LOWER BOUNDS ON PROBABILITY THRESHOLDS FOR RAMSEY PROPERTIES

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Abstract. In this paper we survey recent results on Ramsey properties of random graphs and their deterministic consequences and counterparts. In addition to that, we present two proofs (one for complete graphs and one in general case) of the following result.

Theorem. *For every graph G which is not a star forest there exists a constant $c > 0$ such that if $p = cn^{-1/m_G^{(2)}}$, where $m_G^{(2)} = \max_{H \subseteq G, v_H > 2} \frac{e_H - 1}{v_H - 2}$, then*

$$\lim_{n\to\infty} P(K(n,p) \to (G)_2^2) = 0 \ .$$

The corresponding upper bound, establishing the existence of $C > 0$ such that

$$\lim_{n\to\infty} P(K(n,p) \to (G)_2^2) = 1 \quad \text{whenever} \quad p = Cn^{-1/m_G^{(2)}} \ ,$$

is proved elsewhere.

1.Introduction. Applications of probabilistic methods to Ramsey theory begun with the seminal paper of Erdős [Er 47], long before the theory of random graphs was born. The probabilistic method introduced in [Er 47] in order to set a lower bound on Ramsey numbers was already formulated in terms of a random graph. Over the years, random graphs have become a powerful tool in Ramsey theory, and the theory of random graphs itself has bloomed rapidly. While many graph theoretic concepts have been studied in the probability context , no paper focused directly on investigating Ramsey properties of random graphs. The excellent monograph on random graphs [Bo 85] contains entire chapter devoted to the applications to

The first author supported by NSF grant DUS-9011850. The second author on leave from Department of Discrete Mathematics, Adam Mickiewicz University, Poznań, Poland; supported by grant KBN 2 1087 9101

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Ramsey theory, but the question about Ramsey properties of random graphs in not raised there. Perhaps the first paper that includes both these aspects is [FR 86]. Already in the seventies, E. Szemerédi and the authors of [FR 86] observed that random graphs with n vertices and $cn^{3/2}$ edges can, after certain deletions, serve as examples of K_4-free graphs that arrow K_3 and have reasonably few vertices (see Section 4 for more on this subject). On their way they proved the following result. Let $K(n,p)$ be the binomial random graph obtained from the complete graph K_n by independent deletion of edges, each with probability $1-p$. The standard arrow notation $F \to (G)^i_r$, $i = 1, 2$, means that for every r-coloring of the vertices ($i = 1$) or edges ($i = 2$) of the graph F there is a subgraph of F isomorphic to G such that all its vertices (edges) are colored by the same color.

Theorem 1a. *There exists a constant C such that if $p = p(n) = Cn^{-1/2}$ then*

$$\lim_{n\to\infty} P(K(n,p) \to (K_3)^2_2) = 1 \ .$$

This is the best possible in the sense that, as proved in [ŁRV 92],

Theorem 1b. *There is a constant c such that if $p = p(n) = cn^{-1/2}$ then*

$$\lim_{n\to\infty} P(K(n,p) \to (K_3)^2_2) = 0 \ .$$

Such a sharp threshold behavior is a typical feature of random graphs.

A systematic treatment of Ramsey properties of random graphs appears first in [ŁRV 92]. In fact, it started back in 1986 after H. Lefmann coined that kind of question in a discussion with B. Voigt and the second author. The first goal was to find the threshold for the property $K(n,p) \to (G)^1_r$. It was already known that if $p = p(n)$ goes to zero slower than n^{-1/m_G}, where $m_G = \max_{H\subseteq G} \frac{e_H}{v_H}$ (here and throughout the paper e_G and v_G stand for the number of edges and vertices of a graph G) then almost surely (=with probability approaching 1 as $n \to \infty$) there is at least one copy of G in $K(n,p)$ (see [Bo 81]). But to enjoy the Ramsey property, $K(n,p)$ needs to contain at least about n copies of G. What order of p gives that?

It is not hard to see that when $p = n^{-1/m_G^{(1)}}$, where $m_G^{(1)} = \max_{H \subseteq G, v_H \geq 2} \frac{e_H}{v_H - 1}$, then the expected number of copies of every subgraph of G is of order at least n. And indeed, this turned out to be the right threshold. A matching is a graph whose edges are pairwise disjoint.

Theorem 2 [ŁRV 92]. *For every graph G which is not a matching and for every integer r there are constants C_G and c_G such that*

(a)
$$\lim_{n\to\infty} P(K(n, C_G n^{-1/m_G^{(1)}}) \to (G)_r^1) = 1$$

and

(b)
$$\lim_{n\to\infty} P(K(n, c_G n^{-1/m_G^{(1)}}) \to (G)_r^1) = 0 \ .$$

The proof of a strenghening of part (b) in a special case can be found in Section 2 below. A one-line proof of part (a) relies on the exponential bound from [JŁR 90, p.75]:

$$P(K(n,p) \not\supset G) \leq \exp\{-c\phi_n(G)\} \ ,$$

for some $c > 0$, where $\phi_n(G) = \min_{H \subseteq G} \mathrm{Exp}(X_H)$, the expected number of the "least likely" subgraph of G. (Here and throughout the paper X_H stands for the number of copies of H in $K(n,p)$.) In our case $\phi_n(G)$ is of the order of n and, noticing that each color class is G-free, we have

$$P(K(n,p) \not\to (G)_r^1) < 2^n P(K(n/r, p) \not\supset G) < 2^n e^{-C'n} = o(1) \ ,$$

for C_G big enough, where C' is a function of C_G.

In the edge coloring case, a reasonable prediction was that the threshold should be when the number of copies of the "least likely" subgraphs of G in $K(n,p)$ reaches the order of magnitude of the number of edges in $K(n,p)$. This happens exactly when $p = p(n) = n^{-1/m_G^{(2)}}$, where $m_G^{(2)} = \max_{H \subseteq G, v_H \geq 3} \frac{e_H - 1}{v_H - 2}$.

Recently we proved

Theorem 3. *For every graph G which is not a star forest, there exist constants C_G and c_G such that*

(a) $$\lim_{n\to\infty} P(K(n, C_G n^{-1/m_G^{(2)}}) \to (G)_2^2) = 1$$

and

(b) $$\lim_{n\to\infty} P(K(n, c_G n^{-1/m_G^{(2)}}) \to (G)_2^2) = 0 \ .$$

It is expected that the same threshold holds for an arbitrary number of colors. This is already confirmed in case of triangles (see [RR 94]).

The aim of this paper is to give a proof of part (b) of Theorem 3. The proof of part (a), which has a different flavor and is based on Szemerédi's regularity lemma, will appear elsewhere ([RR **]). For a class of graphs including complete graphs a strengthening of part (b) of Theorem 2 leads to a short proof of Theorem 3b. This will be shown in Section 2. Section 3 contains the proof of general case.

In addition to that, in Section 4 we discuss deterministic consequences of our results with respect to local and global densities of Ramsey graphs.

2. Lower Bound For Complete Graphs.

Here we prove part (b) of Theorem 3 for complete graphs K_k, $k \geq 4$, only. However, the same method can be applied to all graphs G for which $m_G^{(2)}$ is attained by a subgraph H with a vertex of degree v_H in H. Let us introduce a simplified notation $K(n,p) \to k$ for $K(n,p) \to (K_k)_2^2$ and $K(n,p) \stackrel{v}{\to} k$ for $K(n,p) \to (K_k)_2^1$. For complete graphs, Theorem 3b reads as follows.

Proposition 1. *For each $k \geq 3$ there exists $c > 0$ such that if $p = cn^{-\frac{2}{k+1}}$ then*

$$\lim_{n\to\infty} P(K(n,p) \to k) = 0 \ .$$

We will heavily rely on the following lemma which is a refinement of Theorem 2b in case $G = K_k$.

Lemma. *For all $k \geq 3$ there exists $a > 0$ such that if $p = an^{-\frac{2}{k}}$ then*

$$P(K(n,p) \xrightarrow{v} k) = O(n^{-\frac{k+2}{k}}) \ .$$

We shall prove the lemma at the end of this section.

Proof of Proposition 1.

For $k = 3$ Proposition 1 was already proved in [ŁRV 92], so throughout the proof we assume that $k \geq 4$.

For $\varepsilon > 0$, by Chernoff's inequality, a fixed vertex of $K(n,p)$ has degree within the range $(1 \pm \varepsilon)cn^{\frac{k-1}{k+1}}$, with probability $1 - o(\frac{1}{n})$. Denote by $N(v)$ the graph spanned in $K(n,p)$ by the neighbors of v and call v *bad* if $N(v) \xrightarrow{v} (k-1)$ and call it *good* otherwise. The expected number of bad vertices is $nP(N(1) \xrightarrow{v} k-1)$. Observe that a typical neighborhood is itself a random graph $K(m,p)$ with $m \asymp n^{\frac{k-1}{k+1}}$, and $p = p(m) \asymp m^{-\frac{2}{k-1}}$ (the symbol "$\asymp$" reads "asymptotically equal up to a constant factor"), to which our lemma can be applied with $k-1$ instead of k. Conditioning on $N(1)$ and using the lemma, we thus obtain

$$P(N(1) \xrightarrow{v} k-1) = o(1/n) + O((n^{\frac{k-1}{k+1}})^{-\frac{k+1}{k-1}}) = O(1/n) \ .$$

Hence, the expectation of bad vertices is $O(1)$ and, by Markov's inequality, almost surely, there are less than say, $\log\log n$ bad vertices in $K(n,p)$. On the other hand, for every $\varepsilon > 0$, almost surely, no subgraph H on $\log\log n$ or less vertices has its density $\frac{e_H}{v_H}$ bigger than $\frac{k+1}{2} + \varepsilon$. Indeed, the expected number of such subgraphs is

$$O\left(\sum_{t=1}^{\log\log n} 2^{\binom{t}{2}} n^t p^{(\frac{k+1}{2}+\varepsilon)t}\right) = o(1) \ . \tag{1}$$

We shall use (1) to show that almost surely the following coloring procedure colors all the edges of $K(n,p)$ without producing a monochromatic K_k.

Procedure: Order good vertices arbitrarily, $v_1, v_2, \dots, v_t$, $n - \log\log n \leq t \leq n$. We know that there exists a K_{k-1}-free coloring of the vertices of $N(v_1)$. Thus we color the edges incident to v_1 in the same manner, i.e. the edge $\{v_1, u\}$ receives the color of u. Suppose we have already colored all the edges incident to $v_1, \dots, v_i$. We color the yet uncolored edges incident to v_{i+1} following a K_{k-1}-free coloring of the vertices of $N(v_{i+1})$. Our construction guaranties that none of $v_1, ..., v_i$ will ever be a part of a monochromatic K_k, and so the same can be said about v_{i+1}. The first phase of the procedure ends when all the edges incident to $v_1, \dots, v_t$ are colored. So far, no monochromatic K_k has been created. Moreover, we can be sure that no matter how we will color the remaining edges, there is no danger of obtaining a monochromatic K_k with any of the good vertices involved.

