On the maximum number of edges in a triple system not containing a disjoint family of a given size

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Abstract

In 1965 Erdős conjectured a formula for the maximum number of edges in a \( k \)-uniform \( n \)-vertex hypergraph without a matching of size \( s \). We prove this conjecture for \( k = 3 \) and all \( s \geq 1 \) and \( n \geq 4s \).

1 Introduction

A \( k \)-uniform hypergraph, or a \( k \)-graph for short, is a pair \( H = (V,E) \), where \( V := V(H) \) is a finite set of vertices and \( E := E(H) \subseteq \binom{V}{k} \) is a family of \( k \)-element subsets of \( V \). Whenever convenient we will identify \( H \) with \( E(H) \). A matching in \( H \) is a set of disjoint edges of \( H \). The number of edges in a matching is called the size of the matching. The size of the largest matching in a \( k \)-graph \( H \) is denoted by \( \nu(H) \). A matching is perfect if its size equals \( |V|/k \).

In this paper we study the relation between \( |E(H)| \) and \( \nu(H) \).

Definition 1.1 Let integers \( k, s \), and \( n \) be such that \( k \geq 2 \) and \( 0 \leq s \leq n/k \). Define \( m^*(k,n) \) to be the smallest integer \( m \) such that every \( n \)-vertex \( k \)-graph \( H \) with \( |E(H)| \geq m \) contains a matching of size \( s \). In other words,

\[
m^*(k,n) = \min \{ m : |E(H)| \geq m \implies \nu(H) \geq s \}.
\]

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It is quite easy to see that for graphs \( m^{n/2}(2, n) = \binom{n-1}{2} + 1 \). For \( k \)-graphs an analogous result is true: \( m^{n/k}(k, n) = \binom{n-1}{k} + 1 \). Indeed, the example of the clique on \( n - 1 \) vertices plus an isolated vertex yields the lower bound. For the upper bound, let \( k \) divide \( n \), and let \( H \) be an arbitrary \( n \)-vertex \( k \)-graph with at least \( \binom{n-1}{k} + 1 \) edges. Then, the complement \( H^c \) of \( H \) has fewer than \( \binom{n-1}{k} \) edges, and therefore has to miss at least one perfect matching of \( K_n^{(k)} \). (This can be seen by considering the expected number of edges in common with a random perfect matching.)

Erdős and Gallai [4] proved that for all \( 1 \leq s \leq n/2 \)

\[
m^s(2, n) = \max \left\{ \binom{2s-1}{2}, \binom{n}{2} - \binom{n-s+1}{2} \right\} + 1.
\]  

(1)

A few years later Erdős [3] conjectured a generalization to all \( k \geq 2 \) and \( 1 \leq s \leq n/k \):

\[
m^s(k, n) = \max \left\{ \binom{ks-1}{k}, \binom{n}{k} - \binom{n-s+1}{k} \right\} + 1.
\]  

(2)

The two competing \( k \)-graphs yielding the lower bound are \( K_{ks-1}^{(k)} \cup (n - ks + 1)K_1 \), that is, the clique on \( ks - 1 \) vertices appended by \( n - ks + 1 \) isolated vertices, and \( K_n^{(k)} - K_{n-s+1}^{(k)} \), a \( k \)-graph obtained from the complete \( k \)-graph on \( n \) vertices by deleting all edges of a fixed clique of order \( n - s + 1 \). Equivalently, \( K_n^{(k)} - K_{n-s+1}^{(k)} \) is the \( k \)-graph consisting of all \( k \)-element sets intersecting a given subset of vertices of size \( s - 1 \).

Trivially, the conjecture is true for \( s = 1 \), but already for \( s = 2 \) it is equivalent to the celebrated Erdős-Ko-Rado theorem. For larger \( s \), the conjecture has been confirmed by Erdős himself [3], but only for \( n \) sufficiently large with respect to \( k \) and \( s \). Later, Bollobás, Daykin, and Erdős [2], and Frankl and Füredi [6] improved the lower bound on \( n \), to \( 2k^3s \) and \( 100ks^2 \), respectively. We refer the reader to the survey paper [5]. Here we prove the Erdős conjecture for \( k = 3 \) and all \( s \geq 1 \) and \( n \geq 4s \).

