)^{2m} + (2r\delta)^{2m})(r + 1)\omega(\delta; f)_m,$$

(see [6]), where

$$\omega(\delta; f)_m := \sup_{|h| \leq \delta} \|f(\cdot + h) - f(\cdot)\|_m^\circ.$$

Consequently, from Theorem 2.1 (with $\varrho = 0$, $q = 2m + 3$) it follows

**Corollary 3.2.** If the sequence $\gamma = (\gamma_n)_{n=1}^\infty$ satisfies the conditions of Theorem 2.1 and if $f \in C_m(R)$, then

$$\|\widetilde{F}_n f - f\|_m^\circ \leq K(c, m) \left( \omega(\gamma_n; f)_m + \frac{1}{n} \|f\|_m^\circ \right)$$

for all $n \in N$.

Finally, let us suppose that $f \in X_\sigma(R)$ with some $\sigma > 0$ and that at a fixed point $x \in R$ the one-sided limits $f(x+), f(x-)$ exist. Introduce the functions

$$g_x(t) := \begin{cases} f(t) - f(x+) & \text{if } t > x, \\ 0 & \text{if } t = x, \\ f(t) - f(x-) & \text{if } t < x, \end{cases} \qquad \text{sgn}_x(t) := \begin{cases} 1 & \text{if } t > x, \\ 0 & \text{if } t = x, \\ -1 & \text{if } t < x. \end{cases}$$

Then, it is easy to verify that

$$f(t) = \frac{1}{2}(f(x+) + f(x-)) + g_x(t) + \frac{1}{2}(f(x+) - f(x-))\,\text{sgn}_x(t)$$
$$+ \left(f(x) - \frac{1}{2}f(x+) - \frac{1}{2}f(x-)\right)\delta_x(t),$$

where $\delta_x(x) = 1$ and $\delta_x(t) = 0$ if $t \neq x$. Hence

$$\widetilde{F}_n f(x) = \frac{1}{2}(f(x+) + f(x-))(1 + S_n(x)) + \widetilde{F}_n g_x(x)$$
$$+ \frac{1}{2}(f(x+) - f(x-))\widetilde{F}_n\,\text{sgn}_x(x),$$

where $S_n(x)$ is defined by (2.2) and estimated in (2.3). As is shown in [1] (p. 104),

$$\left|\widetilde{F}_n\,\text{sgn}_x(x)\right| = n\left|\sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma)\left(\int_x^\infty - \int_{-\infty}^x\right)p_{n,k}(t;\gamma)dt\right| \leq \frac{4}{n\gamma_n}.$$

In order to estimate $|\widetilde{F}_n g_x(x)| = |\widetilde{F}_n g_x(x) - g_x(x)|$ we use Theorem 2.1 (with $\varrho = 0$ and $q \geq 2$). Consequently, by a simple calculation (cf. e.g. [6], p. 150) we obtain that under the assumptions of Theorem 2.1,

$$\left|\widetilde{F}_n f(x) - \frac{1}{2}(f(x+) + f(x-))\right|$$
$$\leq K_2(q,c)\gamma_n^{q-1}\int_0^{1/\gamma_n} t^{q-2}w_x(1/t;g_x)dt$$
$$+ \frac{1}{2}|f(x+) + f(x-)|(\pi n\gamma_n)^{-2} + 2|f(x+) - f(x-)|(n\gamma_n)^{-1}$$

for all $n \in N$ such that $16\sigma\gamma_n^2 \leq 1$.

In particular, let us consider the class $BV_p(R)$ of all functions $f$ of bounded $p$-th power variation on $R$ and let us denote by $V_p(f; I)$ or $V_p(f; a, b)$ the total $p$-th variation of $f$ on the interval $I = [a, b]$ (defined as in [6]). The obvious inequality

$$w_x(\delta; f) \leq V_p(f; x - \delta, x + \delta) \quad (\delta > 0)$$

and some easy computations lead to

**Corollary 3.3.** *If the sequence* $\gamma = (\gamma_n)_{n=1}^{\infty}$ *satisfies the conditions of Theorem 2.1 and if* $f \in BV_p(R)$, $p \geq 1$, *then for all* $x \in R$ *and* $n \in N$,

$$\left| \tilde{F}_n f(x) - \frac{1}{2}(f(x+) + f(x-)) \right|$$

$$\leq K_3(q, c)\gamma_n^{q-1} \sum_{k=0}^{\mu} k^{q-2} V_p(g_x; I_k)$$

$$+ \frac{1}{2}|f(x+) + f(x-)|(nc)^{-1} + 2|f(x+) - f(x-)|(n\gamma_n)^{-1},$$

where $q \geq 2, \mu = [1/\gamma_n], I_0 = R, I_k = [x - 1/k, x + 1/k]$ if $k = 1, \ldots, \mu$.

