Minimal number of periodic points for $C^1$ self-maps of compact simply-connected manifolds

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Abstract

Let $f$ be a self-map of a smooth compact connected and simply-connected manifold of dimension $m \geq 3$, $r$ a fixed natural number. In this paper we define a topological invariant $D^r_m[f]$ which is the best lower bound for the number of $r$-periodic points for all $C^1$ maps homotopic to $f$. In case $m = 3$ we give the formula for $D^3_3[f]$ and calculate it for self-maps of $S^2 \times I$.

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1 Introduction

In 1983 Albrecht Dold (cf. [D]) formulated the necessary conditions that must be fulfilled by a sequence of indices of iterations. He considered a map $f : U \rightarrow X$, where $U \subset X$ is an open subset of an ENR, satisfying the following fixed-compactness condition. We denote inductively $U_0 = U$, $U_{n+1} = f^{-1}(U_n)$ i.e. $U_n = \{x \in U : x, f(x), ... , f^n(x) \in U\}$. We assume that the fixed point set $\text{Fix}(f^n) = \{x \in U_n : f^n(x) =
$x \}$ is compact for each $n \in \mathbb{N}$. In such situation the fixed point index $\text{ind}(f^n) = \text{ind}(f^n, U_n)$ is well-defined. Dold proved that the sequence of fixed point indices $\{\text{ind}(f^n)\}_{n=1}^{\infty}$ must satisfy the following congruences, (called Dold relations): $\sum_{k|n} \mu(n/k) \text{ind}(f^k) \equiv 0 \pmod{n}$ for each $n \in \mathbb{N}$, where $\mu$ denotes the Möbius function.

The classical issue in periodic point theory is the question of what is the minimal number of periodic points in the homotopy class of a given $f$, a self-map of a compact manifold. It appears that this problem is strictly related to the question what are possible forms of local indices (at an isolated fixed point) of iterations of maps homotopic to $f$. Namely, let $f : U \to \mathbb{R}^m$ be a continuous map defined on a neighborhood of the point $0 \in \mathbb{R}^m$. We define the iterations $f^n$ as above, and assume that 0 is the only fixed point of $f^n$ for each $n \in \mathbb{N}$ in some small neighborhood $V_n \subset U_n$ of 0 and consider $\text{ind}(f^n, 0) = \text{ind}(f^n, 0, V_n)$. Which integer sequences $\{c_n\}_{n=1}^{\infty}$ can be obtained as $\{\text{ind}(f^n, 0)\}_{n=1}^{\infty}$? It is shown in [BaBo] and [GNP2] that once the Dold congruences are satisfied, there is a continuous map $f$ realizing an arbitrary sequence $\{c_n\}_{n=1}^{\infty}$, for $m \geq 2$, as a sequence of indices at a single fixed point.

This fact has a strong impact on the possibility of cancelling periodic points in the given homotopy class. If $f : M \to M$ is a self-map of a compact connected and simply-connected manifold of dimension $\geq 3$, then it is homotopic to a continuous map $g$ with $\text{Fix}(g^r)$ consisting of a single point, where $r$ is a prescribed natural number (cf. [Je2] Thm. 5.1). In the smooth category however, the situation is much different. (We say that $f$ is smooth in the meaning that it is $C^1$.) In 1974 Shub and Sullivan (cf. [SS]) proved that fixed point indices of iterations of a smooth map, at an isolated fixed point, are periodic. This is a very strong restriction making the theory quite different than in the continuous case. Later, in 1981, Chow, Mallet-Paret and Yorke (cf. [CMPY]) showed that in the smooth case a sequence of indices may be represented as the sums of some elementary sequences whose coefficients depend on the derivative of a map at the fixed point. This leads to a system of conditions that any sequence $\{\text{ind}(f^n, 0)\}_{n=1}^{\infty}$, for $f$ being $C^1$ map, must satisfy. It was conjectured in [CMPY] that these conditions are also sufficient to realize given sequence as the sequence of indices of iterations of some $C^1$ map. The conjecture was answered positively by Babenko and Bogaty (cf. [BaBo]) in dimension 2 and by Graff and Nowak-Przygodzki in dimension 3 (cf. [GNP1]).

Basing on the results mentioned above and methods of Nielsen
fixed point theory developed in [Je2] (cf. also [HK1], [HK2]), we are able to deal with the smooth case.

First of all we show in this paper that, unlike in the continuous case, $\text{Fix}(f^r)$ cannot be reduced to a point by a smooth map $g$, homotopic to the given $f$, where $f : M \to M$, $M$ satisfies the same assumptions as in the continuous case, $r$ is a fixed natural number.

Next, we express in terms of local indices of iterations the least cardinality of $\text{Fix}(g^r)$ which can be obtained by a smooth map $g$ in the homotopy class of $f$ (cf. Theorem 3.8). We reduce this problem to the decomposition of the sequence of Lefschetz numbers $L(f^n)$, where $n|r$, into the sum of sequences each of which can be locally realized as a sequence of indices at an isolated periodic orbit for some $C^1$ map. As a result we get the topological invariant $D_m^r[f]$ which is the best lower bound for the number of $r$-periodic points for all $C^1$ maps homotopic to $f$. Let us notice that $D_m^r[f]$ may be interpreted purely in the terms of smooth category, namely we may assume that $f$ is smooth and approximate the homotopy which joins $f$ with $g$ by a smooth one. Then $D_m^r[f]$ gives the minimal number of periodic points in the smooth homotopy class of the given $f$.

In the second part of the paper, we use the classification of sequences of indices of iterations of smooth maps in dimension 3, described in [GNP1], to give the explicit formulae for $D_3^r[f]$. We also calculate $D_3^r[f]$ for a 3-manifold with simple homology groups, namely $S^2 \times I$.

Let us remark that in the non simply-connected case the problem of finding minimal number of periodic points in smooth category is much more complicated. In the possible version of the non simply-connected counterpart of Theorem 3.8 the restrictions for local indices for a smooth map must be combined with topological properties of a space expressed in terms of orbits of Reidemeister classes of the iterated map.

2 Theorem of Chow, Mallet-Paret and Yorke and $DD^m$ sequences

Definition 2.1 Let $f : U \to \mathbb{R}^m$, where $U$ is an open subset of $\mathbb{R}^m$, be a map such that 0 is an isolated fixed point for each iteration of $f$. For each $n$ we define integers (called local Dold coefficients) $i_n(f, 0)$
by the following equality:
\[ i_n(f,0) = \sum_{k|n} \mu(n/k) \text{ind}(f^k,0), \]
where \( \mu \) is the Möbius function, i.e. \( \mu : \mathbb{N} \to \mathbb{Z} \) is defined by three properties: \( \mu(1) = 1 \), \( \mu(k) = (-1)^r \) if \( k \) is a product of \( r \) different primes, \( \mu(k) = 0 \) otherwise.

We may write down Dold relations, mentioned in the introduction, as: for each natural \( n \)
\[ i_n(f,0) \equiv 0 \pmod{n} \tag{1} \]
Dold relations make it possible to decompose a sequence of indices of iterations into a sum of elementary sequences. This decomposition is called a periodic expansion.

**Definition 2.2** For a given \( k \) we define the basic sequence (basic expansion):
\[ \text{reg}_k(n) = \begin{cases} k & \text{if } k|n, \\ 0 & \text{if } k \nmid n. \end{cases} \]

Notice that each basic sequence \( \text{reg}_k \) is a periodic sequence of the form:
\[ (0,\ldots,0,k,0,\ldots,0,k,\ldots,\ldots), \]
where the non-zero entries appear only for indices of the sequence which are multiples of \( k \).