**Theorem 1.1** For all \( s \geq 1 \) and \( n \geq 4s \), if \( H \) is a 3-uniform hypergraph with \( |V(H)| = n \) and \( \nu(H) \leq s - 1 \) then \( |H| \leq \binom{n}{3} - \binom{n-s+1}{3} \). In other words,

\[
m^s(3, n) = \binom{n}{3} - \binom{n-s+1}{3} + 1.
\]

Note that for \( k = 3 \) and \( n \geq 4s \) the maximum in (2) is achieved by the second term. The actual transition point is around \( (3.486)s \), so the question remains if Theorem 1.1 can be extended to all \( n \) for which \( \binom{3s-1}{3} \leq \binom{n}{3} - \binom{n-s+1}{3} \).

The proof of Theorem 1.1 relies on the technique of shifting and is presented in Section 2.

### 1.1 Minimum degree versus perfect matching

There are several results relating the minimum degree of a \( k \)-uniform hypergraph to the existence of a perfect matching (see, e.g., [10]). It has been shown in [1] that in order
to determine asymptotically a minimum degree guaranteeing the existence of a perfect matching in a $k$-graph on $n$ vertices, it suffices to prove a fractional version of Erdős’s conjecture for $(k-1)$-graphs with $n-1$ vertices and $s = \frac{n}{k}$. For $k = 3$ this can be easily deduced from Theorem 1.1 and we obtain the following corollary:

**Corollary 1.1** If $H$ is a 4-uniform hypergraph with the number of vertices $n$ divisible by 4 and with $\delta(H) \geq (1 + o(1)) \frac{37}{64} \binom{n-1}{3}$ then $H$ contains a perfect matching.

This improves a previous bound of $(1 + o(1)) \frac{42}{64} \binom{n-1}{3}$ due to Markström and Ruciński [9]. In [1] the fractional version of Erdős conjecture is proved, by quite different methods, also for 4-graphs with $s = \frac{n}{4}$. This yields an analog of Corollary 1.1 for 5-uniform hypergraphs with $\frac{369}{625} \binom{n-1}{4}$ in place of $\frac{37}{64} \binom{n-1}{3}$. (Both thresholds are asymptotically best possible.)

Recently, the result of Corollary 1.1 has also been proved, by quite different methods, by Khan [7] as well as by Lo and Markström [8].

## 2 Proof of Theorem 1.1

The proof is by induction on $s$, with the case $s = 1$ completely trivial. Before we show the induction’s step, let us recall the operation of shifting. Consider a hypergraph $H$ with the vertex set $V(H)$ ordered linearly, say $V(H) = \{1, 2, \ldots, n\}$. Given $1 \leq i < j \leq n$ and an edge $e \in H$, we define the $(i, j)$-shift $S_{ij}(e)$ of $e$ as follows:

$$S_{ij}(e) = \begin{cases} \{e \setminus \{j\}\} \cup \{i\} & \text{if } i \notin e, j \in e, (e \setminus \{j\}) \cup \{i\} \notin H, \\ e & \text{otherwise}. \end{cases} (3)$$

We define $S_{ij}(H) = \{S_{ij}(e) : e \in H\}$. We call $H$ shifted if $S_{ij}(H) = H$ for all $1 \leq i < j \leq n$. Note that shifting preserves the size of a hypergraph and that it does not increase the size of a largest matching. Formally, $|S_{ij}(H)| = |H|$ and $\nu(S_{ij}(H)) \leq \nu(H)$. Therefore, in what follows we may assume that $H$ is shifted.