Clearly, in view of the continuity of $g_x$ at $x$, the right-hand side of the inequality given in Corollary 3.3 converges to 0 as $n \to \infty$. Moreover, in some classes of functions this inequality cannot be essentially improved. To see this, let us first mention some properties of the functions

$$H_{n,r}(x) := n \sum_{k=-\infty}^{\infty} p_{n,k}(x; \gamma) \int_{-\infty}^{\infty} (x - t)^r p_{n,k}(t; \gamma) dt,$$

where $r \in N_0, n \in N, x \in R$. It is easy to verify that with $S_n(x)$ as defined by (2.2) we have the recursion formula

$$H_{n,0}(x) = 1 + S_n(x), \quad H_{n,1} = \gamma_n^2 S_n'(x),$$
$$H_{n,r+1}(x) = \gamma_n^2 H_{n,r}'(x) - 2r\gamma_n^2 H_{n,r-1}(x).$$

From this formula, by the method of induction, it follows the representation

$$H_{n,2r}(x)$$

$$= \gamma_n^{2r}\left( d_{0,r}(1 + S_n(x)) + \sum_{l=1}^{r-1} d_{l,r}\gamma_n^{2l}(S_n^{(2l-1)}(x) + S_n^{(2l)}(x)) + \gamma_n^{2r} S_n^{(2r-1)}(x) \right),$$

where $d_{l,r}$ $(l = 0, \ldots, r - 1)$ are real numbers independent of $n$ and $x$. Moreover, under the assumption $n\gamma_n^2 \geq c$ for all $n \in N$, the functions $S_n^{(\nu)}(x)$ $(\nu = 0, 1, \ldots, 2r - 1)$ are bounded uniformly in $x \in R$ and $n \in N$. Consequently,

$$H_{n,2r}(x) \leq K_4(c, r)\gamma_n^{2r} \quad \text{for all} \quad x \in R, \ n \in N \tag{3.1}$$

and

$$\lim_{n \to \infty} \gamma_n^{-2r} H_{n,2r}(x) = d_{0,r} \tag{3.2}$$

uniformly in $x \in R$.

Now, let us fix a point $x_0$ and a positive number $\alpha$ and let us denote by $U(\alpha, x_0)$ the class of all functions $f \in BV_p(R)$, continuous at $x_0$ and such that

$V_p(f; x_0 - \delta, x_0 + \delta) \leq \delta^\alpha$ for $\delta \in (0, 1]$. From Corollary 3.3 with $q > \max(2, \alpha+1)$ it follows that for $f \in U(\alpha, x_0)$ and $n \in N$,

$$|\widetilde{F}_n f(x_0) - f(x_0)| \leq K_5(\alpha, c)(1 + V_p(f; R))\gamma_n^\alpha + \frac{|f(x_0)|}{nc}. \tag{3.3}$$

On the other hand, the function $f_\alpha(t) := \frac{1}{2}|t - x_0|^\alpha$ if $|t - x_0| \leq 1, f_\alpha(t) = 1/2$ otherwise on $R$, belongs to $U(\alpha, x_0)$ and

$$\widetilde{F}_n f_\alpha(x_0) - f_\alpha(x_0) \geq \frac{1}{2}n \sum_{k=-\infty}^{\infty} p_{n,k}(x_0; \gamma) \int_{x_0-\delta}^{x_0+\delta} |t - x_0|^\alpha p_{n,k}(t; \gamma) dt$$

for any $\delta \in (0, 1]$. Let $\tau$ be a positive number such that $\tau + \alpha = 2r$, where $r \in N$. Then

$$\widetilde{F}_n f_\alpha(x_0) - f_\alpha(x_0)$$

$$\geq \frac{1}{2}n\delta^{-\tau} \sum_{k=-\infty}^{\infty} p_{n,k}(x_0; \gamma) \int_{x_0-\delta}^{x_0+\delta} (t - x_0)^{2r} p_{n,k}(t; \gamma) dt$$

$$= \frac{1}{2}\delta^{\alpha-2r} \left( H_{n,2r}(x_0) - n \sum_{k=-\infty}^{\infty} p_{n,k}(x_0; \gamma) \int_{|t-x_0|>\delta} (t - x_0)^{2r} p_{n,k}(t; \gamma) dt \right)$$

$$\geq \frac{1}{2}\delta^{\alpha-2r}(H_{n,2r}(x_0) - \delta^{-2}H_{n,2r+2}(x_0)).$$

From (3.2) it follows that

$$H_{n,2r}(x_0) \geq \frac{1}{2}d_{0,r}\gamma_n^{2r}$$

for sufficiently large $n$. Applying inequality (3.1) for $H_{n,2r+2}(x_0)$ and putting $\delta = 2\gamma_n^2\sqrt{K_4(c, r + 1)} \big/ \sqrt{d_{0,r}}$ we obtain that

$$\widetilde{F}_n f_\alpha(x_0) - f_\alpha(x_0) \geq 2^{\alpha-2r-3}(K_4(c, r + 1))^{-r+\alpha/2}(d_{0,r})^{1+r-\alpha/2}\gamma_n^\alpha \tag{3.4}$$

for sufficiently large $n$. Inequalities (3.3) and (3.4) ensure that in the classes $U(\alpha, x_0)$ with $0 < \alpha \leq 2$, or $\overline{U}(\alpha, x_0) := \{f \in U(\alpha, x_0): f(x_0) = 0\}$ with arbitrary $\alpha > 0$, the estimate given in Corollary 3.3 is the best concerning the order.

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