**Theorem 2.3** (cf. [JM]) Any sequence \( \psi : \mathbb{N} \to \mathbb{C} \) can be written uniquely in the following form of a periodic expansion:
\[ \psi(n) = \sum_{k=1}^{\infty} a_k(\psi) \text{reg}_k(n), \]
where \( a_n(\psi) = i_n(\psi)/n = \frac{1}{n} \sum_{k|n} \mu(k)\psi(n/k). \)
Moreover, \( \psi \) takes integer values and satisfies Dold relations iff \( a_n(\psi) \in \mathbb{Z} \) for every \( n \in \mathbb{N} \).

**Definition 2.4** Assume that a periodic expansion for a sequence \( \{\psi(n)\}_{n=1}^{\infty} \) is given. Let \( B(\psi) = \{n \in \mathbb{N} : a_n \neq 0\} \). The set \( B(\psi) \) will be called the basic set for a periodic expansion of \( \psi \).
Now consider $f : U \to \mathbb{R}^m$, where $U$ is an open subset of $\mathbb{R}^m$ containing 0, a $C^1$ map with 0 an isolated fixed point for each iteration. Let us denote by $\Delta$ the set of degrees of all primitive roots of unity which are contained in $\sigma(Df(0))$, the spectrum of derivative at 0. Chow, Mallet-Paret and Yorke showed that the basic set $B(\{\text{ind}(f^n, 0)\}_{n=1}^\infty)$, for a $C^1$ map $f$, is finite and depends on the set $\Delta$.

We denote by $\sigma_+$ the number of real eigenvalues of $Df(0)$ greater than 1, $\sigma_-$ the number of real eigenvalues of $Df(0)$ less than $-1$, in both cases counting with multiplicity. Let $O = \{\text{LCM}(K) : K \subset \Delta\}$, where by $\text{LCM}(K)$ we mean the least common multiple of all elements in $K$ with the convention that $\text{LCM}(\emptyset) = 1$. Let $O_{\text{odd}} = O \cap \{n \in \mathbb{N} : n = 2k - 1, k = 1, 2, \ldots\}$.

We present below the result of Chow, Mallet-Paret and Yorke ([CMPY]) expressed in the language of periodic expansions (cf. [GNP1], [MP], [JM]).

**Theorem 2.5** Let $U \subset \mathbb{R}^m$ be an open neighborhood of 0, $f : U \to \mathbb{R}^m$ be a $C^1$ map having 0 as an isolated fixed point for each iteration. Then:

$$\text{ind}(f^n, 0) = \sum_{k \in O} a_k \text{reg}_k(n),$$

where

$$O = \begin{cases} \overline{O} & \text{if } \sigma_- \text{ is even}, \\ \overline{O} \cup 2O_{\text{odd}} & \text{if } \sigma_- \text{ is odd}. \end{cases}$$

$$(*) \quad \text{If } \sigma_- \text{ is odd and } k \in 2O_{\text{odd}} \setminus \overline{O}, \text{ then } a_k = -a_{k/2}.$$  

$$(***) \quad \text{The bounds for the coefficients } a_1 \text{ and } a_2 \text{ are the following:}$$

$$(1) \quad a_1 = (-1)^{\sigma_+} \text{ if } 1 \not\in \sigma(Df(0)).$$

$$(2) \quad a_1 \in \{-1, 0, 1\} \text{ if } 1 \text{ is an eigenvalue of } Df(0) \text{ with multiplicity 1.}$$

$$(3) \quad a_2 \in \{0, (-1)^{\sigma_+ + 1}\} \text{ if } 1 \not\in \sigma(Df(0)) \text{ and } -1 \text{ is the eigenvalue of } Df(0) \text{ with multiplicity 1.} \quad \square$$

**Remark 2.6** Let us notice that there are three different types of restrictions described in the theorem. Namely: bounds for the set $O$ (which contains the basic set $B$), relations between coefficients $a_k$ and $a_{k/2}$ and conditions for $a_1$ and $a_2$. 
Theorem 2.5 gives the description of indices for a map $f$ for which $0$ is an isolated fixed point for all iterations. In the further considerations we take some fixed $r$ and consider more general situation, in which $0$ is an isolated fixed point only for $n|r$. Nevertheless, for a finite sequence $\{\text{ind}(f^n, 0)\}_{n|r}$ the formula and conditions for indices of iterations given in Theorem 2.5 remain true.

Remark 2.7 Let $r$ be a fixed natural number, $U \subset \mathbb{R}^m$ be an open neighborhood of $0$, $f : U \to \mathbb{R}^m$ be a $C^1$ map having $0$ as an isolated fixed point for $f^n$, where $n|r$. Then $\{\text{ind}(f^n, 0)\}_{n|r} = \sum_{k|n, k \in \mathcal{O}} a_k \text{reg}_k(n)$ and the coefficients $a_k$ (if they appear, so if $k|r$) satisfy the conditions $(\ast)$ and $(\ast\ast)$ of Theorem 2.5.

Notice that by the assumption $\{\text{ind}(f^n, 0)\}_{n|r}$ is well-defined. Remark 2.7 may be relatively easily obtained by analyzing the original proof of Theorem 2.5. Namely, it results from two facts. First, so-called Virtual Period Proposition (Prop. 3.2 [CMPY]) remains true without the assumption that $0$ is an isolated fixed point. Second, if we choose as in [CMPY] the approximation of $f$ by a map with simple fixed points, the contribution to the sequence $\{\text{ind}(f^n, 0)\}_{n|r}$ comes only from such $n$-orbits, for which $n|r$ (by an $n$-orbit we understand an orbit whose elements are periodic points with the minimal period equal to $n$).

Definition 2.8 In the rest of the paper we will use the following notation (cf. Definition 2.4):

\[
\begin{align*}
B(f, x_0) &= B(\{\text{ind}(f^n, x_0)\}_{n=1}^\infty), \\
B_r(f, x_0) &= B(\{\text{ind}(f^n, x_0)\}_{n|r}), \\
B(f) &= B(\{L(f^n)\}_{n=1}^\infty), \\
B_r(f) &= B(\{L(f^n)\}_{n|r}).
\end{align*}
\]

Definition 2.9 A sequence of integers $\{c_n\}_{n=1}^\infty$ is called $DD^m(p)$ sequence if there is an isolated $p$-orbit $P$, its neighborhood $U \subset \mathbb{R}^m$ and a $C^1$ map $\phi : U \to \mathbb{R}^m$ such that $c_n = \text{ind}(\phi^n, P)$ (notice that $c_n = 0$ if $n$ is not a multiple of $p$). The finite sequence $\{c_n\}_{n|r}$ will be called $DD^m(p|r)$ sequence if this equality holds for $n|r$, where $r$ is fixed. In other words, $DD^m(p)$ sequence is a sequence that can be realized as a sequence of indices of iterations on an isolated $p$-orbit for some smooth map $\phi$. 
The following lemma gives the description of $DD^m(p)$ sequences.

**Lemma 2.10** A sequence $\{c_n\}_{n=1}^{\infty}$ is $DD^m(1)$ sequence if and only if
\[ \tilde{c}_n = \begin{cases} 0 & \text{for } p \nmid n, \\ p \frac{c_n}{p} & \text{for } p \mid n. \end{cases} \]
is $DD^m(p)$ sequence. We will say that $DD^m(p)$ sequence $\{\tilde{c}_n\}_{n=1}^{\infty}$ comes from $DD^m(1)$ sequence $\{c_n\}_{n=1}^{\infty}$.