Let us fix $s \geq 2$ and assume that Theorem 1.1 is true for all $s' < s$. In the proofs of the next two claims we are going to use the following notation: for all $v \in V(H)$, let

$$H_{\not\geq v} = \{e \in H : v \not\in e\}, \quad H_{\geq v} = \{e \in H : v \in e\}.$$

We will first show that it suffices to restrict the proof of Theorem 1.1 to the sole case of $n = 4s$.

**Claim 2.1** For all $s \geq 2$ and $n \geq 4s + 1$ if Theorem 1.1 holds for $n - 1$ then it also holds for $n$.

**Proof:** Let $H$ be a shifted 3-uniform hypergraph on $n$ vertices with $\nu(H) \leq s - 1$. Then, clearly, $|V(H_{\not\geq n})| = n - 1$, $\nu(H_{\not\geq n}) \leq s - 1$ and so, by our assumption that Theorem 1.1 holds for $n - 1$, we have $|H_{\not\geq n}| \leq \binom{n-1}{3} - \binom{n-s}{3}$. 

3
Let $H' = \{e \setminus \{n\} : e \in H_{3n}\}$. We claim that $\nu(H') \leq s - 1$. Suppose not. Then there are $s$ disjoint edges in $H'$ and, because $H$ is shifted, each of them forms an edge of $H$ with any vertex $v \in V(H)$. There are, however, at least $n - 2s \geq 2s + 1 \geq s$ vertices $v$ available and a matching of size $s$ exists in $H$, a contradiction.

Since $\nu(H') \leq s - 1$, by Erdős-Gallai Theorem, cf. (1), $|H'| \leq (n-1)/2 - (n-s)/2$. Hence,

$$|H| = |H_{3n}| + |H'| \leq \left(\frac{n-1}{3}\right) - \left(\frac{n-s}{3}\right) + \left(\frac{n-1}{2}\right) - \left(\frac{n-s+1}{2}\right) = \left(\frac{n}{3}\right) - \left(\frac{n-s+1}{3}\right).$$

We can therefore assume that $n = 4s$ throughout the rest of the proof.

**Claim 2.2** If $\{1, 3s - 1, 3s\} \in H$ and $\nu(H) \leq s - 1$ then $|H| \leq \binom{n}{3} - \binom{n-s+1}{3}$.

**Proof:** Suppose $\nu(H_{\geq 1}) = s - 1$. Then, because $H$ is shifted, there is a matching $M$ of size $s - 1$ in $H_{\geq 1}$ such that $V(M) = \{2, 3, \ldots, 3s - 2\}$. This matching, together with the edge $\{1, 3s - 1, 3s\}$ forms a matching of size $s$ in $H$, which contradicts the assumption that $\nu(H) \leq s - 1$. Consequently, $\nu(H_{\geq 1}) \leq s - 2$ and, hence, by induction on $s$, $|H_{\geq 1}| \leq (n-1)/3 - (n-s+1)/3$. On the other hand, trivially, $|H_{\geq 1}| \leq (n-1)/2$ and we conclude that

$$|H| \leq |H_{\geq 1}| + |H_{\geq 1}| \leq \left(\frac{n-1}{3}\right) + \left(\frac{n-1}{2}\right) - \left(\frac{n-s+1}{3}\right) = \left(\frac{n}{3}\right) - \left(\frac{n-s+1}{3}\right).$$

In view of Claim 2.2, we assume that $\{1, 3s - 1, 3s\} \notin H$. As a consequence, no edge of $H$ intersects the set $\{3s - 1, 3s, \ldots, 4s\}$ in 2 or 3 vertices. Let

$$F_0 = \left\{e \in \binom{\{4s\}}{3} : |e \cap \{3s - 1, 3s, \ldots, 4s\}| \geq 2 \right\}.$$

Then the complement $H^c$ of $H$ contains $F_0$ and

$$|F_0| = \left(\frac{s+2}{3}\right) + \left(\frac{s+2}{2}\right)(3s-2).$$

Hence, in order to prove Theorem 1.1, it is enough to verify that

$$|H^c \setminus F_0| \geq \left(\frac{3s+1}{3}\right) - |F_0| = \frac{1}{3} \left(17s^3 - 24s^2 - 5s + 12 \right) := W(s). \quad (4)$$