Let B be the graph spanned in $K(n,p)$ by the bad vertices. We need to properly color the edges of B. By (1) we know that

$$m_B = \max_{H \subseteq B} \frac{e_H}{v_H} \leq \frac{k+1}{2} + \varepsilon < \frac{1}{2}(k-1)^2 \ ,$$

since $k \geq 4$ and ε is arbitrarily small. Proposition 2 of section 4, with $\chi = k$ and $r = 2$, completes the proof of Proposition 1. $\square$

Proof of the Lemma. *A path* in a hypergraph is a sequence of edges $E_1, ..., E_t$ such that $E_i \cap E_j \neq \emptyset$ if and only if $|i-j| < 2$. A path is called *simple* if $|E_i \cap E_j| \leq 1$ for all $i \neq j$. A hypergraph is connected if for every two vertices $x \neq y$ there is a path containing x and y.

Define the hypergraph

$$\mathcal{G} = ([n], \{V(G) \ : \ G \subset K(n,p),\, G \text{ is a copy of } K_k\}) \ .$$

Properties $K(n,p) \overset{v}{\to} k$ and $\chi(\mathcal{G}) \geq 3$ are equivalent. Suppose that $\chi(\mathcal{G}) \geq 3$ and let $\mathcal{G}_0$ be a 3-critical subhypergraph of $\mathcal{G}$. In order to have random variables referring to $\mathcal{G}_0$ well defined we may think of $\mathcal{G}_0$ as of the lexicographically first 3-critical subhypergraph of $\mathcal{G}$. Out of several properties of $\mathcal{G}_0$ we shall utilize three:

(i) $\mathcal{G}_0$ is connected,

(ii) $\mathcal{G}_0$ has no cut vertex, where by a cut vertex we mean a vertex v for which there is a partition $V(\mathcal{G}_0) \setminus \{v\} = V_1 \cup V_2$ into two nonempty classes such that every edge of $\mathcal{G}_0$ is contained in $V_i \cup \{v\}$ for some $i = 1, 2$,

(iii) every vertex belongs to at least two edges.

Our strategy is to show that the existence of $\mathcal{G}_0$ implies that $K(n,p)$ contains subgraphs whose expectation is bounded by $O(n^{-\frac{k+2}{k}})$.

For a hypergraph $\mathcal{H}$ define the cluster hypergraph $clus(\mathcal{H})$ in the following way. Let $A = A(\mathcal{H})$ be an auxiliary graph whose vertices are edges of $\mathcal{H}$, and two vertices E_1 and E_2 are adjacent if and only if $|E_1 \cap E_2| \geq 2$. Let $\mathcal{C}$ be the set of connected components of A. Then each $C \in \mathcal{C}$ corresponds to a maximal set of edges $E_1, ..., E_t$ such that for every partition $[t] = I \cup J$ there are $i \in I$ and $j \in J$ with $|E_i \cap E_j| \geq 2$. The unions over these sets become edges of the cluster hypergraph, i.e.

$$clus(\mathcal{H}) = (V(\mathcal{H}), \{ \bigcup_{E \in V(C)} E \ : \ C \in \mathcal{C}\}) \ .$$

Set $\mathcal{H} = clus(\mathcal{G}_0)$.

Remark 1. Since every edge of $\mathcal{G}_0$ is contained in some edge of $\mathcal{H}$, properties (i) and (ii) of $\mathcal{G}_0$ are preserverd in $\mathcal{H}$, whereas (iii) still holds for vertices belonging to edges of size k, i.e. edges in the set $E(\mathcal{H}) \cap E(\mathcal{G}_0)$. $\Box$

As every edge of $\mathcal{G}_0$ is a vertex set of a copy of K_k and every edge E of $\mathcal{H}$ is a union $\bigcup_{i \in I} E_i$ of some edges of $\mathcal{G}_0$, we define *the underlying graph of* E as $G[E] = \bigcup_{i \in I} G_i$, where $E_i = V(G_i)$. For a subhypergraph $\mathcal{H}'$ *the underlying graph* is defined as $G[\mathcal{H}'] = \bigcup_{E \in \mathcal{H}'} G[E]$.

Remark 2. By the definition of $\mathcal{H}$, subgraphs underlying its edges are pairwise edge-disjoint. $\Box$

Let us introduce the graph function

$$f(H) = v_H + \frac{k+2}{k} - \frac{2}{k} e_H .$$

As a linear function of v_H and e_H, f is modular, i.e.

$$f(H_1 \cup H_2) = f(H_1) + f(H_2) - f(H_1 \cap H_2) \ . \tag{2}$$

The function f has been defined so to take care of all “small” subgraphs of $K(n,p)$. This feature of f is exhibited in

Claim 1. *For every graph H,*

(a) *if, for at least one $H' \subseteq H$, $f(H') \leq 0$ then $P(K(n,p) \supset H) = O(n^{-\frac{k+2}{k}})$,*

(b) *if $f(H') > 0$ for all $H' \subseteq H$ then $H \stackrel{v}{\not\to} k$.*

Proof.

If $f(H') \leq 0$ and $H' \subseteq H$ then, recalling that X_H is the number of copies of H in $K(n,p)$, and applying Markov's inequality, $P(X_H > 0) \leq P(X_{H'} > 0) \leq Exp(X_{H'}) = O(n^{v_{H'}} p^{e_{H'}}) = O(n^{v_{H'} - \frac{2}{k} e_{H'}}) = O(n^{-\frac{k+2}{k}})$.

Assume that $f(H') > 0$ for all $H' \subseteq H$ and consider two cases. If $v_{H'} \leq 2k-2$ then, trivially, $\frac{e_{H'}}{v_{H'}} \leq \frac{1}{2}\Delta(H') \leq \frac{1}{2}(2k-3)$. If $v'_H \geq 2k-1$ then

$$\frac{e_{H'}}{v_{H'}} < \frac{k}{2} + \frac{k+2}{2v_{H'}} \leq k-1 = \delta(K_k),$$

and so $m_H < k-1 = \delta(K_k) = \max_{H' \subseteq K_k} \delta(H')$. Consequently, by [ŁRV 92, Lemma 1] (see also Theorem 5 in section 4 of this paper) $H \stackrel{v}{\not\to} k$. □

Thus, according to Claim 1, for every natural t, with probability $1 - O(n^{-\frac{k+2}{k}})$, every subgraph H of $K(n,p)$ with $v_H < t$ satisfies $H \stackrel{v}{\not\to} k$. Subgraphs whose size depends on n may not satisfy the above bound on the expected number of their copies in $K(n,p)$ and therefore Claim 1 is not applicable to them.

Claim 2. *With probability $1 - O(n^{-\frac{k+2}{k}})$, for every $E \in \mathcal{H}$, $|E| \leq 3k+1$.*

Proof. Let $E \in \mathcal{H}$ and $L = G[E] = \bigcup_{i=1}^{l} G_i$ where the G_i's are copies of K_k labeled so that for every $i = 2, \ldots, l$ $E(G_i \cap L_{i-1}) \neq \emptyset$, where $L_i = \bigcup_{j=1}^{i} G_j$. If $V(L_i) \not\subseteq V(L_{i-1})$ then we set $H = G_i \cap L_{i-1}$, where $2 \leq v_H \leq k-1$. As $f(H) \geq f(K_r)$, where $r = v_H$, we infer by (2) that

$$f(L_i) = f(L_{i-1} \cup G_i) = f(L_{i-1}) + f(G_i) - f(H) \leq f(L_{i-1}) + f(K_k) - f(K_r) .$$

Since, in the range $2 \leq r \leq k-1$, $f(K_r)$ attains minimum at $r = 2$, and $f(K_2) = 3$, $f(K_k) = 2 + \frac{2}{k}$, we have

$$f(L_i) - f(L_{i-1}) \leq f(K_k) - f(K_2) \leq 2 + \frac{2}{k} - 3 = -\frac{k-2}{k}. \tag{3}$$

Let $x = |\{i : V(L_i) \not\subseteq V(L_{i-1}), i = 2, ..., l\}|$ and suppose that $v_L \geq 3k + 2$. Then, since $|V(L_i) \setminus V(L_{i-1})| \leq k - 2$ and $v_{L_1} = k$, x must be so large that $k + x(k-2) \geq 3k + 2$, giving

$$x \geq \frac{2k+2}{k-2} \ . \tag{4}$$

On the other hand, if $V(L_i) = V(L_{i-1})$ then, by the definition of function f,

$$f(L_i) \leq f(L_{i-1}) \ . \tag{5}$$

By (3), (4) and (5),

$$f(L) = f(L_1) + \sum_{i=2}^{l} (f(L_i) - f(L_{i-1})) \leq f(K_k) - x\frac{k-2}{k} \leq 2 + \frac{2}{k} - \frac{2k+2}{k} = 0 \ . \tag{6}$$

We cannot apply Claim 1.a directly to L as it may grow with n. However, let i be the smallest index for which $f(L_i) \leq 0$. Then

$$v_{L_i} \leq k + \left\lceil \frac{2k+2}{k-2} \right\rceil (k-2) \leq 4k$$

and the existence of L_i in $K(n,p)$ has, by Claim 1.a, probability at most $O(n^{-\frac{k+2}{k}})$. □

Now we shall analyse the cyclic structure of $\mathcal{H}$. We define a*cycle* as a sequence of edges $E_1, ..., E_t$, such that

for $t = 2$, $|E_1 \cap E_2| \geq 2$ (these will be called *2-cycles*),

for $t = 3$, $E_{i-1} \cap E_i \neq \emptyset$, $i = 1, ..., 3$ ($E_0 = E_3$), and $E_1 \cap E_2 \cap E_3 = \emptyset$,

for $t \geq 4$, $E_{i-1} \cap E_i \neq \emptyset$, $i = 1, ..., t$ ($E_0 = E_t$), and $E_i \cap E_j = \emptyset$ for all $|i - j| \geq 2$ (modulo t).

A set of cycles is *independent* if the edge set of none of them is covered by the union of edge sets of the other cycles in the set. An edge of $\mathcal{H}$ is *complex* if its underlying graph is not K_k. Otherwise, it is called *pure*.

Claim 3. *With probability* $1 - O(n^{-\frac{k+2}{k}})$,

(a) *each cycle of* $\mathcal{H}$ *contains at most* $d = \lceil \frac{k+2}{k-2} \rceil$ *complex edges,*

(b) $\mathcal{H}$ *contains at most two independent cycles.*

Before we turn to the proof of Claim 3, let us see how Claims 1-3 complete the proof of the Lemma. All statements below hold with probability $1 - O(n^{-\frac{k+2}{k}})$. Properties (i) and (ii) of $\mathcal{G}_0$ are preserved in $\mathcal{H}$ – see Remark 1 above. Hence, if $\mathcal{H}$ is a hypertree then, by prope rty (ii), it may only have one edge and the underlying graph $G[\mathcal{G}_0] \overset{v}{\not\to} k$ by Claims 2 and 1.b. Otherwise $\mathcal{H}$ has a cycle. We claim that $\mathcal{H}$ is either a single cycle or a union of two independent intersecting cycles.