**Proof.** "⇒" Let $\phi : \mathbb{R}^m \to \mathbb{R}^m$ be a smooth map with the isolated fixed point $x_1$, such that $\text{ind}(\phi^k, x_1) = c_k$. We fix different points $x_1, \ldots, x_p \in \mathbb{R}^m$. Let us choose small enough mutually disjoint neighborhoods $U_i \ni x_i$ (for $i = 1, \ldots, p$), and diffeomorphisms $\phi_i : U_i \to U_{i+1}$ (for $i = 1, \ldots, p-1$). At last we define $\phi_p : U_p \to U_1$ by the formula:
\[ \phi_p(x) = \phi(\phi^{-1}_1(\phi^{-1}_2(\cdots \phi^{-1}_{p-1}(x)))) \]
for $x$ near $x_p$. We define $\Phi : \bigcup_{i=1}^p U_i \to \mathbb{R}^m$ putting
\[ \Phi(x) = \phi_i(x) \text{ for } x \in U_i. \]
Then
\[ \Phi^p(x) = \phi_p(\phi_{p-1}(\cdots \phi_1(x))) = \phi(x) \]
in a neighborhood of $x_1$. What is more,
\[ \text{ind}(\Phi^k, x_1) = \text{ind}(\phi^k; x_1) = c_k. \]
Since the indices in all points of the orbit are the same,
\[ \text{ind}(\Phi^k, \{x_1, \ldots, x_p\}) = p \frac{c_n}{p} \]
if $p \mid n$, or zero otherwise, which proves that $\tilde{c}_n$ is a $DD^m(p)$ sequence.

"⇐" Obvious, as $\phi^p$ is the $C^1$ map which gives the needed $DD^m(1)$ sequences. \qed

**Remark 2.11** Let us notice that the same construction, as in the proof of "⇒", can be done for an arbitrary orbit of points in a manifold.

**Remark 2.12** Lemma 2.10 gives an easy procedure which enables one to obtain all forms of $DD^m(p)$ sequence once we know the forms of $DD^m(1)$ sequences. Namely by the relation between respective basic sets: $B(\tilde{c}_n) = pB(c_n) = \{ pk : k \in B(c_n) \}$, we find that in order to get any $DD^m(p)$ sequence it is enough to replace all basic sequences $a_k \text{reg}_k$ by $a_k \text{reg}_{pk}$ in the periodic expansion of some $DD^m(1)$ sequence. \qed
3 Minimal number of periodic points

Let $f : X \to X$ be a self-map of a topological space and $r \in \mathbb{N}$ a fixed number. What is the minimal number of periodic points $\#\text{Fix}(g^r)$, where $g$ is a map homotopic to $f$? The Nielsen periodic point theory gives an invariant $\text{NF}_r(f)$, introduced by Boju Jiang in [Ji], which is a homotopy invariant and the lower bound for the number of periodic points:

$$\text{NF}_r(f) \leq \#\text{Fix}(g^r)$$

for each $g$ homotopic to $f$. In [Je1] it is shown that, for $X$ - a compact PL-manifold of dimension not less than 3, $\text{NF}_r(f)$ is the best such lower bound, i.e. for each $f$ there is a map $g$ homotopic to $f$ such that $\#\text{Fix}(g^r) = \text{NF}_r(f)$.

In this paper we confine ourselves to the simply-connected case. Then Nielsen theory is trivial and $\text{NF}_r(f)$ is equal either 0 or 1. In such a case Theorem 5.1 in [Je2] gives:

**Theorem 3.1** Any continuous self-map $f$ of a compact connected simply-connected PL-manifold of dimension $\geq 3$ is homotopic to a map $g$ such that

$$\text{Fix}(g^r) = \begin{cases} \emptyset & \text{if } L(f^n) = 0 \text{ for all } n \mid r, \\ \{*\} & \text{otherwise}, \end{cases}$$

where $\{*\}$ - $\text{Fix}(g)$ denotes the set which consists of one point. \(\square\)

Now we study a similar question in the smooth category. We are given a manifold $M$. We will assume through the rest of the paper that $M$ is a smooth compact connected and simply-connected. Let $f$ be a continuous self-map of $M$ and $r \in \mathbb{N}$ a given number. If the boundary $\partial M \neq \emptyset$, then we assume that all considered maps have no periodic points on the boundary. The question we ask is: what is the minimal number of periodic points $\#\text{Fix}(g^r)$, where $g$ is a smooth map homotopic to $f$ ?

The results presented in the previous section show that here the situation is drastically different. It is easy to notice that a self-map of a simply-connected manifold is not homotopic, in general, to a smooth map $g$ with $\text{Fix}(g^r) = \{*\}$, a point. In fact:

**Remark 3.2** If a self-map $f$ of $M$ is homotopic to a smooth map $g$ with $\text{Fix}(g^r) = \{*\}$, then the sequence of Lefschetz numbers $\{(L(f^n))_{n \mid r}\}$ is a $DD^m(1|r)$ sequence.
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Proof. For every $n | r$ we have:

$$L(f^n) = L(g^n) = \text{ind}(g^n) = \text{ind}(g^n, *)$$

$\square$

In the next section we will see that the set of periodic points can be reduced to a single point in rather exceptional cases. On the other hand the following lemma holds:

Lemma 3.3 If $f$, a self-map of $M$, is homotopic to a smooth map $g$ with $\#\text{Fix}(g^r) = N$ finite, then the sequence of Lefschetz numbers of $f$ decomposes as

$$L(f^n) = c_1(n) + \ldots + c_s(n),$$

for $n | r$, where $c_i$ is a $DD^m(l_i | r)$ sequence and $l_1 + \ldots + l_s = N$.

Proof. If $\#\text{Fix}(g^r) = N$, then

$$\text{Fix}(g^r) = a_1 \cup \ldots \cup a_s,$$

where $a_i$ is an orbit of the length $l_i$. We get for $n | r$:

$$L(f^n) = L(g^n) = \text{ind}(g^n) = \sum_{i=1}^{s} \text{ind}(g^n, a_i) = \sum_{i=1}^{s} c_i(n),$$

where $c_i(n) = \text{ind}(g^n, a_i)$ is a $DD^m(l_i | r)$ sequence, since $g$ is smooth.

It remains to notice that $l_1 + \ldots + l_s = \#a_1 + \ldots + \#a_s = \#\text{Fix}(g^r) = N$.

$\square$

The above lemma suggests the following definition.

Definition 3.4 Let $\{\xi_n\}_{n | r}$ be a sequence of integers satisfying Dold relations. Assume that we are able to decompose it into the sum:

$$\xi(n) = c_1(n) + \ldots + c_s(n),$$

where $c_i$ is a $DD^m(l_i | r)$ sequence for $i = 1, \ldots, s$. Each such decomposition determines the number $l = l_1 + \ldots + l_s$. We define the number $D^m_r[\xi]$ as the smallest $l$ which can be obtained in this way.
We will denote briefly $D^m_r[f] = D^m_r[\{L(f^n)\}_{n \geq r}]$. It is easy to see that $D^m_r[f]$ is a homotopy invariant. On the other hand, by Kupka-Smale Theorem (cf. for example [R]) every $f$ is homotopic to some smooth map $g$ which has a finite number of $r$-periodic points ($r$ is fixed). Thus, Lemma 3.3 shows that $D^m_r[f]$ is a lower bound of the number of $r$-periodic points: $\#\text{Fix}(f^r) \geq D^m_r[f]$.

The aim of this section is to prove Theorem 3.8 which states that $D^m_r[f]$ is the best such lower bound, provided the dimension of the manifold is not less than 3. The main idea of the proof may be described as follows. We fix a decomposition $L(f^n) = c_1(n) + \cdots + c_s(n)$ realizing $D^m_r[f]$ i.e. the sum $l = l_1 + \cdots + l_s$ in Definition 3.4 is the smallest. Then we deform $f$ to a map $\tilde{f}$ adding an isolated orbit of points $O_i$ satisfying

$$\text{ind}(\tilde{f}^n; O_i) = c_i(n)$$

for each $i = 1, \ldots, s$ (Creating Procedure, Theorem 3.5). It turns out that the remaining periodic points $\text{Fix}(\tilde{f}^r) \setminus (O_1 \cup \ldots \cup O_s)$ can be removed (Canceling Lemma, Lemma 3.7). The map $f_1$, obtained after this cancelation, satisfies

$$\#\text{Fix}(f_1^r) = \#(O_1 \cup \ldots \cup O_s) = \#O_1 + \ldots + \#O_s = l_1 + \cdots + l_s = D^m_r[f],$$

hence $f_1$ realizes the lower bound of the number of periodic points $D^m_r[f]$.