Consider an auxiliary bipartite graph $B = (X, Y, E(B))$ where $X = \{2s + 1, \ldots, 3s\}$, $Y = [s] := \{1, \ldots, s\}$ and, for $w \in X$ and $i \in Y$, the pair $\{w, i\} \in E(B)$ if $\{i, 2s + 1 - i, w\} \in H$. Since $\nu(H) \leq s - 1$, the graph $B$ does not have a perfect matching, and by Hall’s Theorem, there is a set $T \subseteq X$, $1 \leq t := |T| \leq s$, such that its neighborhood $N := N_B(T)$ has size $|N| = t - 1$. Because $H$ is shifted, we may assume that $T = \{3s - t + 1, \ldots, 3s\}$. 


Because $H$ is shifted, if $\{x, y, z\} \not\in H$ and $x \leq u, y \leq v$, and $z \leq w$, with all $u, v, w$ distinct, then also $\{u, v, w\} \not\in H$. In such case we say that triple $\{u, v, w\} \not\in H$ is forbidden by triple $\{x, y, z\} \not\in H$.

By the definition of the set $N$, for every $w \in T$ and $i \in [s] \setminus N$, we have $\{i, 2s+1-i, w\} \not\in H$. For each $i = 1, \ldots, s$, let $A_i$ be the set of all triples of $H^c \setminus F_0$ forbidden by $\{i, 2s+1-i, 3s-t+1\}$, that is,

$$A_i = \{1 \leq u < v < w \leq 4s : i \leq u, 2s+1-i \leq v \leq 3s-2, 3s-t+1 \leq w\}.$$  

Then

$$|H^c \setminus F_0| \geq |\bigcup_{i \in [s] \setminus N} A_i|. \tag{5}$$

Since our goal is to prove (4), it will be sufficient to show that $|\bigcup_{i \in [s] \setminus N} A_i| \geq W(s)$. We will consider two cases: $t = 1$ and $t \geq 2$, with the latter split into two subcases: $t = s$ and $2 \leq t \leq s - 1$.

**Case $t = 1$.** In this case $N = \emptyset$ and, hence, none of the triples $\{i, 2s+1-i, 3s\}, i = 1, \ldots, s$, belongs to $H$. In order to estimate $|\bigcup_{i \in [s] \setminus N} A_i|$ we count triples which are not forbidden by any of the triples $\{i, 2s+1-i, 3s\}$.

For each $w = 3s, 3s+1, \ldots, 4s$ only the pairs $\{i, j\}, i = 1, \ldots, s-1, j = i+1, \ldots, 2s-i$, can form an edge with $w$, altogether, at most $s(s-1)$ edges for each $w$. This means that

$$|\bigcup_{i \in [s] \setminus N} A_i| \geq (s+1) \left(\binom{3s-2}{2} - s(s-1)\right) = \frac{1}{2} (7s^3 - 6s^2 - 7s + 6)$$

and inequality (4) can be easily verified for every $s \geq 1$.

**Case $t \geq 2$.** In order to calculate $|A_i|$, we define four segments of vertices:

- $A = \{i, \ldots, 2s-i\}, a := |A| = 2s - 2i + 1,$
- $B = \{2s-i+1, \ldots, 3s-t\}, b := |B| = s - t + i,$
- $C = \{3s-t+1, \ldots, 3s-2\}, c := |C| = t - 2,$
- $D = \{3s-1, \ldots, 4s\}, d := |D| = s + 2.$

Observe that

$$A_i = \{1 \leq u < v < w \leq 4s : u \in A \cup B \cup C, v \in B \cup C, w \in C \cup D\},$$

and, moreover,

- $ab(c + d)$ triples in $A_i$ satisfy $u \in A, v \in B,$
- $a(cd + \binom{d}{2})$ triples in $A_i$ satisfy $u \in A, v \in C,$
• \( \binom{b}{3} (c + d) \) triples in \( A_i \) satisfy \( u \in B, v \in B \),
• \( b(cd + \binom{c}{2}) \) triples in \( A_i \) satisfy \( u \in B, v \in C \),
• \( \binom{c}{2} d + \binom{c}{3} \) triples in \( A_i \) satisfy \( u \in C, v \in C \).