To see this define $\mathcal{H}_0$ as a subhypergraph of $\mathcal{H}$ which achieves the maximum in

$$\max_{\mathcal{H}^* \subseteq \mathcal{H}} \{|E(\mathcal{H}^*)| \ : \ \mathcal{H}^* = \mathcal{C}_1 \cup \mathcal{C}_2\} \ ,$$

where $\mathcal{H}^*$ is connected and $\mathcal{C}_1, \mathcal{C}_2$ are independent (if there are no two independent intersecting cycles in $\mathcal{H}$ we let $\mathcal{C}_1 = \mathcal{C}_2$ to be any cycle). Suppose that $\mathcal{H}_1 = (V(\mathcal{H}), E(\mathcal{H}) \backslash E(\mathcal{H}_0))$ is not empty. If there is a path $\mathcal{P} \in \mathcal{H}_1$ with $|V(\mathcal{P}) \cap V(\mathcal{H}_0)| \geq 2$, then, by the connectivity of $\mathcal{H}_0$, $\mathcal{H}_0 \cup \mathcal{P}$ contains a cycle $\mathcal{C}_3$ which is not contained in $\mathcal{H}_0 = \mathcal{C}_1 \cup \mathcal{C}_2$. As $\mathcal{C}_i$, $i = 1, 2, 3$, are not independent (Claim 3.b), two of them must cover the third one, and without loss of generality we have $\mathcal{C}_1 \cup \mathcal{C}_3 \supseteq \mathcal{C}_2$ and hence $\mathcal{C}_1 \cup \mathcal{C}_3 \supseteq \mathcal{C}_1 \cup \mathcal{C}_2 = \mathcal{H}_0$. As $\mathcal{C}_3 \setminus (\mathcal{C}_1 \cup \mathcal{C}_2) \neq \emptyset$, we infer that $|\mathcal{C}_1 \cup \mathcal{C}_3| > |\mathcal{H}_0|$ violating the choice of $\mathcal{H}_0$. So, if there is any edge in $\mathcal{H}_1$, it must intersect $V(\mathcal{H}_0)$ in at most one vertex, which is impossible by property (ii) of $\mathcal{H}$. Thus $\mathcal{H} = \mathcal{C}_1 \cup \mathcal{C}_2$. It is possible that either of these cycles, or both, contains a 2-cycle. If the longer cycle, $\mathcal{C}_1$ say, has at least $d + 30k + 15$ edges, then by Claim 3.a it contains at least $30k + 15$ pure edges and, as $k \geq 3$, each of them, except for those involved in 2-cycles, contains a vertex of degree 1 in $\mathcal{C}_1$. By Claim 2 and by Fact 1 below, $\mathcal{C}_2$ intersects $\mathcal{C}_1$ in at most $5b = 15k + 5$ components. Let us discard all edges of $\mathcal{C}_1$ which are intersected by, but not contained in $\mathcal{C}_2$ (at most $30k + 10$ edges), and also those which are involved in a 2-cycle (at most 4 edges). There will be still at least one pure edge in $\mathcal{C}_1$ with a vertex of degree 1 in $\mathcal{H}$. But then the degree of that vertex in $\mathcal{G}_0$ is also 1, contradicting property (iii) of $\mathcal{G}_0$. Thus, the longer cycle of $\mathcal{H}$ has at most $d + 30k + 14$ edges, and so $|E(\mathcal{H})| \leq 2d + 60k + 28$. Hence, the underlying graph $H = G[\mathcal{H}] = G[\mathcal{G}_0]$ has at most $(2d + 30k + 28)(3k + 1)$ vertices. Anyway, all what matters is that v_H is bounded and therefore, by Claim

1.b, $H \overset{v}{\not\to} k$, contradicting the fact that $\chi(\mathcal{G}_0) = 3$. $\Box$

Proof of Claim 3. Let $C(L_1, ..., L_t)$ be the number of cycles $\mathcal{C} = (E_1, ..., E_t)$ of $\mathcal{H}$ such that $G[E_i] = L_i$ and $|E_i| \le b = 3k + 1$, $i = 1, ..., t$. Let us count how many subgraphs of K_n underlie cycles counted here. There are at most $n^{v_{L_1}}$ choices of L_1, then $bn^{v_{L_2}-1}$ choices of L_2 and so on, until we reach the choice of the closing graph L_t which can be picked in at most $b^2 n^{v_{L_t}-2}$ ways. As each such a graph has precisely $\sum_{i=1}^{t} e_{L_i}$ edges (see Remark 2), we have

$$ExpC(L_1, \dots , L_t) \le \prod_{i=1}^{t} (bn^{v_{L_i}-1} p^{e_{L_i}}) = (ab)^t n^{\sum_{i=1}^{t}(f(L_i)-(2+\frac{2}{k}))}, \tag{7}$$

since

$$v_{L_i} - 1 - \frac{2}{k} e_{L_i} = f(L_i) - (2 + \frac{2}{k}) \ . \tag{8}$$

By (6) with $x \ge 1$, for each complex edge $E \in \mathcal{H}$, $f(G[E]) \le 1 + \frac{4}{k}$. If E is pure then $f(G[E]) = f(K_k) = 2 + \frac{2}{k}$. Thus, if there are at least d complex edges among the $E_1, ..., E_t$, then the exponent of n in (7) is not bigger than $d(1 + \frac{4}{k} - (2 + \frac{2}{k})) \le -\frac{k+2}{k}$ and

$$ExpC(L_1, ..., L_t) \le (ab)^t n^{-\frac{k+2}{k}} \ .$$

Let X be the number of all cycles in $\mathcal{H}$ with at least d complex edges and with all edges of size not bigger than b. Furthermore, let c be the size of the family $\mathcal{U}$ of all pairwise nonisomorphic graphs on up to b vertices which may underlie edges of $\mathcal{H}$. Note that $c < 2^{\binom{b}{2}}$. Then

$$\begin{aligned} P(X > 0) &\le Exp(X) = \sum_t \sum_{L_1,\dots,L_t} ExpC(L_1, ...L_t) \\ &= O(n^{-\frac{k+2}{k}}) \sum_t (abc)^t = O(n^{-\frac{k+2}{k}}) \ , \end{aligned}$$

for $a < (bc)^{-1}$, where the inner sum extends over t-tuples of graphs from $\mathcal{U}$, each t-tuple containing at least d graphs which are not just plain K_k.

To prove part b of Claim 3 we need a couple of statements about hypergraphs.

Fact 1. *Suppose that $\mathcal{F}$ is a connected hypergraph with edges of size at most b, containing at most l independent cycles. Then for any two cycles $\mathcal{C}_i = (V_i, \mathcal{E}_i)$, $i = 1, 2$, in $\mathcal{F}$, their intersection $\mathcal{C}_1 \cap \mathcal{C}_2 = (V_1 \cap V_2, \mathcal{E}_1 \cap \mathcal{E}_2)$ consists of at most $(2l + 1)b$ connected components.*

Proof. Fix an orientation of $\mathcal{C}_2$ and set $E(\mathcal{C}_2) = \{E_1, ..., E_t\}$, where the edges are listed in the cyclical order. Call a path $\mathcal{Q} = E_i, ..., E_j$, $i \le j$, *good* if $E(\mathcal{Q}) \subseteq E(\mathcal{C}_2) \setminus E(\mathcal{C}_1)$ and

$$|V(\mathcal{Q}) \cap V(\mathcal{C}_1)| \ge 2 \ , \tag{9}$$

while neither $\mathcal{Q}' = E_{i+1}, ..., E_j$ nor $\mathcal{Q}'' = E_i, ..., E_{j-1}$ satisfies (9). (Note that any edge of $\mathcal{C}_2 \setminus \mathcal{C}_1$ intersecting $V(\mathcal{C}_1)$ in at least 2 vertices is a good path on its own). Suppose to the contrary that $\mathcal{C}_1 \cap \mathcal{C}_2$ consists of at least $(2l + 1)b + 1$ connected components. For each component $\mathcal{K}$ define the exit edge $E_{\mathcal{K}}$ as the latest edge of $\mathcal{C}_2$ (in the cyclical order) satisfying $E_{\mathcal{K}} \cap V(\mathcal{K}) \neq \emptyset$ and $E_{\mathcal{K}} \notin E(\mathcal{K})$. Thus, if $\mathcal{K}$ is just an isolated vertex then $E_{\mathcal{K}}$ is the latest edge of $\mathcal{C}_2$ containing it. In fact, several isolated vertices may have the same exit edge, but as the edges have size at most b, there are, by the Pigeon-Hole Principle, $2l + 2$ different exit edges $E_{i_1}, ..., E_{i_{2l+2}}$ (listed in the cyclical order). Let us extend each E_{i_j}, j odd, to a good path $\mathcal{Q}_j$. By taking every other E_{i_j} we guarantee that

$$E_{i_j} \in \mathcal{Q}_j \setminus (\mathcal{Q}_{j-2} \cup \mathcal{Q}_{j+2}) \ . \tag{10}$$

For $j = 1, 3, ..., 2l + 1$ there are paths $\mathcal{P}_j$, $E(\mathcal{P}_j) \subset E(\mathcal{C}_1)$, such that $\mathcal{C}^*_j = \mathcal{P}_j \cup \mathcal{Q}_j$ forms a cycle. By (10), $E_{i_j} \in \mathcal{C}^*_j \setminus \bigcup_{s \neq j, s \text{ odd}} \mathcal{C}^*_s$, $j = 1, 3, ..., 2l + 1$, and so the cycles $\mathcal{C}^*_j$, $j = 1, 3, ..., 2l + 1$, form a set of $l + 1$ independent cycles contradicting our assumption on $\mathcal{F}$. $\square$

Definitions. For a hypergraph $\mathcal{F}$, $ic(\mathcal{F})$ is the largest number of independent cycles in $\mathcal{F}$. For $E \in \mathcal{F}$, we denote by $\mathcal{F} - E$ the hypergraph of $\mathcal{F}$ obtained by removing the edge E together with all vertices which belong only to E. We say that $\mathcal{F}$ is *l-minimal* if $\mathcal{F}$ is connected and $ic(\mathcal{F}) = l$ but for every $E \in \mathcal{F}$ either $ic(\mathcal{F} - E) < l$ or $\mathcal{F} - E$ is disconnected.