The following two results i.e. Creating Procedure and Canceling Lemma, proved in [Je2] (valid not only for smooth but also PL-manifold $M$) will be crucial in the proof of Theorem 3.8.

The Creating Procedure enables one to create an additional orbit in the homotopy class of $f$, by a use of homotopy $f_t$ which is constant near periodic points of $f$ (up to the given period $r$) and such that $f_t^r$ near the created orbit may be given by an arbitrarily prescribed formula.

**Theorem 3.5** (Creating Procedure, [Je2] Theorem 3.3) Given numbers $p, r \in \mathbb{N}$, $p|r$ and a map $f : M \to M$, where $\dim M \geq 3$, such that $\text{Fix}(f^r)$ is finite and a point $x_0 \notin \text{Fix}(f^r)$. Then there is a homotopy $\{f_t\}_{0 \leq t \leq 1}$ satisfying:

1. $f_0 = f$.
2. $\{f_t\}$ is constant in a neighborhood of $\text{Fix}(f^r)$.
3. $f_t^r(x_0) = x_0$ and $f_t^r(x_0) \neq x_0$ for $i = 1, \ldots, p - 1$. 
4. The map $f_1^p$ is given near $x_0$ by an arbitrarily prescribed formula $\phi$ with the property $\phi(x) = x \iff x = x_0$.

5. The orbit $x_0, f_1(x_0), \ldots, f_1^{p-1}(x_0)$ is isolated in $\text{Fix}(f_1^p)$. \hfill $\Box$

**Proposition 3.6** The map $f_1$ in Theorem 3.5 can realize an arbitrarily prescribed $DD^m(p|r)$ sequence $\tilde{c}_n$ on the created $p$-orbit $O$, i.e. $f_1$ can be smooth in a neighborhood of $O$ and $\tilde{c}_n = \text{ind}(f_1^n, O)$.

**Proof.** The statement may be obtained by a slight modification of the proof of Creating Procedure from [Je2]. Namely, it results from the fact that $f_1$ may be chosen in such a way that $f_1(x) = \Phi(x) \text{ near } O$, where $\Phi$ has the form from the formula (3) in Lemma 2.10, i.e. $\phi_i$ are some fixed diffeomorphisms and $\phi$ is an arbitrary smooth map that realizes any $DD^m(1)$ sequence (for more details cf. [Je2]).

Now we take as $\phi$ a map that realizes $DD^m(1|\tilde{c}_p)$ sequence $c_n = \frac{1}{p}\tilde{c}_np$ and by Lemma 2.10 obtain that:

$$\text{ind}(f_1^n, O) = \text{ind}(\Phi^n, O) = \tilde{c}_n.$$

The next lemma makes possible to cancel subsets of periodic points which have indices of iterations equal to zero.

**Lemma 3.7** (Cancelling Lemma, [Je2] Lemma 5.2) Let $f$ be a continuous self-map of $M$. Suppose that $S \subset \text{Fix}(f^r)$ satisfies:

1. $S$ is finite and $f$-invariant i.e. $f(S) = S$.
2. $\text{Fix}(f^n) \setminus S$ is compact.
3. $\text{ind}(f^n, \text{Fix}(f^n) \setminus S) = 0$ for all $n|r$.

Then there is a homotopy $f_t$, starting from $f_0 = f$, constant near $S$ and such that $\text{Fix}(f_1^1) = S$. \hfill $\Box$

Now we are in a position to prove the main result of this section.

**Theorem 3.8** (Main Theorem). Let $M$ be a smooth compact connected and simply-connected manifold of dimension $m \geq 3$ and $r \in \mathbb{N}$ a fixed number, then

$$\min\{\#\text{Fix}(g^r) : g \text{ is } C^1 \text{ and is homotopic to } f\} = D^m_r[f].$$

In other words $D^m_r[f]$ is the best lower bound for $\#\text{Fix}(g^r)$ for smooth $g$ in the homotopy class of $f$. \hfill $\Box$
Proof.

The fact that the minimum is greater or equal to $D^m_r[f]$ follows from Lemma 3.3. Let us decompose

$$L(f^n) = \sum_{i=1}^{s} c_i(n),$$

(5)

where $n|r$, $c_i$ is a $DD^m_r(l_i|r)$ sequence and the sum $\sum_{i=1}^{s} l_i$ is minimal i.e. $D^m_r[f] = \sum_{i=1}^{s} l_i$. We have to show that $D^m_r[f]$ is reachable by some map in the homotopy class of $f$ i.e. find a smooth map $f_1$ homotopic to $f$ with $Fix(f_1^s) = \sum_{i=1}^{s} l_i$.

First we use Creating Procedure and Proposition 3.6, deforming $f$ to $\bar{f}$ in such a way that we subordinate to every $i \in \{1, \ldots, s\}$ an isolated orbit $O_i$ of the length $l_i$ for which $\text{ind}(\bar{f}^n, O_i) = c_i(n)$. Each $c_i$ is a $DD^m_r(l_i|r)$ sequence, so by Proposition 3.6 $\bar{f}$ is smooth near the created orbits.

If the boundary of $M$ is nonempty we take $O_i$ inside $M$. We denote $S = \bigcup_{i=1}^{s} O_i$. We will show that there is a deformation of $\bar{f}$, rel. a neighborhood of $S$, to a smooth map $f_1$ satisfying $Fix(f_1^s) = S$. Then $f_1$ is the desired map, since

$$\#Fix(f_1^s) = \#S = \sum_{i=1}^{s} l_i = D^m_r[f].$$

It remains to perform the above deformation of $\bar{f}$. We will apply Lemma 3.7. We check that the assumptions of this lemma are satisfied. $S$, as the sum of orbits, is $\bar{f}$-invariant. Since $S$ is isolated, $Fix(\bar{f}^n) \setminus S$ is compact. Now we notice that for each $n|r$:

$$\text{ind}(\bar{f}^n) = \text{ind}(\bar{f}^n, Fix(\bar{f}^n) \setminus S) + \text{ind}(\bar{f}^n, S)$$

$$= \text{ind}(\bar{f}^n, Fix(\bar{f}^n) \setminus S) + \sum_{i=1}^{s} \text{ind}(\bar{f}^n, O_i)$$

$$= \text{ind}(\bar{f}^n, Fix(\bar{f}^n) \setminus S) + \sum_{i=1}^{s} c_i(n) = \text{ind}(\bar{f}^n, Fix(\bar{f}^n) \setminus S) + \text{ind}(f^n),$$

where the last equality results from the equation (5) and from the fact that Lefschetz number and global index coincide.
Thus
\[ \text{ind}(\bar{f}_n) = \text{ind}(\bar{f}_n, \text{Fix}(\bar{f}_n) \setminus S) + \text{ind}(f^n). \]

Since \( \text{ind}(\bar{f}_n) = \text{ind}(f^n) = L(f^n), \text{ind}(\bar{f}_n, \text{Fix}(\bar{f}_n) \setminus S) = 0 \), so the third assumption is also satisfied. The map \( f_1 \) is smooth in some neighborhood \( W \) of \( S \) and \( f_1^r \) has no fixed points outside \( W \). Thus, if \( f_1 \) is not smooth as the global map, we may easily approximate it by a smooth map constantly equal to \( f_1 \) on \( W \) without adding any new \( r \)-periodic points in the compact set \( M \setminus W \).

\[ \square \]

**Remark 3.9** Let us consider the set
\[ \text{MP}_r(f) = \min \{ \#P_r(g) : g \text{ is homotopic to } f \}, \]
where \( P_r(g) \) denotes the set of periodic points of \( g \) with the minimal period \( r \). In the simply-connected case we have, by Remark 3.1, that \( \text{MP}_r(f) = 0 \) for \( r > 1 \). If we define its smooth counterpart:
\[ \text{MP}^s_r(f) = \min \{ \#P_r(g) : g \text{ is smooth and homotopic to } f \}, \]

then it follows from the proof of Theorem 3.8 that the same statement holds: \( \text{MP}^s_r(f) = 0 \) for \( r > 1 \). Namely, we may find smooth \( g \) homotopic to \( f \) such that all its \( r \)-periodic points are fixed points (which however, does not imply that \( \text{Fix}(g^r) = D^m_m(f) \)).

