Additive combinatorics

Winter Semester 2016/2017

SERIES I

Problem 1.1. Let $A, B \subseteq \mathbb{Z}$, $|A| = |B|$ and $|A + B| = |A| + |B| - 1$. Show that A and B are arithmetic progressions with the same common difference.

Problem 1.2. Provide an example of an arbitrary large set A such that $|A+A| = |A|^{\alpha}$, $|A-A| = |A|^{\beta}$ and $\alpha > \beta$.

Problem 1.3. Show that if $A, B \subseteq G$ and $|A| + |B| > |G|$, then $A + B = G$.

Problem 1.4.* Prove that if $|A + A| \leqslant K|A|$, then

$$|kA| \leqslant \binom{K^4 + k - 2}{k - 1} K^2 |A|.$$

Hint: Apply Ruzsa's covering lemma for $A = A$ and $B = 2A - A$.

Problem 1.5. Suppose that $|A + B| \leqslant K|A|$. Prove that for each $\varepsilon > 0$ there exists $X, |X| \geqslant (1 - \varepsilon)|A|$ such that $|X + kB| \leqslant (K/\varepsilon)^k |X|$ for every $k \geqslant 1$.

Problem 1.6. For every N give an example of sets $A, B \subseteq \mathbb{Z}^2$ such that $|A| \sim N^2$, $|B| \ll N$, $|A + B| \ll N^2$, $|A + 2B| \gg N^3$.
Hint: Take $B = \{(j, 0) : j \in [N]\} \cup \{(0, j) : j \in [N]\}$ and choose appropriately A.

Problem 1.7. Show that there exist arbitrary large sets S such that A such that $|A+A| = K|A|$ and $|3A| \sim K^3|A|$ (for every $K < |A|^{1/3}$).

Problem 1.8. Prove that $|A + 2B| \leqslant \frac{|A+A||A+B|^2}{|A|^2} \leqslant \frac{|A+B|^4}{|A|^2|B|}$.

Problem 1.9. Let $A = \{(x_1, \ldots, x_{2n}) : x_1 + \cdots + x_{2n} = n, \; x_i \geqslant 0\}$ and $B = \{e_1, \ldots, e_{2n}\}$. Show that $|A + B| \ll |A|$ and $|A - B| \geqslant n|A|$.

Problem 1.10. Show that for every $k \in \mathbb{N}$ there exist sets A, B, C, each of cardinality k with $|A - B| \sim k^2$, $|A - C| \sim k^{3/2}$ and $|B - C| \sim k^{3/2}$.

Problem 1.11. Let $A = \{(x_1, \ldots, x_d) \in \mathbb{Z}^d : \sum_i x_i \leqslant n, \; x_i \geqslant 0\}$. Find $|A|$ and limits $|A+A|/|A|$ i $|A - A|/|A|$ as $n \to \infty$ and for fix d.

Problem 1.12. Suppose that $|A-A| = K|A|$. Prove that for each k we have $|kA| \geqslant K^{1-2^{-k+1}}|A|$.

Problem 1.13. Check the following properties of *Dyson's transform*:

a) $A_x + B^x \subseteq A + B$,

b) $|A| + |B| = |A_x| + |B^x|$,

c) $A_x \setminus A = x + (B \setminus B^x)$.

Problem 1.14. Assume that $|A + A| = K|A|$. Show that for each $k \in \mathbb{N}$ we have

$$|A + 2^k \cdot A| \leqslant K^{3k}|A|.$$

Problem 1.15. Suppose that $|A + A| = K|A|$ and $|3A| = K^{3-\varepsilon}|A|$. Show that there is a set $X \subseteq A$ such that $|X| \geqslant |A|/K$ and for every k

$$|kX| \leqslant K^{\varepsilon k+3}|A| \leqslant K^{\varepsilon k+4}|X|.$$

Problem 1.16. Suppose that $|A + A| = K|A|$ and $|A + B_i| = K_i|A|$ for $i = 1, \ldots, k$. Show that

$$|B_1 + \cdots + B_k| \leqslant K^k K_1 \cdots K_k |A|.$$

Problem 1.17. Assume that $|A + A| = K|A|$. Show that for each l we have

$$|A + l \cdot A| \leqslant K^{3(\log_2 2l)^2}|A|.$$

Problem 1.18. Let $A \subseteq [N]$ be a set with at least $N/2 + 1$ elements. Prove that:
a) there exist $a, b \in A$ such that $(a, b) = 1$,
b) there exist $c, d \in A$ such that $c|d$,
c) there exist $e, f, g \in A$ such that $e + f = g$.

Problem 1.19. Suppose that $A \subseteq \mathbb{Z}/p\mathbb{Z}$ and A does not contain any solution to the equation $x + y = z$. Prove that $|A| \leqslant \frac{p+1}{3}$.