Remark 3. Every connected hypergraph $\mathcal{F}$ with $ic(\mathcal{F}) \geq l$ contains an l-minimal subhypergraph. $\square$

Remark 4. If $\mathcal{F}$ is l-minimal then there are l independent cycles $\mathcal{C}_1, ..., \mathcal{C}_l$ in $\mathcal{F}$ such that every edge of $\mathcal{F}$ which lies on a cycle belongs to $\mathcal{F}_0 = \bigcup_{i=1}^{l} \mathcal{C}_i$ and every edge of $\mathcal{F}$ intersects at leas t two other edges (no pendant edges). In other words, all edges outside $\mathcal{F}_0$ are cut edges. When $l = 2$ or $l = 3$ the structure of l-minimal hypergraphs can be described in a more transparent way. Namely, if $\mathcal{F}_0$ is connected then $\mathcal{F} = \mathcal{F}_0$. If $\mathcal{F}_0$ has two components then there is a simple path P joining them and $\mathcal{F} = \mathcal{F}_0 \cup P$. (We say that a path joins two disjoint subhypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$ if it shares exactly one vertex with each $\mathcal{H}_i$, $i = 1, 2$, and is minimal with this property.) Finally, if $\mathcal{C}_i$, $i = 1, 2, 3$, are pairwise disjoint (i.e. $\mathcal{F}_0$ consists of 3 components) there are two simple paths $\mathcal{P}_i$, $i = 1, 2$, such that, say, $\mathcal{P}_1$ joins $\mathcal{C}_1$ with $\mathcal{C}_2$ and $\mathcal{P}_2$ joins $\mathcal{C}_3$ with $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{P}_1$. In fact, $\mathcal{P}_1$ may coincide with $\mathcal{P}_2$. This happens when all three cycles are connected via a single edge. $\square$We shall state Fact 2 in full generality, but due to the restricted range of app lication and in order to clarify the exposition we shall prove it only in cases $l = 2$ and $l = 3$.

Fact 2. *The number of pairwise nonisomorphic l-minimal hypergraphs on v vertices is at most C^v, for some constant $C > 1$.*

Proof. According to Remark 4 every l-minimal hypergraph on v vertices, $l \leq 3$, can be obtained by "gluing together" at most 3 cycles $\mathcal{C}_i$, $i = 1, 2, 3$, and at most two paths $\mathcal{P}_i$, $i = 1, 2$. As types of cycles and paths are dete rmined by the cardinalities of their edges (ranging from 3 to b) and the car dinalities of consecutive edge-intersections (from 1 to $b-1$), there are at most $\sum_{j \leq v}((b-2)(b-1))^j < b^{2v}$ types of cycles and at most $\sum_{j \leq v}(b-2)^j < b^v$ types of paths on at most v vertices . In view of Fact 1, the intersection of any two cycles consists of at most $7b$ components, some of them are paths and others are isolated vertices. Each path-component is determined by choosing the two end-edges (in at most v^2 ways). This has to be done on both "to-be-intersected" cycles. Hence, the number of nonisomorphic ways

$\mathcal{C}_1$ intersects $\mathcal{C}_2$ on t components, s of which are isolated vertices, is bounded by $(v^{s+2(t-s)})^2 < v^{28b}$ and the total number of ways $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect is at most $\sum_{t=0}^{7b}\sum_{s=0}^{t}\binom{t}{s}v^{28b} = O(v^{28b})$. Repeating the same estimates for intersections of $\mathcal{C}_3$ with $\mathcal{C}_i$, $i = 1,2$, gives the total number of types of $\mathcal{F}_0$ (see Remark 4) not bigger than $O(b^{6v}v^{84b})$. Finally, in case when $\mathcal{F}_0$ is disconnected, we need to connect its components by paths $\mathcal{P}_i$, $i = 1,2$, which brings the factor of $b^{2v}v^4$ as there are at most v^2 ways to "hook up" each path. (In case $\mathcal{P}_1 = \mathcal{P}_2 = E$, which happens when all 3 cycles are connected via a single edge, this estimate goes down to $O(v^3)$.) Altogether, the number of all nonisomorphic types of l-minimal hypergraphs is bounded by $O(b^{8v}v^{84b+4}) < C^v$, for some $C > 0$. □

Finally, we will need

Fact 3. *If a connected hypergraph $\mathcal{F}$ contains at least 3 independent cycles, then*

$$|V(\mathcal{F})| \leq \sum_{E\in\mathcal{F}}(|E|-1) - 2 \ .$$

Proof. Consider an auxiliary bipartite graph B with $V(\mathcal{F})$ and $E(\mathcal{F})$ as the two sets of vertices, and edges representing the incidences between verti ces and edges of $\mathcal{F}$. B is connected and every cycle of $\mathcal{F}$ correspo nds to a cycle of B. Moreover, every 3 independent cycles of $\mathcal{F}$ correspond to 3 inde pendent cycles of B, as any "private" edge of a cycle of $\mathcal{F}$ becomes a "privat e" vertex of the corresponding cycle in B. Thus the cyclomatic number of B, $e_B - v_B + 1$, is at least 3 (see [Be 73] for the definition). However, $v_B = |V(\mathcal{F})| + |\mathcal{F}|$ and $e_B = \sum_{v\in V(\mathcal{F})} deg_{\mathcal{F}}(v) = \sum_{E\in\mathcal{F}}|E|$, which completes the proof. □

Now we can complete the proof of Claim 3. If $ic(\mathcal{H}) \geq 3$ then $\mathcal{H}$ contains a 3-minimal subhypergraph (see Remark 3). Let X be the number of subgraphs of $K(n,p)$ underlying 3-minimal subhypergraphs of $\mathcal{H}$. Let $\mathcal{Q}$ be the event that every edge of $\mathcal{H}$ consists of at most $3k+1$ vertices. By Claim 2 we know that $P(\mathcal{Q}) = 1 - O(n^{-\frac{k+2}{k}})$. Let X' be defined as X with the additional restriction that all edges are smaller than $3k+2$. Then, by Markov's inequality we have $P(X > 0) < O(n^{-\frac{k+2}{k}}) + ExpX'$. To estimate $ExpX'$ we compute the number of subgraphs of the complete graph K_n which may underlie subhypergraphs $\mathcal{F}$ of $\mathcal{H}$ counted by X'. There are no more than

$$\sum_t \sum_{\mathcal{F}:|E(\mathcal{F})|=t} \sum_{L_1,\dots,L_t} n^{|V(\mathcal{F})|} \tag{11}$$

such subgraphs. Here we sum over all sizes $t = |E(\mathcal{F})|$, all nonisomorphic types of $\mathcal{F}$, and all choices of graphs $L_1, ..., L_t$ to "fill" the edges of $\mathcal{F}$. By Fact 3, $|V(\mathcal{F})| \le \sum_{i=1}^{t}(v_{L_i} - 1) - 2$. To obtain an upper bound on $Exp(X')$ we have to multiply (11) by $p^{|E(G[\mathcal{F}])|}$, where the exponent, $|E(G[\mathcal{F}])| = \sum_{i=1}^{t} e_{L_i}$, is the total number of edges in the graph $G[\mathcal{F}]$ (recall Remark 2 here). As $p = an^{-\frac{2}{k}}$, this gives a factor of at most $a^{\binom{k}{2}t} < a^t$ (assuming $a < 1$), and the exponent of n becomes by (8)

$$|V(\mathcal{F})| - \frac{2}{k}|E(G[\mathcal{F}])| = -2 + \sum_{i=1}^{t}(v_{L_i} - 1 - \frac{2}{k}e_{L_i}) = -2 + \sum_{i=1}^{t}(f(L_i) - (2 + \frac{2}{k}))\ ,$$

which by (6) with $x = 0$ is not bigger than -2. Hence

$$Exp(X') \le n^{-2}\sum_t a^t \sum_{\mathcal{F}:|E(\mathcal{F})|=t} 1 \sum_{L_1,\dots,L_t} 1\ .$$

As $v_{L_i} \le b = 3k + 1$ for all $i = 1, ..., t$, $\sum_{L_1,\dots,L_t} 1 < 2^{\binom{b}{2}t}$. Also, by Fact 2, $\sum_{\mathcal{F}} 1 \le C^{|V(\mathcal{F})|} < C^{bt}$. Summarizing, for $a = a(b, C)$ small enough,

$$Exp(X') < n^{-2}\sum_t (2^{\binom{b}{2}}C^b a)^t = O(n^{-2}) = o(n^{-\frac{k+2}{k}})\ . \quad \square$$

3.General Case.

Let G be a graph on at least 3 vertices and recall that

$$m_G^{(2)} = \max_{H\subseteq G, v_H\ge 3} \frac{e_H - 1}{v_H - 2}\ .$$

A star forest is an acyclic graph whose every component is a star, i.e. a graph with at most one vertex of degree bigger than 1. In this section we shall prove the following result.

Theorem 3b. *For every graph G which is not a star forest there is a positive constant c_G such that*

$$\lim_{n\to\infty} Prob(K(n, c_G n^{-1/m_G^{(2)}}) \to (G)_2^2) = 0 \ .$$

If G is a star forest with maximum degree d then the above threshold coincides, by the Pigeon-Hole Principle, with that for apperance of vertices of degree $2d-1$ and, in general, of degree $r(d-1)+1$ when r colors are used. This threshold was already found in [ER 60] to be $n^{-\frac{r(d-1)+1}{r(d-1)}}$. This is the only known instance when the threshold of a Ramsey property depends on the number of colors. All other forests contain P_4, a path of length 3. It is known (see again [ER 60]) that for $c < 1$, almost surely, every component of $K(n, \frac{c}{n})$ is either a tree or a unicyclic graph and therefore its edges can be 2-colored without producing a monochromatic P_4 and the theorem is true for forests. From now on we assume that G contains a cycle, or equivalently, that $m_G^{(2)} > 1$. By Proposition 1, Theorem 3b is already proved for complete graph s. In particular, we assume that $G \neq K_3$ and $G \neq K_4$. As a further simplification we may assume without loss of generality that G is *strictly balanced*, i.e. the maximum $m_G^{(2)}$ is achieved only by G itself. We may do so, since we can always replace G by its smallest subgraph H with $\frac{e_H-1}{v_H-2} = m_G^{(2)}$. Such an H is clearly strictly balanced. It can be easily verified that a strictly balanced graph must have minimum degree at least 2 and that it cannot have a 2-vertex cut set which is an edge.

With no loss of generality we may further assume that for every edge of G there is another edge in G which is vertex-disjoint from it. We shall call graphs with this property *spacious*. It is not hard to see that the only strictly balanced graph which is not spacious is the triangle K_3.

It is convenient to view the copies of G that appear in $K(n,p)$ as edges of a hypergraph. Set $\mathcal{G} = (X, \mathcal{E})$, where $X = E(K(n,p))$ and

$$\mathcal{E} = \{E(G') \ : \ G' \subset K(n,p) \ , G' \text{ is isomorphic to } G\} \ .$$

But there is a danger associated with such a representation. Namely, the vertices of $K(n,p)$ become invisible and so the hypergraphic picture may be very misleading. Also, the terms "edge" or "vertex" now become ambiguous. To avoid confusion we will be often using the term "graph edge". To distingiush between graph and hypergraph vertices, the latter will be called elements and the set of elements of a hypergraph $\mathcal{H}$ will be denoted by $X(\mathcal{H})$.