The problem of determining the least number of \( \text{Fix}(f^r) \), i.e. \( D^m_m(f) \), reduces to the decomposition of the sequence of Lefschetz numbers \( \{L(f^n)\}_{n|r} \) into the sum of some elementary sequences i.e. \( DD^m(m|r) \) sequences. This becomes a combinatorial task, once we know what they are. In the next section we will use the complete description of \( DD^3(1) \) sequences from \([GNP1]\) to give explicit formulae for the least number of periodic points in dimension 3.

**4 3-dimensional case**

First of all we remind the reader of the result from \([GNP1]\) (cf. Theorem 4.1 below) which describes all \( DD^3(1) \) sequences (and thus, by Lemma 2.10, all \( DD^3(p) \) sequences).

**Theorem 4.1** There are seven kinds of \( DD^3(1) \) sequences:
(A) \( c_A(n) = a_1\text{reg}_1(n) + a_2\text{reg}_2(n) \),
(B) \( c_B(n) = \text{reg}_1(n) + a_d\text{reg}_d(n) \),
\((C) \ c(n) = -\text{reg}_1(n) + a_d\text{reg}_d(n),\)
\((D) \ c(n) = a_d\text{reg}_d(n),\)
\((E) \ c(n) = \text{reg}_1(n) - \text{reg}_2(n) + a_d\text{reg}_d(n),\)
\((F) \ c(n) = \text{reg}_1(n) + a_d\text{reg}_d(n) + a_2\text{reg}_2d(n), \text{ where } d \text{ is odd},\)
\((G) \ c(n) = \text{reg}_1(n) - \text{reg}_2(n) + a_d\text{reg}_d(n) + a_2\text{reg}_2d(n), \text{ where } d \text{ is odd}.\)

In all cases \(d \geq 3\) and \(a_i \in \mathbb{Z}.\)

**Proposition 4.2** Each \(DD^3(p|r)\) sequence is also \(DD^m(p|r)\) sequence (for \(m \geq 3\), as a consequence \(D^3_r[\xi] \geq D^m_r[\xi].\)

**Proof.** Let \(\{c_n\}_{n|r}\) be a \(DD^3(p|r)\) sequence, and let \(f\) be a map whose existence is guaranteed by Definition 2.9. Then \(\{c_n\}_{n|r}\) may be realized as the sequence of indices of the map \(f \times g\), where \(g: \mathbb{R}^{m-3} \to \mathbb{R}^{m-3}\) is a sink type map for which \(\text{ind}(g^n, 0) = 1\), for example \(g(x) = \frac{1}{2} x.\)

**Lemma 4.3** ([BaBo] Corollary 3.6) Let \(f\) be a \(C^1\) self-map of a smooth \(m\)-dimensional manifold and let \(x_0\) be a fixed point of \(f\). Then the basic set \(B(f, x_0)\) for the periodic expansion of indices of iterations at \(x_0\) satisfies:

\[\#B(f, x_0) \leq 2^{\left\lfloor \frac{m+1}{2} \right\rfloor}.\]

The same inequality holds for each orbit of periodic points.

**Corollary 4.4** Let \(O_{(p)} = \{x_1, \ldots, x_p\}\) be an \(p\)-orbit of a smooth map \(f: M \to M\) (dim \(M = m\). By \(B(f, O_{(p)})\) we denote \(B(\{\text{ind}(f^n, O_{(p)})\}_{n=1}^{\infty})\) (cf. Definitions 2.4 and 2.8). Let us consider the map \(f^p\). Notice that

\[\{\text{ind}((f^p)^n, x_1)\}_{n=1}^{\infty} = \sum_{k \in B(f^p, x_1)} a_k\text{reg}_k(n)\]

implies

\[\{\text{ind}(f^n, O_{(p)})\}_{n=1}^{\infty} = \sum_{s \in B(f, O_{(p)})} a_{s/p}\text{reg}_s(n)\]

and \(B(f, O_{(p)}) = pB(f^p, x_1) = \{pk : k \in B(f^p, x_1)\}\) (cf. [BaBo]). Thus \(\#B(f, O_{(p)}) = \#B(f^p, x_1)\). As a consequence by Lemma 4.3 applied to \(f^p\) we get:

\[\#B(f, O_{(p)}) \leq 2^{\left\lfloor \frac{m+1}{2} \right\rfloor}.\]
Theorem 4.5 Let $f$ be a self-map of a smooth compact simply-connected $m$-dimensional manifold. Then:

$$D^m_r[f] \geq \frac{\#B_r(f)}{2^{\left\lfloor \frac{m+1}{2} \right\rfloor}}.$$ 

Proof. Each basic sequence $reg_k$ which appears with a non-zero coefficient in the periodic expansion of Lefschetz numbers of $f$ must also appear in expansions of some periodic orbits of $f$, thus:

$$B_r(f) \subset \bigcup_{\{O(p) : p|r\}} B_r(f, O(p)).$$

As a consequence, by Corollary 4.4:

$$(6) \quad \#B_r(f) \leq \left( \sum_{p|r} \#\text{Orb}_p(f) \right) \cdot 2^{\left\lfloor \frac{m+1}{2} \right\rfloor},$$

where $\text{Orb}_p(f)$ denotes the set of all $p$-orbits of $f$.

Assume now that $g$ is a smooth map homotopic to $f$ for which $\#\text{Fix}(g^r)$ is minimal. We get: $B_r(f) = B_r(g)$, next we apply the inequality (6) for $g$ and obtain:

$$\frac{\#B_r(f)}{2^{\left\lfloor \frac{m+1}{2} \right\rfloor}} = \frac{\#B_r(g)}{2^{\left\lfloor \frac{m+1}{2} \right\rfloor}} \leq \sum_{p|r} \#\text{Orb}_p(g) \leq \sum_{p|r} \#\text{Orb}_p(g) \cdot p = D^m_r[f].$$

This ends the proof. □

Corollary 4.4 leads to the following observation:

Remark 4.6 In the definition of $D^m_r[f]$ it is enough to consider only $DD^m(p|r)$ sequences such that $p < 2^{\left\lfloor \frac{m+1}{2} \right\rfloor}$. In fact, by Corollary 4.4 any $DD(p|r)$ sequence, giving the contribution $p$ to $D^m_r[f]$, can be replaced with at most $2^{\left\lfloor \frac{m+1}{2} \right\rfloor}$ expressions of the type (D) (so $DD^m(1|r)$ sequences). Notice that, in particular for $m = 3$, we have $p < 2^2$, hence $p \leq 3$. □

Definition 4.7 Let us list three $DD^3(2)$ sequences which come from $DD^3(1)$ sequences of the form (E), (F) and (G):

(E') $c_{E'}(n) = \text{reg}_2(n) - \text{reg}_d(n) + a_{2d}\text{reg}_{2d}(n)$, where $d \geq 3$,

(F') $c_{F'}(n) = \text{reg}_2(n) + a_{2d}\text{reg}_{2d}(n) + a_{4d}\text{reg}_{4d}(n)$, where $d \geq 3$ is odd,

(G') $c_{G'}(n) = \text{reg}_2(n) - \text{reg}_4(n) + a_{2d}\text{reg}_{2d}(n) + a_{4d}\text{reg}_{4d}(n)$, where $d \geq 3$ is odd.
Let us emphasize that the contribution to $D^3_r[f]$ of each sequence (A)-(G) is 1, while of each (E')-(G') is 2.

