Additive combinatorics

Winter semester 2016/2017

SERIES II

2.1. Find $(I * I)$, $(I * I * I)$, $(I \circ I)$ and $(I * J)$ where $I = [N]$, $J = [M]$.

2.2. Let A, B be subsets of a finite abelian group. Show that there exists $x \in G$ such, that $(A * B)(x) \geq \frac{|A||B|}{|G|}$.

2.3. Prove that

$$\mathsf{E}(A, B) = \sum_{a \in A, b \in B} |(b + A) \cap (a + B)|.$$

2.4. Show that for every $x \in A + B$

$$(A * B)(x) \leq |(A - A) \cap (B - B)|.$$

In particular, $|A \pm B||(A - A) \cap (B - B)| \geq |A||B|$.

2.5. Put $P = \{x : (A * A)(x) \geq \varepsilon|A|\}$. Prove that $|P| \leq \varepsilon^{-1}|A|$.

2.6. Suppose that $(A \circ A)(x) \geq (1 - \varepsilon)|A|$ and $(A \circ A)(y) \geq (1 - \delta)|A|$. Show that $(A \circ A)(x - y) \geq (1 - \varepsilon - \delta)|A|$.

2.7. Let A be a finite subset of an abelian group and suppose for every $d \in A - A$ we have $(A \circ A)(d) > |A|/2$. Show that $A - A$ is a subgroup of G.

2.8. Give an example of a set A such that $(A \circ A)(d) \geq |A|/2$, but $A - A$ is not a subgroup.

2.9. Let $A_s = A \cap (A + s)$. Show that

$$((A - A) \circ (A - A))(s) \geq |A - A_s|, \quad \text{and} \quad ((A + A) \circ (A + A))(s) \geq |A + A_s|.$$

2.10. Prove that $\mathsf{E}(A - A) \geq |A - A||A|^2$.

2.11.* Show that

$$\sum_s |A + A_s| \leq |A + A|^2.$$

2.12.* Show that $\sum_{s,t} \mathsf{E}(A_s, A_t) = \sum_x (A \circ A)(x)^4$.

2.13. Suppose that $\sum_{s \in S} (A \circ B)(s) \geq \varepsilon|B||S|$. Prove that

$$\varepsilon^2 |B||S|^2 \leq \mathsf{E}(A, S) \leq \mathsf{E}(A)^{1/2} \mathsf{E}(S)^{1/2}.$$

2.14.* Let $\varepsilon > 0$ and assume that $|A + B| \leq K|A|$. Show that there is a set X of size $O(K/\varepsilon)$ such that $|(X + A) \cap B| \geq (1 - \varepsilon)|B|$.

2.15. Let N be an odd number and let $A \subseteq [N]$ be a *sum-free set* of size $\frac{N+1}{2}$. Prove that $A = [(N+1)/2, N]$ or $A = (2\mathbb{N}+1) \cap [N]$.
Hint: Use 1.1.

2.16. What can you say on the size of a set $A \subseteq [N]$ such that:
a) $A - A$ does not contain a power of 2,
b) $A + A$ does not contain a power of 2.