The property $K(n,p) \rightarrow (G)_2^2$ is equivalent to $\chi(\mathcal{G}) \geq 3$. For our proof we shall need the following elementary property of 3-edge- critical hypergraphs. A hypergraph is *3-edge-critical* if it is 3-chromatic but removal of any edge decreases the chromatic number.

Exercise. *Show that if $\mathcal{H}$ is a 3-edge-critical hypergraph then for every $E \in \mathcal{H}$ and every $x \in E$ there exists $E' \in \mathcal{H}$ such that $E \cap E' = \{x\}$.* $\square$

To proceed we need a couple of hypergraph definitions. A *path* is a sequence of edges $E_1, ..., E_k$, $k \geq 1$, with $E_i \cap E_j \neq \emptyset$ if and only if $|i-j| \leq 1$. A *cycle* is a sequence $E_1, ..., E_k$, $k \geq 3$, with $E_i \cap E_j \neq \emptyset$ if and only if $|i-j| \leq 1$ (mod k) and, when $k = 3$, $E_1 \cap E_2 \cap E_3 = \emptyset$. (Note that we do not consider pairs of edges E_1, E_2 with $|E_1 \cap E_2| \geq 2$ as cycles as we did in Section 2.) A hypergraph (path, cycle) is called *simple* if no two edges intersect in more than one element. A subhypergraph $\mathcal{H}_0$ of $\mathcal{H}$ is said to have a *handle* if there is an edge E in $\mathcal{H}$ such that $2 \leq |E \cap X(\mathcal{H}_0)| < |E|$. $\mathcal{H}_0$ is said to have a *detour* if there are $x, y \in X(\mathcal{H}_0)$, $x \neq y$, and a simple path $\mathcal{D} = E_1, ..., E_k$ such that $X(\mathcal{H}_0) \cap X(\mathcal{D}) = \{x, y\}$, $x \in E_1 \setminus E_2$ and $y \in E_k \setminus E_{k-1}$. (In case $k = 1$, we set $E_0 = E_2 = \emptyset$.) If $\mathcal{H}_0$ is a union of two disjoin cycles, one containing x and the other y, we call such a detour *a bridge*.

The next two definitions refer to the hypergraph $\mathcal{G}$ of copies of G in $K(n,p)$. Let $\mathcal{C} = E(G_1), ..., E(G_k)$ be a cycle in $\mathcal{G}$. Denote by H_i the graph spanned in $K(n,p)$ by the graph edges belonging to $E(G_i) \cap E(G_{i+1})$, $i = 1, ..., k$, (here $G_{k+1} = G_1$). We call $\mathcal{C}$ *normal* if $V(H_1) \cap ... \cap V(H_k) = \emptyset$ and we call it *bad* if it is normal, simple and there exist i, j, $i \neq j$ and a vertex v belonging to $V(G_i) \cap V(G_j)$ but not belonging to $V(H_{i-1}) \cup V(H_i)$ (here $H_0 = H_k$).

The idea of the proof is to show that $\chi(\mathcal{G}) \geq 3$ implies existence of subgraphs of $K(n,p)$ which, on the other hand, are almost surely not there. So, naturally, our proof splits into a deterministic and a probabilistic part.

Deterministic Lemma. *If $\chi(\mathcal{G}) \geq 3$ then $\mathcal{G}$ contains either*

(i) *a normal non-simple cycle, or*

(ii) *a normal cycle with a handle, or*

(iii) *a normal cycle with a detour, or*

(iv) *a pair of normal cycles sharing exactly one element, or*

(v) *a pair of disjoint normal cycles with a bridge, or*

(vi) *a bad cycle, or*

(vii) *(in case when $G = C_4$) $K(n,p)$ contains a subgraph H with $v_H \leq 8$ and $e_H > \frac{3}{2}v_H$.*

Proof. Let $\mathcal{P} = (E_1, ..., E_k)$ be the longest simple path of $\mathcal{G}_0$, a 3-edge-critical subhypergraph of $\mathcal{G}$. By Exercise, $k \geq 2$. Denote by e_i the common element of E_i and E_{i+1}, $i = 1, ..., k-1$ and by G_i the copy of G in $K(n,p)$ for which $E(G_i) = E_i$. Because G is spacious, there exist $e_0 \in E_1$ and $e_k \in E_k$ such that both $\{e_0, e_1\}$ and $\{e_{k-1}, e_k\}$ are vertex-disjoint pairs of graph edges. By Exercise, there are $E', E'' \in \mathcal{G}_0$ such that $E' \cap E_1 = \{e_0\}$ and $E'' \cap E_k = \{e_k\}$. Let

$$I' = \{2 \leq i \leq k \ : \ E' \cap (E_i \setminus \{e_{i-1}\}) \neq \emptyset\}$$

and

$$I'' = \{1 \leq i \leq k-1 \ : \ E'' \cap (E_i \setminus \{e_i\}) \neq \emptyset\} \ .$$

Both I' and I'' must be nonempty since otherwise we would obtain a simple path in $\mathcal{G}$, longer than $\mathcal{P}$. Let $i' = \min I'$ and $i'' = \max I''$. Note that both $\mathcal{C}' = E_1, ..., E_{i'}, E'$ and $\mathcal{C}'' = E_{i''}, ..., E_k, E''$ are normal cycles, with the pairs $\{e_0, e_1\}$ and $\{e_{k-1}, e_k\}$ responsible for that.

Case 1. $|I'| \geq 2$ or $|I''| \geq 2$.

By symmetry, we assume that $|I'| \geq 2$ and set $j' = min(I' \setminus \{i'\})$. If $|E' \cap E_{i'}| \geq 2$ then $\mathcal{C}'$ is a normal non-simple cycle. Otherwise, if $|E' \cap E_{j'}| \geq 2$, we obtain a normal cycle with a handle, since $E_{j'} \not\subset X(\mathcal{C}')$. If $E' \cap E_{j'} = \{z\}$, we obtain a normal cycle $\mathcal{C}'$ with a detour $E_{i'+1}, ...E_{j'}$, with $x = e_{i'}$ and $y = z$.

Case 2. $|I'| = |I''| = 1$.

As before we may assume that $|E' \cap E_{i'}| = |E'' \cap E_{i''}| = 1$, since otherwise we obtain configuration (i). We will distinguish two cases:

a) $i' < k$ or $i'' > 1$;

b) $i' = k$ and $i'' = 1$.

To discuss a) we assume by symmetry that $i' < k$. Set $E' \cap E_{i'} = \{a\}$, $E'' \cap E_{i''} = \{b\}$ and consider 4 subcases.

α) Assume first that $i' > i''$. If $E' \cap E'' \neq \emptyset$ then, as $b \notin E'$, $|E'' \cap X(\mathcal{C}')| \geq 2$. Since $e_k \in E'' \setminus X(\mathcal{C}')$, we infer that E'' is a handle of $\mathcal{C}'$ (which is our configuration (ii)). If $E' \cap E'' = \emptyset$ then, for $x = e_{i'}$ and $y = b$, the path $E_{i'+1}, ..., E_k, E''$ is a detour of $\mathcal{C}'$.

β) Assume now that $i' = i''$. Note that by the definition of I', $a \neq e_{i-1}$ and thus $E_{i'} \in \mathcal{C}'$. If $(E' \setminus \{a\}) \cap (E'' \setminus \{b\}) \neq \emptyset$ then E'' intersects $\mathcal{C}'$ in at least two elements (but is not contained in $\mathcal{C}'$) and thus E'' is a handle of $\mathcal{C}'$. If $(E' \setminus \{a\}) \cap (E'' \setminus \{b\}) = \emptyset$ then, again $e_{i'}, E_{i'+1}, ..., E_k, E'', b$ is a detour of $\mathcal{C}'$. (Note that $e_{i'} \neq b$.)

γ) Assume further that $i'' = i' + 1$ and that there is an element $z \in (E' \setminus \{a\}) \cap (E'' \setminus \{b\})$. If $b = e_{i'}$ then E'' is a handle of $\mathcal{C}'$. If $b \neq e_{i'}$ then $E_{i'+1}, E''$ is a detour of $\mathcal{C}'$ with $x = e_{i'}$ and $y = z$. In case $(E' \setminus \{a\}) \cap (E'' \setminus \{b\}) = \emptyset$, there are four possibilit ies: $a = e_{i'} \neq b$, $a \neq e_{i'} = b$, $a \neq e_{i'} \neq b$, and $a = e_{i'} = b$, all leading to configuration (iv).

δ) Finally, assume that $i'' > i' + 1$. Case $E' \cap E'' = \emptyset$ gives configuration (v). If, on the other hand, there is $z \in E' \cap E''$, then $z \notin X(\mathcal{P}) = \bigcup_{i=1}^{k} E_i$ and so $E_{i'+1}, ..., E_{i''}, E''$ is a detour of $\mathcal{C}'$ with $x = e_{i'}$ and $y = z$.

Let us turn now to case **2b**. For notational ease set $E' = E_0$ and consider the normal, simple cycle $\mathcal{C} = E_1, ..., E_k, E_0$. For $i = 0, ..., k$, let G_i be the copy of G with $E(G_i) = E_i$. Two cases will be discussed.

α) $\exists i\ \exists v \in V(G_i) \setminus (e_{i-1} \cup e_i)$

(Note that α) holds for every i whenever $v_G \geq 5$.) Let f be an edge incident to v in G_i. By Exercise, there is a graph G_f isomorphic to G such that $E(G_f) = E_f$ satisfies $E_f \cap E_i = \{f\}$. If $|E_f \cap X(\mathcal{C})| = 1$ then $E_{i+2}, ..., E_k, E_0, ..., E_i, E_f$ is a simple path longer than $\mathcal{P}$ – a contradiction. Thus, we have $|E_f \cap X(\mathcal{C})| \geq 2$. As $\delta(G) \geq 2$, there exists $g \in E(G_f) \setminus \{f\}$ incident to v. If $g \in X(\mathcal{C})$, there is an index $j \neq i$ such that $g \in E_j$. But then $v \in V(G_j)$ and so by the choice of v, $\mathcal{C}$ is a bad cycle. If $g \notin X(\mathcal{C})$ then $E_f \not\subset X(\mathcal{C})$ and E_f is a handle of $\mathcal{C}$.

Before we turn to case β), we first prove the following claim, which is true under general assumptions of case 2b, but was not needed until now.