**Lemma 4.8** Let $f : M \to M$ be a $C^1$ map, dim $M = 3$, then in the definition of $D^3_r[f]$ it is enough to consider only $DD^3(1|r)$ sequences i.e. sequences which, for $n|r$, are of the forms (A)-(G); and $DD^3(2|r)$ sequences of the forms (E'), (F') and (G').

**Proof.** By Remark 4.6 it is enough to use only $DD^3(p|r)$ sequences with $p \leq 3$. Let us now consider a $DD^3(3|r)$ sequence. If it comes from one of $DD^3(1|r)$ sequences of the type (A) - (F), then the basic set $B_r(f, \mathcal{O})$ for the sequence of indices of its realization on some 3-orbit has no more than 3 elements, thus we can replace it by at most three $DD^3(1|r)$ sequences of the types (D) or (A). If the $DD^3(3|r)$ sequence comes from (G), then it has the expansion: $g(n) = \text{reg}_3(n) - \text{reg}_6(n) + a_d \text{reg}_{3d}(n) + a_{2d} \text{reg}_{6d}(n)$, where $d$ is odd. It can be replaced by three $DD^3(1|r)$ sequences, namely: $c_1^1(n) = \text{reg}_1(n) + a_d \text{reg}_{3d}(n) + a_{2d} \text{reg}_{6d}(n)$,
$c_2^1(n) = \text{reg}_1(n) + \text{reg}_3(n) - \text{reg}_6(n)$,
$c_3^1(n) = -2\text{reg}_1(n)$.

Finally, $DD^3(2|r)$ sequence which comes from one of the $DD^3(1|r)$ sequences of the types (A) - (D) may be replaced by two $DD^3(1|r)$ sequences of the types (A) or (D).

4.1 Formula for $D^3_r[f]$

In this section we add the assumption that $M$ is 3-dimensional, thus we consider $f$, a self-map of $M$, where $M$ is a smooth compact connected simply-connected 3-manifold. We will give a formula for the least number of periodic points, up to $r$ periodic, for a smooth map homotopic to $f$. By Theorem 3.8 the number is equal to $D^3_r[f] = D^3_r[\{L(f^n)\}_{n|r}]$.

We will apply Lemma 4.8 which allows us, during the calculation of $D^3_r[f]$, to use only sequences (A)-(F), (E'), (F') and (G').

Let $r$ be a fixed natural number. The sequence of Lefschetz numbers $\{L(f^n)\}_{n|r}$ satisfies Dold relations, hence it can be written in the form of a periodic expansion, for $n|r$:

\begin{equation}
L(f^n) = b_1\text{reg}_1(n) + b_2\text{reg}_2(n) + b_4\text{reg}_4(n) + \sum_{k \in G} b_k \text{reg}_k(n),
\end{equation}

where $b_1, b_2, b_4$ are arbitrary integers,

\[ G = \{k \in \mathbb{N} : k \notin \{1, 2, 4\} \text{ and } b_k \neq 0\} = B_r(f) \setminus \{1, 2, 4\}. \]
Let us define $H$, the subset of $G$:

$$H = \{ k \in G : k \text{ is odd and } b_k \neq 0, \ b_{2k} \neq 0 \}.$$  

**Definition 4.9** We will say that a sequence $\psi$ of one of the types (A)-(G), (E')-(G') reduces a basic sequence $b_k$ in the periodic expansion of Lefschetz numbers (7) if $k \in B_r(\psi)$ and $a_k = b_k$, i.e. $b_k$ appears in the periodic expansion of $\psi$.

**Theorem 4.10** If $r$ is odd, then:

(*) \[ D^3_r[f] = \begin{cases} \#G & \text{if } |L(f)| \leq \#G, \\ \#G + 1 & \text{otherwise.} \end{cases} \]

If $r$ is even and $r > 4$, then:

(**) \[ D^3_r[f] \in [\#G - \#H, \#G - \#H + 2]. \]

**Proof.** Let $r$ be odd so $b_2$ and $b_4$ are not needed and the sequences of Definition 4.7 are not relevant. Then, seeking for the exact value of $D^3_r[f]$, it is enough to use only the sequences $c_i(n)$ of the types (A)-(D). Each of the sequences (B), (C), (D) reduces exactly one basic sequence $b_k$ in the periodic expansion of $\{L(f^n)\}_{n|r}$ for $k \geq 3$. Thus we need at least $\#G$ basic expansions to get the equality (7):

(8) \[ L(f^n) = c_1(n) + ... + c_{\#G}(n) \]

for $n|r$, $n > 1$.

On the other hand notice that to get the above equality for $n > 1$ we can use $t_B, t_C, t_D$ sequences of the types (B), (C), (D) respectively, where $t_B, t_C, t_D$ are prescribed non-negative integers satisfying $t_B + t_C + t_D = \#G$. Since the contribution of each of these sequences to $b_1 = L(f)$ is $+1, -1, 0$ respectively, we may choose them to get the equality (8) also for $n = 1$ if and only if $-\#G \leq L(f) \leq \#G$. Otherwise, i.e. for $|L(f)| > \#G$, we have to use one additional sequence of the type (A) with $a_2 = 0$ to get the equality for $n = 1$. This ends the proof of part (*).

Now we assume that $r$ is even. We need at least $\#G - \#H$ basic expansions to get the equality (7) for $r \neq 2, 4$. In fact, only sequences (F) and (G) reduce two $b_k$’s, $k \neq 1, 2, 4$, and we may use them
in such a way at most \( \#H \) times ((F') and (G') also reduces two \( \text{reg}_k \)'s but each of them is counted twice). To reduce the remaining \( \text{reg}_k \)'s we will need \( \#G - 2\#H \) of them. Thus \( D_3^3(f) \geq \#H + (\#G - 2\#H) = \#G - \#H \). On the other hand notice that we may always realize \( \{L(f^n)\}_{n|r} \) by \( \#H \) sequences (F), \( \#G - 2\#H \) sequences (D), one sequence (A) with the coefficients \( a_1 = b_1 - \#H, a_2 = b_2 \) and one sequence (D) with \( d = 4 \) and \( a_4 = b_4 \). Thus \( D_3^3(f) \leq \#G - 2\#H + 2 \).

Now we consider some particular cases, which are not included in Theorem 4.10

**Proposition 4.11** 1. If \( \{L(f^n)\}_{n|r} \) is constantly equal to 0, then \( D_3^3[f] = 0 \).

Assume that \( \{L(f^n)\}_{n|r} \) is not constantly equal to 0, then

2. \( D_2^3(f) = 1 \).

3. \( D_4^3(f) = \begin{cases} 1 & \text{if (a) or (b) or (c) is satisfied,} \\ 2 & \text{otherwise,} \end{cases} \)

where
(a) \( L(f) = 1 \) and \( L(f^2) = -1 \),
(b) \( L(f^4) = L(f^2) \),
(c) \( L(f^2) = L(f) \) and \( L(f) \in \{-1, 0, 1\} \).

**Proof.**

1. \( L(f^n) \equiv 0 \) iff the sequence \( \{L(f^n)\}_{n|r} \) is the sum of zero basic sequences iff \( D_3^3[f] = 0 \).