Claim 4. *Under the assumptions of case 2b, either $\mathcal{C}$ has a handle or the set $X(\mathcal{G}_0)$ of elements of the critical subhypergraph $\mathcal{G}_0$ coincides with $X(\mathcal{C})$.*

Proof. Suppose there is $z \in X(\mathcal{G}_0) \setminus X(\mathcal{C})$. Due to the connectivity of $\mathcal{G}_0$ there is a path connecting $\mathcal{C}$ to z. Thus, there is an edge $E \in \mathcal{G}_0$, $1 \leq |E \cap X(\mathcal{C})| < |E|$. But, by the maximality of the path $\mathcal{P}$, $|E \cap X(\mathcal{C})| \geq 2$, which means that E is a handle of $\mathcal{C}$. $\square$

β) $\forall i \,:\, V(G_i) = e_{i-1} \cup e_i$

This case implies that $v_G \leq 4$ and, because G is strictly balanced and contains a cycle, it can be only K_3, K_4 or C_4, the 4-cycle. The cases $G = K_3$ and $G = K_4$ are covered by Proposition 1 and thus we further assume that $G = C_4$. Consequently, for every $i = 0, ..., k$, we now have $e_{i-1} \cap e_i = \emptyset$. Take $f \in E_1 \setminus \{e_0, e_1\}$. By Exercise, there is E_f, $E_f \cap E_1 = \{f\}$. By Claim 4, $E_f \subset X(\mathcal{C})$. Observe that either we end up at configuration (vii), or

$$E_0 \setminus \{e_k, e_0\} \not\subseteq E_f \quad \text{and} \quad E_2 \setminus \{e_1, e_2\} \not\subseteq E_f \ . \tag{12}$$

To see this suppose that $E_0 \setminus \{e_k, e_0\} \subseteq E_f$ and set $e_k = \{v_1, v_2\}$, $e_0 = \{v_3, v_4\}$, $e_1 = \{v_5, v_6\}$, and, without loss of generality, $f = \{v_3, v_5\}$. By our assumption $\{v_1, v_3\}$ and $\{v_2, v_4\}$ belong to E_f and thus $v_5 = v_2$. But now $\{v_1, v_3\}$, $\{v_3, v_2\}$, and $\{v_2, v_4\}$ all belong to the same copy of C_4 (underlying E_f) forcing the pair $\{v_1, v_4\}$ to be an edge of $K(n,p)$. Hence the vertices v_1, v_2, v_3, v_4, v_6 induce in $K(n,p)$ a subgraph with at least 8 edges yielding configuration (vii).

(i) Assume $k \geq 4$. Then there exists i, $0 \leq i \leq k$, such that $E_i \setminus \{e_{i-1}\}$ or $E_i \setminus \{e_i\}$ is disjoint from E_f. (Here and below all indices are modulo $k+1$.) Let $\mathcal{P}_0 = E_u, ..., E_i, ..., E_v$ be such that $u \neq v$, $E_u \cap E_f \neq \emptyset$, $E_v \cap E_f \neq \emptyset$, but for all w, $u < w < v$, $E_w \cap E_f = \emptyset$. Note that one of u or v may equal i, and that, by definition, the length of $\mathcal{P}_0$, $|\mathcal{P}_0|$, is at most k, i.e. $\mathcal{P}_0$ is indeed a path. Set $\mathcal{C}_0 = E_u, ..., E_v, E_f$. Since the neighbors of E_i intersect E_i in e_{i-1} and e_i, $\mathcal{C}_0$ is a normal cycle. If $|\mathcal{P}_0| = k$ then the only edge of $\mathcal{C}$ not in $\mathcal{C}_0$ is E_0 or E_2. Say it is E_0. By (12), we have $2 \leq |E_0 \cap X(\mathcal{C}_0)| \leq 3$ and E_0 is a handle of $\mathcal{C}_0$. Assume now that the path $\mathcal{C} \setminus \mathcal{P}_0 = E_{v+1}, ..., E_{u-1}$ (edges listed in the same cyclical order as in $\mathcal{P}_0$) consists of at least two edges. If each of them intersects $X(\mathcal{C}_0)$ in at most one element, a detour of $\mathcal{C}_0$ can easily be found. Indeed, let E_w satisfy $v+2 \leq w \leq u-1$, $E_w \cap X(\mathcal{C}_0) \neq \emptyset$, and $E_j \cap X(\mathcal{C}_0) = \emptyset$ for $v+2 \leq j \leq w-1$. (The existence of w is guaranteed by $E_{u-1} \cap E_u \neq \emptyset$.) Then $E_{v+1}, ..., E_w$ is a detour of $\mathcal{C}_0$. If there exists $v+1 \leq w \leq u-1$ such that $|E_w \cap X(\mathcal{C}_0)| \geq 2$ then E_w is a handle of $\mathcal{C}_0$. Indeed, then $|E_w \cap X(\mathcal{P}_0)| \leq 1$ and $|E_f \setminus X(\mathcal{P}_0)| \leq 2$, giving $|E_f \cap X(\mathcal{C}_0)| \leq 3$.

(ii) $k \leq 3$. Here one could apply a tedious case by case analysis leading each time to one of the configurations (ii), (iii), or (vii). Alternately, we choose to apply the above Claim 4 together with Theorem 6 from section 4. The graph F underlying $\mathcal{G}_0$ has in this case no more than 8 vertices. If $m_F > 3/2$ we have configuration (vii). If $m_F \leq 3/2$ then by Theorem 6 one c an color the edges of F so that no C_4 becomes monochromatic. This, however, contradicts the definition of $\mathcal{G}_0$.

This completes the proof of the Deterministic lemma. □

Probabilistic Lemma. *Almost surely $\mathcal{G}$ does not have any of the structures (i)-(vii) described in the statement of the Deterministic Lemma above.*

Proof. Our only tool in this proof is Markov's inequality $P(X > 0) \leq Exp(X)$, valid for every nonnegative integer-valued random variable X. We shall show that the expected number of all subgraphs of $K(n, p)$ giving rise to the structures described in (i)-(vii) tends to 0 as $n \to \infty$, forcing the probab ility of their presence in $K(n, p)$ tend to 0 as well. In that case we will say that the configurations in question are *rare*. We begin by showing that the expected number of simple cycles S is $O(n)$. Let S_k be the number of simple cycles $E(G_1), ..., E(G_k)$ of length k in $\mathcal{G}$. Each such a cycle has precisely $\alpha = k(e_G - 1)$ graph edges. Thus $Exp(S_k) = N_k p^\alpha$, where N_k is the number of subgraphs of K_n which are unions of k copies of G, $G_1, ..., G_k$ which form a simple cycle in $\mathcal{G}$. To bound N_k observe that there are at most n^{v_G} choices of G_1. Having G_1 chosen, we may pick G_2 in no more than $e_G n^{v_G - 2}$ ways. Here we also cover the possibility that G_1 and G_2 intersect in more than two vertices. Continuing this way, having chosen $G_1, ..., G_{k-2}$, we choose G_{k-1} in at most $e_G n^{v_G-2}$ ways and finally we choose G_k in no more than $e_G^2 n^{v_G-3}$ ways. This estimate corresponds to the worse case when the edges being the intersections of $E(G_{k-1})$ and $E(G_k)$, and of $E(G_k)$ and $E(G_1)$ share one endpoint. (They cannot coincide by the definition of a cycle.) Altogether,

$$Exp(S) \leq \sum_{k=3}^{\infty} e_G^k n^{k(v_G-2)+1} p^\alpha = O(n)$$

since $p = cn^{-\frac{v_G-2}{e_G-1}}$ and $\sum_{k=3}^{\infty} (e_G c)^k < \infty$ for $c < 1/e_G$.

Now let us see how this estimate will be affected if we count all cycles, simple or not. This time the graphs H_i spanned by the edge sets $E(G_i) \cap E(G_{i+1})$ may be any subgraphs of G and not just single edges. Neverthless, similar estimates lead to the bound

$$\sum_{k=3}^{\infty} \sum_{H_1,\dots,H_k} 2^{ke_G} n^\beta p^\alpha \ ,$$

where $\alpha = ke_G - \sum_{i=1}^k e_{H_i}$ and

$$\beta = v_G + (v_G - v_{H_1}) + ... + (v_G - v_{H_{k-2}}) + (v_G - v_{H_{k-1}} - v_{H_k} + v_{H_{k-1} \cap H_k}) \leq kv_G - \sum_{i=1}^k v_{H_i} + v_G \, .$$

Note that it is possib le now that $V(H_{k-1}) = V(H_k) = V(G_1)$ and thus for general graphs the last inequality cannot be improved. Note also that

$$n^{v_G - v_H} p^{e_G - e_H} \leq n^{v_G - 2} p^{e_G - 1} = O(1) \tag{13}$$

by the strict balance of G. This gives the bound of

$$O(n^{v_G}) \sum_{k=3}^{\infty} \sum_{H_1,\dots,H_k} 2^{ke_G} \prod_{i=1}^k n^{v_G - v_{H_i}} p^{e_G - e_{H_i}} \leq O(n^{v_G}) \sum_{k=3}^{\infty} 2^{ke_G} c^k \tag{14}$$

for the expected number of all cycles in $\mathcal{G}$, which we will not use. However, the same estimate tells us that cycles, almost surely, cannot be longer than $O(\log n)$. More precisely, let L be the number of all cycles longer than $b \log n$. Then, for any a and $b = b(a)$ sufficiently large, and for $c < 2^{-e_G}$,

$$Exp(L) = O(n^{v_G}) \sum_{k=b\log n}^{\infty} 2^{ke_G} c^k = O(n^{-a}) \ . \tag{15}$$

We shall use this fact later. But now let us look closer at our estimates and see if we can do better. For clarity let us again assume that our cycle is simple and denote by e_i the edge shared by G_i and G_{i+1}, $i = 1,,,k$, $(G_{k+1} = G_1)$. The last exponent in the estimates, $v_G - 3$, was due to the possibility that e_{k-1} and e_k share a vertex and thus we still need $v_G - 3$ new vertices to build G_k. But this means that we only needed $v_G - 3$ new vertices for G_{k-1} provided e_{k-2} is disjoint from $e_{k-1} \cap e_k$. If not, we move backward one more step, and here we arrive at the origin of the notion of a normal cycle. For a normal cycle we shall finally find a moment where we need only $v_G - 3$ new vertices to build the next copy of G and the overall outcome will be just $O(1)$ and not $O(n)$. The only type of a simple non-normal cycle is one with the edges $e_1, ..., e_k$ forming a star. For non-simple cycles the picture is more complex and we need a very careful analysis. Let $Y = Y^{(s)} + Y^{(ns)}$ be the number of normal cycles broken into simple and non-simple ones. Given $H_1, ..., H_k$, all proper subgraphs of G without isolat ed vertices, and given two integer sequences $\mathbf{u} = (u_2, ..., u_{k-1})$ and $\mathbf{u}' = (u'_2, ..., u'_{k-1})$ with $u'_i \leq u_i$ and $u_i, u'_i \in \{0, ..., v_G\}$, $i = 2, ..., k-1$, let $Y(H_1, ..., H_k, \mathbf{u}, \mathbf{u}')$ be the number of normal cycles $E(G_1), ..., E(G_k)$ such that H_i is the graph spanned by $E(G_i) \cap E(G_{i+1})$, $i = 1, ..., k$ (here $G_{k+1} = G_1$) , $|V(H_i) \cap V(H_k)| = u_i$ and $|V(H_{i-1}) \cap V(H_i) \cap V(H_k)| = u'_i$, $i = 2, ..., k-1$. As before, each such a cycle has precisely

$$\alpha = ke_G - \sum_{i=1}^{k} e_{H_i}$$

edges and

$$Exp(Y(H_1, ..., H_k, \mathbf{u}, \mathbf{u}')) = N(H_1, ..., H_k, \mathbf{u}, \mathbf{u}')p^{\alpha} \ ,$$

where $N = N(H_1, ..., H_k, \mathbf{u}, \mathbf{u}')$ is the number of subgraphs of K_n giving rise to the cycles in question. As before there are no more than n^{v_G} choices of G_1 and no more than 2^{e_G} choices of H_k in G_1. Now, there are at most $2^{2v_G(k-2)}$ ways to fix all sets $U_i = V(H_i) \cap V(H_k)$ and $U_i' = U_i \cap V(H_{i-1})$, $i = 2, ..., k-1$. Once this is done, we proceed and select G_2 in at most $2^{e_G} n^{v_G - v_{H_1} - u_2 + u_2'}$ ways. Here the first factor takes care of the choice of H_1 and the exponent of n, being an upper bound on the number of new vertices of G_2, follows the Inclusion-Exclusion Principle. Continuing this way, for each $3 \le i \le k-1$ we choose G_i in at most

$$2^{e_G} n^{v_G - v_{H_{i-1}} - u_i + u_i'} \tag{16}$$

ways. Finally, we close the cycle with G_k chosen in no more than

$$2^{e_G} n^{v_G - v_{H_{k-1}} - v_{H_k} + u_{k-1}} \tag{17}$$

ways. After multiplying, the exponents of n sum up to $\beta + \gamma$, where $\beta = kv_G - \sum_{i=1}^k v_{H_i}$ and

$$\gamma = \sum_{i=2}^{k-1} (-u_i + u_i') + u_{k-1} = u_2' + \sum_{i=2}^{k-2} (-u_i + u_{i+1}') \ . \tag{18}$$

We shall now show that due to the normality of cycles in count, $\gamma \le 0$. For $k = 3$ the range of the above sum is empty, and indeed, then $\gamma = u_2'$ which is 0 by the definition of a normal cycle. To get this inequality for $k \ge 4$, observe that, for each $i = 2, ..., k-2$,

$$U_i' \subseteq (U_i \setminus U_{i+1}') \cup (U_i' \cap U_{i+1}') \ .$$

By subsequent substitutions one obtains

$$U_2' \subseteq \bigcup_{i=2}^{k-2} (U_i \setminus U_{i+1}') \cup \bigcap_{i=2}^{k-1} U_i'$$

but, by normality, $\bigcap_{i=2}^{k-1} U_i' = \emptyset$ and, as we also have $U_{i+1}' \subseteq U_i$, we infer that

$$u_2' \le \sum_{i=2}^{k-2} (u_i - u_{i+1}') \ . \tag{19}$$

Combining (18) and (19) we conclude that $\gamma \leq 0$. Hence, for some constant c', $N \leq (c')^k n^\beta$ and, by (13), using the same argument as for (14), we obtain

$$Exp(Y) \leq \sum_{k=3}^{\infty} \sum_{H_1,...,H_k} \sum_{\mathbf{u},\mathbf{u}'} (c')^k n^\beta p^\alpha = O(1) \ , \tag{20}$$

for c sufficiently small. However, for $Y^{(ns)}$ we can do even better. The reason is that then, among $H_1, ..., H_k$, there must be at least one proper subgraph H of G with at least 2 edges. Because G is strictly balanced, there is an $\varepsilon > 0$ such that

$$v_G - v_H - (e_G - e_H)\frac{v_G - 2}{e_G - 1} < -\varepsilon \ . \tag{21}$$

Therefore for at least one i we can use (13) with the extra factor of $n^{-\varepsilon}$ on the right-hand side, obtaining

$$Exp(Y^{(ns)}) = O(n^{-\varepsilon}) \ . \tag{22}$$

Thus normal non-simple cycles are rare and we have just excluded the structure (i) from $K(n,p)$. We are now well equipped to deal with all others quickly. Consider the number of bad cycles B. They are normal and simple, but in additi on, they have an extra vertex overlapping, not taken into account in the estimat es leading to (20). More precisely, there exists an i and $j \neq i$, and a ver tex $v \in V(G_i) \cap V(G_j) \setminus (V(H_{i-1}) \cup V(H_i))$. By throwing in a factor of k we may assume that $j = 1$. Hence, for $2 \leq i \leq k$, the estimates (16) and (17) can be now replaced by $2^{e_G} n^{v_G - v_{H_{i-1}} - u_i + u_i' - 1}$ and $2^{e_G} n^{v_G - v_{H_{k-1}} - v_{H_k} + u_{k-1} - 1}$, respectively. As there are $k-1$ choices of i we have an extra term of $k(k-1)/n$ in (20). Since $\sum k^2 c_0^k < \infty$ for $c_0 < 1$, the $k(k-1)$ factor "disappears" giving

$$Exp(B) = O(n^{-1}) \ ,$$

which means that also configurations of type (vi) are rare.

To attack normal cycles with handles we will apply an abbreviated technique. By (15) and (22), we can restrict our attention to normal, simple and short cycles.

By (20), the expected number of such cycles is $O(1)$ and the expected number of handles intersecting a fixed short cycle on a subgraph H is, by (21),

$$O((\log n)^{e_H-1} n^{v_G-v_H} p^{e_G-e_H}) = o(n^{-\frac{1}{2}\varepsilon}) = o(1)\ . \tag{23}$$

To make this argument more formal, let $Z = \sum_{H:2\le e_H<e_G} Z_H$, where Z_H is the number of subgraphs F of $K(n,p)$ which give rise to hypergraph structures consisting of a normal simple short cycle C and a handle G' intersecting C on a subg raph isomorphic to H. (Here C is the subgraph of $K(n,p)$ underlying the cyc le of $\mathcal{G}$ rather than the cycle itself. This is why we use the block letter C and not the script $\mathcal{C}$.) Then, by (20) and (23),

$$\begin{aligned} Exp(Z_H) = \sum_F p^{e_F} &\le \sum_C \sum_{G'\cap C=H} p^{e_C+e_G-e_H} \le \sum_C p^{e_C} \sum_{G'\cap C=H} p^{e_G-e_H} \\ &= Exp(Y) O((\log n)^{e_H-1} n^{v_G-v_H}) p^{e_G-e_H} = o(n^{-\varepsilon/2})\ . \end{aligned}$$

For disproving (iii)-(v) we will use the same trick, but as a preparation we need bounds on the expected numbers of rooted paths and cycles.

Given two elements of $X(\mathcal{H})$, x and y, a path of $\mathcal{H}$ is said to be *rooted at* x, y if only its first edge contains x and only its last edge contains y. Let now x, y be two pairs of vertices of $K(n,p)$ and consider the conditional space $K(n,p)\cup\{x,y\}$ with x and y present as an edge with probability 1. Let $R_{x,y}$ be the number of simple paths rooted at x, y in the corresponding hypergraph $\mathcal{G}_{x,y}$ of copies of G. A cycle *rooted at* x is a cycle containing the element x, and the definition of R_x is analogous to that of $R_{x,y}$. Estimates almost identical to those for $Exp(S)$ give $Exp(R_{x,y}) = O(\frac{1}{np})$ and $Exp(R_x) = O(\frac{1}{np})$. Indeed, the only diff erence is that now we have n^{v_G-2} choices of G_1 (since x is fixed) and the number of edges to appear with probability p is one less than before (sam e reason). This gives the extra n^2p factor in the denominator and the overall order of $O(\frac{1}{np})$ as we had $Exp(S) = O(n)$ before.

Again, by (15) and (22), in all (iii)-(v) we may assume that the cycles in count are simple and short. To rule out (iii), set Z to be the number of subgraphs F of

$K(n,p)$ which induce in $\mathcal{G}$ a normal simple short cycle C with a detour $D_{x,y}$ rooted at some $x, y \in E(C)$. (We consistently follow the convention of using the block letters to denote subgraphs of $K(n,p)$ inducing subhypergraphs of $\mathcal{G}$ designated by corresponding script letters.) We have by (20)

$$\begin{aligned} Exp(Z) &= \sum_F p^{e_F} \le \sum_C \sum_{x,y\in E(C)} \sum_{D:E(D)\cap E(C)=\{x,y\}} p^{e_C+e_D-2} \\ &\le \sum_C p^{e_C} O(\log^2 n) \sum_D p^{e_D-2} = Exp(Y)O(\log^2 n)Exp(R_{x_0,y_0}) \\ &= O\left(\frac{\log^2 n}{np}\right) = o(1) \ , \end{aligned}$$

where the last sum $\sum_D$ ranges over all simple paths rooted at two fixed edges of $K(n,p)$, x_0 and y_0, and is independent of the sum $\sum_C$.

To show that configurations (iv) are rare too, we argue similarly, obtaining

$$\begin{aligned} Exp(Z) &= \sum_F p^{e_F} \le \sum_{C_1} \sum_{x\in E(C_1)} \sum_{C_2:E(C_2)\cap E(C_1)=\{x\}} p^{e_{C_1}+e_{C_2}-1} \\ &\le \sum_{C_1} p^{e_{C_1}} O(\log n) \sum_{C_2} p^{e_{C_2}-1} = Exp(Y)O(\log n)Exp(R_{x_0}) \\ &= O\left(\frac{\log n}{np}\right) = o(1) \ . \end{aligned}$$

In case of (v), let Z be the number of subgraphs F of $K(n,p)$ inducing in $\mathcal{G}$ a pair of normal, simple and short cycles C_1, C_2, joined by a bridge M rooted at $x \in C_1$ and $y \in C_2$. (The use of letter M to designate a bridge is not accidental. Both, the Czech and Polish word for bridge is "most".) Then

$$\begin{aligned} Exp(Z) &= \sum_F p^{e_F} \le \sum_{C_1} p^{e_{C_1}} \sum_{C_2} p^{e_{C_2}} \sum_{x\in E(C_1)} \sum_{y\in E(C_2)} \sum_{M:E(M)\cap E(C_1\cup C_2)=\{x,y\}} p^{e_M-2} \\ &\le (Exp(Y))^2 O(\log^2 n) Exp(R_{x_0,y_0}) = O\left(\frac{\log^2 n}{np}\right) = o(1) \ . \end{aligned}$$

■

Configuration (vii) is the easiest case to disprove. For $G = C_4$, $m_G^{(2)} = \frac{3}{2}$ and the expected number of subgraphs of $K(n,p)$ with $k \le 8$ vertices and $l > \frac{3}{2}k$ edges is $O(\sum_k n^k p^{\frac{3}{2}k+\frac{1}{2}}) = O(\sqrt{p}) = o(1)$. This completes the proof of probabilistic lemma. □

4.Locally and globally sparse Ramsey graphs.