2. Each nonzero sequence of two elements \( b_1, b_2 \) is of the type (A).

3. Let us find when the sequence of three elements \( b_1 = L(f), b_2 = L(f^2) - L(f), b_4 = \frac{L(f^4) - L(f^2)}{4} \) is one of the types (A)-(G):
   (A) \( \iff b_4 = 0 \), which gives the condition (b).
   (B), (C) or (D) \( \iff \langle |b_1| \leq 1 \text{ and } b_2 = 0 \rangle \iff \langle |L(f)| \leq 1 \text{ and } L(f) = L(f^2) \rangle \) which gives the condition (c).
   (E) \( \iff (b_1 = 1 \text{ and } b_2 = -1) \iff (L(f) = 1 \text{ and } L(f^2) = -1) \)
   which leads to condition (a).
   (F) and (G) are reduced to (A).

It remains to notice that each sequence \( b_1, b_2, b_4 \) is the sum of two sequences of the types (A), (D) which implies \( D_3^3(f) = 2 \) in the remaining cases. \( \square \)
**Corollary 4.12** Let $r = 1$. As $D^3_1[f] \leq 1$, each map $f$ is homotopic to a smooth map with (at most) one fixed point, in other words $\text{MP}_1(f) \leq 1$ (compare Remarks 3.9 and 3.2 for $r = 1$).

Let us remind the reader that by $\zeta(r)$ we denote the number of all divisors of $r$.

**Corollary 4.13** Let $r \in \mathbb{N}$. Each map $f$ is homotopic to a smooth map $g$ with

$$\frac{\# B_q(f)}{2^{\frac{m+1}{2}}} \leq \# \text{Fix}(g^r) \leq \zeta(r).$$

**Proof.** We begin with the proof of the second inequality. By the definition, $G = B_r(f) \setminus \{1, 2, 4\}$, on the other hand $B_r(f)$ is the number of basic sequences $b_k \text{reg}_k$ in the expansion of Lefschetz numbers and thus $\# B_r(f) \leq \zeta(r)$. By Proposition 4.2 and Theorem 4.10 $D^m_r[f] \leq D^3_r[f] \leq \zeta(r)$. Finally we apply Theorem 3.8 and get the desired inequality. The first inequality in Corollary results from Theorem 4.5 and Theorem 3.8. \(\square\)

**Corollary 4.14** Let $r = q$ be a prime number. Then the map $f$ is homotopic to a smooth map $g$ with $\# \text{Fix}(g^r) \leq 1$ if and only if at least one of the conditions holds:

1. $q = 2$,
2. $|L(f)| \leq 1$,
3. $L(f^q) = L(f)$.

**Proof.** For $q = 2$ the equivalence results from part 2 of Proposition 4.11. Let us consider now prime $q > 2$. By Corollary 4.13 $D^3_r[f] \leq 2$. We must find all conditions in Theorem 4.10 part (\ast) under which $D^3_r[f] \leq 1$. We get that $D^3_r[f] \leq 1$ is equivalent to the following alternative: either $\#G \leq 1$ and $L(f) \in [−\#G, \#G]$, or $\#G = 0$. The first possibility leads to $|L(f)| \leq 1$ and the second, by the definition of $G$, forces $b_q = \frac{L(f^q) - L(f)}{2} = 0$. \(\square\)

### 4.2 Calculation of $D^3_r[f]$ for self maps of $S^2 \times I$

As in the previous sections we assume that $M$ is a smooth compact connected and simply-connected 3-dimensional manifold. Now, assume that additionally $M$ has non-empty boundary. We remind the reader that for a manifold with boundary we consider self-maps which
have no periodic points on the boundary. We take homology with the coefficients in the field of rational numbers \( \mathbb{Q} \).

Notice that connectivity of \( M \) implies that \( H_0(M) = \mathbb{Q} \), furthermore \( H_1(M) = 0 \) as \( M \) is simply-connected. Because \( M \) is a manifold with boundary, \( H_3(M) = 0 \). Thus the homology spaces of \( M \) have the following form: \( H_0(M) = \mathbb{Q} \), \( H_1(M) = 0 \), \( H_2(M) = \mathbb{Q}^i \), \( H_3(M) = 0 \). This is equivalent that the boundary of \( M \) consists of \( i + 1 \) pairwise disjoined 2-spheres (cf. [Gr]).

In this section we will give the complete description of the least number of periodic points of smooth maps, up to homotopy, in case \( i = 1 \), i.e. for \( M = S^2 \times I \), the only connected and simply-connected 3-manifold with boundary and \( H_2(M) = \mathbb{Q} \).

We define global Dold coefficients \( i_n(f) \) by the formula:

\[
i_n(f) = \sum_{k|r} \mu(n/k)L(f^k).
\]

Recall that \( i_n(f)/n = b_n(f) \), where \( b_n(f) = b_n \) is the \( n \)th coefficient of the periodic expansion of the sequence \( \{L(f^n)\}_{n=1}^{\infty} \) in the formula (7) (cf. also Theorem 2.3).

Let \( f : M \to M \). Then homomorphism \( f_2 : H_2(M; \mathbb{Q}) \to H_2(M; \mathbb{Q}) \) is multiplication by a number \( \beta \in \mathbb{Z} \) (called the degree of \( f \)). Assume \( r \in \mathbb{N} \) is fixed. Then, finding the least cardinality of \( \text{Fix}(g^r) \), where \( g \) is a smooth map homotopic to \( f \), is equivalent to determining the number \( D^3_r[f] = D^3_r[\{L(f^n)\}_{n|r}] \). We will need the exact form of the basic set \( B_r(f) \) (cf. Definitions 2.4 and 2.8). This knowledge is provided by the following result proved in [LPR].

**Lemma 4.15** ([LPR] Theorem 1.2)

(a) \( b_1(f) = 1 + \beta \).

(b) \( b_2(f) = 0 \) if and only if \( \beta \in \{0, 1\} \).

(c) If \( n > 2 \), then \( b_n(f) = 0 \) if and only if \( \beta \in \{-1, 0, 1\} \).

**Corollary 4.16** If \( |\beta| > 1 \), then \( B(f) = \mathbb{N} \), so \( B_r(f) = \{n \in \mathbb{N} : n|\beta\} \), or equivalently \( \#B_r(f) = \zeta(r) \), where \( \zeta(r) \) is the number of all divisors of \( r \).

Under the assumption that \( |\beta| > 1 \) Corollary 4.16 implies:

**Corollary 4.17** For \( r \) odd \( G = B_r(f) \setminus \{1\} \), thus \( \#G = \zeta(r) - 1 \).
Corollary 4.18 For $r$ even there is:
if $4|r$, then $G = B_r(f) \setminus \{1, 2, 4\}$, thus $\#G = \zeta(r) - 3$,
if $4 \not| r$, then $G = B_r(f) \setminus \{1, 2\}$, thus $\#G = \zeta(r) - 2$. □

Notice that $\#H$ is the number of pairs $k, 2k$, where $k|r$, $k > 1$ is odd, in $G$. By Lemma 4.15 each $n$, such that $n|r$, belongs to $G$, thus we obtain for even $r$:

Corollary 4.19 $\#H = \eta(r) - 1$, where $\eta(r)$ is the number of all odd
divisors of $r$.

Taking into account that $L(f) = 1 + \beta$ we get, by Theorem 4.10
and Corollary 4.17, the following description of $D^3_r[f]$ for odd $r$ and
all $f$ with the absolute value of the degree greater than $1$:

Theorem 4.20 If $r$ is odd, $|\beta| > 1$ then:

$$D^3_r[f] = \begin{cases} 
\zeta(r) - 1 & \text{if } -\zeta(r) \leq \beta \leq \zeta(r) - 2, \\
\zeta(r) & \text{otherwise.} 
\end{cases}$$

Example 4.21 Let $f : M \to M$ have the degree $\beta \not\in \{-1, 0, 1\}$ and
$r = 15$. The Lefschetz numbers $L(f^n)$ for $n|15$ are: $1 + \beta$, $1 + \beta^3$,
$1 + \beta^5$, $1 + \beta^{15}$. We represent this sequence (for $n = 1, 3, 5, 15$) in the
form of a periodic expansion.