In this section we discuss some deterministic results which follow from or are directly related to our "random" results. We begin with recalling the story of the existence of graphs F not containing K_{k+1} and such that $F \to (K_l)^2_2$. Answering a question of Erdős and Hajnal [EH 67], several people (see [Gr 81] for an account) established the existence of such graphs. Graham [Gr 68] proved that the smallest such graph for $l = 3$ and $k = 5$ has 8 vertices and Irving [Ir 73] found a graph on 18 vertices satisfying the property for $l = 3$ and $k = 4$. In 1970 Folkman [Fo 70] proved the existence of K_{l+1}-free graphs F which arrow K_l, for every $l \geq 3$, providing an enormous upper bound on v_F. Everyone believed that such graphs are really rare. In [RR**] we prove our Theorem 3a in a stronger form with "almost surely" replaced by probability $1 - n^{-cn}$. This allows us to derive the following.

Corollary 1. *Let $l \geq 3$ and C_{K_l} be the constant from Theorem 3a. If $N = C_{K_l} n^{\frac{2l}{l+1}}$ then almost all K_{l+1}-free graphs F with n vertices and N edges satisfy $F \to (K_l)^2_2$.*

This is, in fact, a case of a more general result.

Theorem 4. *Let H be a graph with $m^{(1)}_H \geq m^{(2)}_G$ and $N = C_G n^{2-1/m^{(2)}_G}$, where C_G is the constant from Theorem 3a. Then almost all H-free (n, N)-graphs F satisfy $F \to (G)^2_2$.*

As another application, almost all n-vertex graphs with girth at least l and with $N = C_{C_l} n^{2-\frac{l-2}{l-1}}$ edges arrow C_l, the cycle of length l. In each case, the number of H-free graphs is about $\binom{\binom{n}{2}}{N} e^{-n}$. This can be proved by the FKG inequality (lower bound) and by the JŁR inequality [JŁR 90] (upper bound).

The graphs discussed above are examples of locally sparse Ramsey graphs. How sparse can they be? There exist constructive results addressing this question (see [NR 76,89]). Here we just mention some immediate corollaries from the proofs of Theorems 2a and 3a. First, in the vertex coloring case, the following result was obtained in [ŁRV 92].

Corollary 2. *For all graphs G and all positive integers r and k there exists a* *graph F such that $F \to (G)^1_r$ and for each subgraph H of F with $1 < v_H < k$, the inequality $m^{(1)}_H \le m^{(1)}_G$ holds.*

Please note that the above inequality is best possible, since there must be copies of G around.

In case of 2 colors the edge analog of Corollary 2 is given in [RR 95].

Corollary 3. *For all graphs G and all integers k there exists a graph F such that $F \to (G)^2_2$ and for each subgraph H of F with $2 < v_H < k$, the inequality $m^{(2)}_H \le m^{(2)}_G$ holds.*

We believe that the conclusion of Corollary 3 is independent of the number of colors. This was already confirmed in case of $G = K_3$ in [RR 94].

If we slightly relax the conditions imposed on small subgraphs in Corollaries 2 and 3, and instead require only that they satisfy the inequality $m_H \le m^{(1)}_G$ and $m_H \le m^{(2)}_G$, respectively, then, by the first moment method, and by Theorems 2a and 3a, for almost all $(n, C_G n^{2-1/m^{(i)}_G})$-graphs F, their small subgraphs obey the above inequalities, and at the same time, F enjoys the respective Ramsey properties, $i = 1, 2$. Conversely, every graph H with $m_H \le m^{(i)}_G$, by Bollobás' Threshold Theorem ([Bo 81]) and by [Ru 90, Cor.2], is a subgraph of a positive fraction of $(n, c_G n^{2-1/m^{(i)}_G})$-graphs, and since by Theorems 2b and 3b, almost all s uch graphs do not arrow G, we have $H \not\to (G)^1_r$, and $H \not\to (G)^2_2$, respectively.

In fact, the statements that $m_H \le m^{(i)}_G$ implies $H \not\to (G)^i_r$, $i = 1, 2$, were part of the argument in the proofs of Theorems 2b and 3b and therefore require independent proofs.

In view of the above, it is reasonable to define the following parameters measuri ng the global density of graphs which are Ramsey with respect to a given graph (as opposed to previously discussed locally sparse Ramsey graphs.) Define

$$m^{(i)}_{\inf}(G, r) = \inf\{m_F \ : \ F \to (G)^i_r\} \ ,$$

$i = 1, 2$. Several results on $m^{(1)}_{\inf}(G, r)$ can be found in [KR 93], among them general lower and upper bounds.

Theorem 5. *For all G and r*

$$\frac{1}{2}D(G)r \leq m^{(1)}_{inf}(G,r) \leq D(G)r \ ,$$

where $D(G) = \max_{H\subseteq G} \delta(H)$.

In the edge coloring case, if the graph G is not bipartite, the lower bound is already exponential in r. This stands in striking contrast to the bipartite case with an upper bound linear in r. Indeed, we have obtained the following preliminary result.

Proposition 2. *If $\chi(G) = \chi \geq 3$ and $F \to (G)^2_r$ then $m_F \geq \frac{1}{2}(\chi-1)^r$. If G is bipartite there exists an F such that $F \to (G)^2_r$ and $m_F \leq \Delta(G)r$.*

Proof. We note that

$$m_F \geq \frac{1}{2}D(F) \geq \frac{1}{2}(\chi(F)-1) \ , \tag{24}$$

the first inequality due to the fact that the minimum is never bigger than the average, the second inequality being the known upper bound on the chromatic number. But if $\chi(F) \leq (\chi-1)^r$ then one can partition the edge set of F into r $(\chi-1)$-partite subgraphs (see [Zy 49] or [Or 62]). Hence, if $F \to (G)^2_r$ then $\chi(F) \geq (\chi-1)^r + 1$ and consequently $m_F \geq \frac{1}{2}(\chi-1)^r$.

Let G be a bipartite graph with maximum degree d on one side and x vertices on the other. Let $n = R_d[r(d-1)+1, x]$ and $N = R_{r(d-1)+1}[n; r]$ be two Ramsey numbers in standard notation. Let F be a bipartite graph with bipartition (X,Y), where $|X| = N$, $|Y| = \binom{N}{r(d-1)+1}$ and each vertex v of Y is joined to a different $r(d-1)+1$-subset X_v of X. An r-coloring of the edges of F induces an r-coloring of the $r(d-1)+1$-subsets of X, by assigning to subset X_v the most frequent color occuring on the edges going from v to X_v. By the definition of N, there is an n-element subset Z of X, such that in every $r(d-1)+1$-subset of Z the dominant color is blue, sa y. Now define a 2-coloring of the d-subsets of Z by giving color 1 to those which are not joined to a single vertex of Y by edges colored all blue. By the definition of n, there must be an x-element subset of Z with all d-subsets colored by color 2. A blue copy of G can now be found and hence $F \to (G)^2_r$. It is easy to check that $m_F \le r(d-1)+1$. $\square$

Finally, to provide a proof of $m^{(2)}_{\inf}(G,2) \ge m^{(2)}_G$ independent of our Theorem 3b.

Theorem 6. *If $m_F \le m^{(2)}_G > 1$ then $F \not\to (G)^2_2$.*

Proof. Without loss of generality we may assume that G is strictly balanced. Therefore, with the exception of K_3, $m^{(2)}_G = \frac{e_G-1}{v_G-2} < \delta(G) \le D(G)$. By Theorem 5, $F \not\to (G)^1_2$, but it is well known that for a non-bipartite connected graph G, $F \not\to (G)^1_2$ implies $F \not\to (G)^2_2$. Thus for nonbipartite graphs (except for triangles) our theorem follows. Let us leave triangles for later and turn to the bipartite case. Consider 2 cases.

Case 1: there is an integer k such that $k \le m^{(2)}_G < k + \frac{1}{2}$.

Then $D(G) \ge k+1$ and $D(F) \le 2m_F \le 2k$ by (24). Thus, one can order the vertices of F so that the degree "to the right" is never bigger than $2k$ and coloring by each color no more than k edges going from a vertex "to the right" prevents us from obtaining a monochromatic G.

Case 2: there is an integer k such that $k + \frac{1}{2} \le m^{(2)}_G < k+1$.

Here we apply both, the Nash-Williams Arboricity Theorem and its analog for the bicyclic graph matroid (see [S-P 72, Be 73, Pa 86]). The former says that $\lceil m^{(1)}_F \rceil$ is the smallest number of forests which cover the edge set of F. The latter says that $\lceil m_F \rceil$ plays the same role with forests replaced by subgraphs whose each component has at most one cycle. We have $\lceil m_F \rceil \le k+1$. On the other hand, if $e_G < \frac{1}{4}v^2_G$ then $m_G \ge \frac{e_G}{v_G} > \frac{e_G-1}{v_G-2} - \frac{1}{2} \ge k$, and consequently $\lceil m_G \rceil \ge k+1$. Thus $\lceil m_F \rceil \le 2\lceil m_G \rceil - 2$, and, after partitioning F into $\lceil m_F \rceil$ appropriate subgraphs, we color at most $\lceil m_G \rceil - 1$ of them by each color, leaving no room for a monochromatic G. In the remaining case G is a complete bipartite graph with, say, l vertices on each side. For such graphs, $m^{(2)}_G = \frac{l+1}{2}$ and due to our assumptions $l = 2k$. So, $m^{(2)}_G = k + \frac{1}{2}$ and $m^{(1)}_G > k$. But also $m^{(1)}_F \le m_F + \frac{1}{2} \le k+1$ and we can repeat the previous argument with m replaced by $m^{(1)}$.

Finally, we deal with triangles. Assume that $G = K_3$ and $m_F \le 2$. Then, by (24),

$D(F) \leq 4$ which means that we can order the vertices $x_1, ..., x_f$, $f = v_F$, so that x_i has minimum degree in the subgraph H_i spanned by $x_i, ..., x_f$. Consequently, each vertex has at most 4 neighbors "to the right" and if x_i is the first vertex with exactly 4 neighbors "to the right" then the graph H_i is 4-regular. Indeed, $\delta(H_i) = 4$ but $\frac{e_{H_i}}{v_{H_i}} \leq m_F \leq 2$. Let us now reorder the vertices $x_i, ..., x_f$ so that, as long as we can, we avoid choosing vertices whose current neighborhood "to the right" spans K_4. Let x_j be the first vertex whose neighbors "to the right" form K_4. Then the graph H_j is a union of vertex disjoint K_5's (since H_i is 4-regular) and can be properly 2-colored. Moreover, all the edges between x_{i-1} and H_i, $i = j, j-1, ..., 2$, can be colored so that no monochromatic triangle is created. This can be easily checked as each v_i has at most 4 neighbors "to the right" and they never span a K_4. Thus all edges of F are properly colored. (This proof, by the way, resembles that of Theorem 1b in [ŁRV 92].) □

Among many questions that remain, what is $m^{(2)}_{\inf}(K_3, 2)$? We only know it is somewhere between 2 and 2.5.

Added in Proof. Very recently the authors proved Theorem 3 for an arbitrary number of colors.

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