(9) $L(f^n) = (1 + \beta)\text{reg}_1(n) + \left(\frac{\beta^3 - \beta}{3}\right)\text{reg}_3(n) + \left(\frac{\beta^5 - \beta}{5}\right)\text{reg}_5(n)$

$$+ \left(\frac{\beta^{15} - \beta^5 - \beta^3 + \beta}{15}\right)\text{reg}_{15}(n).$$

Now we apply Theorem 4.20: $\zeta(15) = 4$, $r = 15$ is odd, hence:

$$D^3_{15}[f] = \begin{cases} 
3 & \text{for } \beta \in \{-4, -3, -2, 2\}, \\
4 & \text{for } \beta < -4 \text{ or } \beta > 2. 
\end{cases}$$

Proposition 4.2 allows us to derive some additional information in
the case of higher dimensional manifolds.
Example 4.22 Let $f$ be a self-map of a smooth compact connected and simply-connected $m$-dimensional manifold $\tilde{M}$, such that $\{L(f^n)\}_{n=15}$ is the same as in Example 4.21, i.e. $L(f^n)$ are expressed by the right hand-side of the formula (9). By Proposition 4.2 there is a $C^1$ map $g$ homotopic to $f$ such that $\text{Fix}(g^{15})$ has at most 3 points if $\beta \in \{-4, -3, -2, 2\}$ and at most 4 points if $\beta < -4$ or $\beta > 2$. Moreover, if $m = 3$ then $\text{Fix}(g^{15})$ cannot have less points than this number. 

Now we analyze the case of $r$ even.

Theorem 4.23 Assume that $r$ is even and $|\beta| > 1$. Then

$$D^3_r[f] = \zeta(r) - \eta(r).$$

Proof. For $r = 2, 4$ the thesis follows from Proposition 4.11. Assume $r > 4$. First we consider the case $4 | r$.

By Corollaries 4.18 and 4.19, we may replace in the thesis $\zeta(r) - \eta(r)$ by $#G - #H + 2$. On the other hand, by Theorem 4.10, $D^3_r[f] \in [#G - #H, #G - #H + 2]$.

Let

$$L(f^n) = b_1\text{reg}_1(n) + b_2\text{reg}_2(n) + b_4\text{reg}_4(n) + \sum_{k \in G} b_k\text{reg}_k(n) = \sum_i c_i(n),$$

(10)

where $c_i$ are $DD^3(p_i|r)$ sequences which give minimal decomposition of Lefschetz numbers of iterations, i.e. $\sum_i p_i = D^3_r[f]$. We remind the reader that by Lemma 4.8 $c_i$ have one of the forms (A)-(G), (E')-(G') (thus $p_i \leq 2$).

Suppose that:

$$D^3_r[f] = #G - #H.$$  

(11)

We show that this cannot happen. The equation (11) forces that the coefficients $b_1, b_2, b_4$, of the periodic expansion of $\{L(f^n)\}_{n|r}$, must be obtained as a sum of coefficients $a_1, a_2, a_4$, respectively; of $#G - #H$ sequences (B)-(G) and (E')-(G'), where the last are counted twice and where each involve $a_d\text{reg}_d$ with $d \in G$. This is impossible because any sum of such sequences gives always non-positive contribution to $b_4$ ((E') and (G') allow to produce $-\text{reg}_4$), while $b_4 = \frac{\beta_1^4 - \beta_2^4}{4} > 0$. As a result we must use an additional sequence, say $c_\alpha$, of the type (B),
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(C), (D), or (E), with the coefficient $a_4 \neq 0$, which makes possible to reduce $b_4 \text{reg}_4$.

Thus we get:

$$D^3_r[f] \in [\#G - \#H + 1, \#G - \#H + 2].$$

Suppose now that $D^3_r[f] = \#G - \#H + 1$.

As we require minimal realization, (E') is not valid, because it reduces only one basic sequence $b_k \text{reg}_k, k \geq 6$, but is counted twice. By the same reason (F'), (G') could be used only under the condition that each of them reduces two basic sequences. Now we show that (F'), (G') may always be replaced in the minimal realization by sequences of the types (A)-(G). For odd $d (d > 1, 4d|\#r)$ we consider a triple:

$$b_d \text{reg}_d(n) + b_2d \text{reg}_{2d}(n) + b_4d \text{reg}_{4d}(n),$$

which appears in the periodic expansion of Lefschetz numbers of iterations. By Lemma 4.15 each coefficient near the basic sequences in the formula (13) is nonzero. Notice that (F'), (G') may be used only if each of them reduces a part of such triple (two last terms). Then we have to use one sequence (B)-(E) to reduce $b_d \text{reg}_d$, which gives the contribution $2 + 1 = 3$ to $D^3_r[f]$. On the other hand, we may use one (F) or (G) to reduce two first terms and one sequence (B)-(E) to reduce the last and one sequence (A) with $a_1 = b_1$ and $a_2 = b_2$, getting the same contribution $1 + 1 + 1 = 3$, independently of the values of $b_1$ and $b_2$. Thus in the minimal realization, we may consider only sequences (A)-(G).

We need at least $\#G - \#H$ sequences (B)-(G) to reduce $b_k \text{reg}_k$ for $k \neq 1, 2, 4$ (cf. also proof of Theorem 4.10), and one $c_a$ of the type (B)-(E) to reduce $b_4 \text{reg}_4$. Finally, we need also one sequence of the type (A) to reduce $b_2 \text{reg}_2$ with $b_2 = \frac{\alpha^2 - \beta}{2} > 0$, because (B)-(G) have coefficients $a_2$ non-positive. As a result $D^3_r[f] = \#G - \#H + 2$.

Now consider the case $4 \nmid \#r$. By Corollary 4.17 and the second part of Corollary 4.18 we know that $\zeta(\#r) - \eta(\#r) = \#G - \#H + 1$. Thus, to finish the proof, it is enough to state that $D^3_r[f] = \#G - \#H + 1$.

Because $b_4 = 0$, we see that $D^3_r[f] \in [\#G - \#H, \#G - \#H + 1]$ (compare the proof of Theorem 4.10). On the other hand, if $D^3_r[f] = \#G - \#H$, then $b_2 > 0$ must be a sum of $a_2$-coefficients of $\#G - \#H$ sequences of minimal realization. This is impossible, because the only sequences which may be used to realize $b_2$ in such a way, are of the
types (E'), (F') or (G'), but in the case $4 \nmid r$ each of them reduces only one sequence (for $k \neq 1, 2$) in the expansion of $\{L(f^n)\}_{n|r}$, but is counted twice. This ends the proof. \hfill $\square$

**Proposition 4.24** In case $\beta \in \{-1, 0, 1\}$ $D^3_\beta[f] = 1$ as we can use one $DD^3(1|r)$ sequence of the type (A). Namely:

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>${L(f^n)}_{n=1}^\infty$</th>
<th>$D^3_\beta[f]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\text{reg}_1(n)$</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$2\text{reg}_1(n)$</td>
<td>1</td>
</tr>
<tr>
<td>$-1$</td>
<td>$\text{reg}_2(n)$</td>
<td>1</td>
</tr>
</tbody>
</table>

Theorems 4.20 and 4.23 together with Proposition 4.24 give the complete description of the least number of periodic points of smooth self-maps, up to homotopy, for $S^2 \times I$.

**References**


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