Hence,

\[
|A_i| = ab(c + d) + a\left(cd + \binom{c}{2}\right) + \binom{b}{2} (c + d) + b\left(cd + \binom{c}{2}\right) + \binom{c}{2} d + \binom{c}{3}.
\]

After plugging in the formulas for \( a, b, c, d \) and collecting together all terms involving \( a \) we obtain the following formula.

\[
|A_i| = a[b(c + d) + q] + r_i = (2s - 2i + 1)[(s - t + i)(s + t) + q] + r_i,
\]

where

\[
q = \binom{c}{2} + cd = \binom{t - 2}{2} + (t - 2)(s + 2)
\]

and

\[
r_i = \binom{b}{2} (c + d) + bq + \binom{c}{2} d + \binom{c}{3}.
\]

**Subcase** \( t = s \). For \( t = s \) the above formula simplifies to

\[
|A_i| = (2s - 2i + 1)(2si + q) + si(i - 1) + qi + \binom{s - 2}{2}(s + 2) + \binom{s - 2}{3}
\]

which is a quadratic function of \( i \) with the main term \(-3si^2\). So, the minimum is achieved at either \( i = 1 \) or \( i = s \).

We have

\[
|A_1| = \frac{1}{3} (11s^3 - 12s^2 - 5s + 6)
\]

and

\[
|A_s| = \frac{1}{6} (19s^3 - 18s^2 - 7s + 6).
\]

It can be easily checked that for \( s \geq 1 \) both these quantities are greater than or equal to \( W(s) \), and so, the inequality (4) holds.

**Subcase** \( 2 \leq t \leq s - 1 \). In this case, we refine our estimates by considering unions \( |A_i \cup A_j| \).

Given \( 1 \leq i < j \leq s \), observe that

\[
A_i \cap A_j = \{1 \leq u < v < w \leq 4s : j \leq u, 2s + 1 - i \leq v \leq 3s - 2, 3s - t + 1 \leq w\},
\]

and thus, the formula for \( |A_i \cap A_j| \) can be obtained from that for \( |A_i| \) by replacing the set \( A = \{i, \ldots, 2s - i\} \) with \( \tilde{A} = \{j, \ldots, 2s - i\} \), in other words, replacing \( a = |A| \) with \( \tilde{a} = |	ilde{A}| = 2s - i - j + 1. \)
we estimate

\[ i, j \]

Since the second derivatives with respect to \( i, j, t \) at

\[ \text{Using (6) we can check that} \]

\[ s \]

because \( Q \) depends also on \( s \), but we suppress this dependence here. Since \( |s| \in [s] \setminus N \) such that \( s + t + 1 \leq j \leq s \) and \( 1 \leq i \leq j + s \). For every \( 2 \leq t \leq s - 1 \), we estimate

\[ \left| \bigcup_{i \in [s] \setminus N} A_i \right| \geq \max_{i,j \in [s] \setminus N} |A_i \cup A_j| \geq \min\{|A_i \cup A_j| : s - t + 1 \leq j \leq s, 1 \leq i \leq j + s\}. \]

Since \( Q(i, j, t) \) is a quadratic function of \( i \) with a negative coefficient at \( i^2 \), we have in the
given range of \( i \)

\[ Q(i, j, t) \geq \min\{Q(1, j, t), Q(j - s + t, j, t)\} \geq Q(j - s + t, j, t), \]

where the last inequality comes from direct comparison:

\[ Q(1, j, t) - Q(j - s + t, j, t) = (j + t - s - 1)(q + s + t) \geq 0 \]

because \( j \geq s - t + 1 \). Consequently, combining (4)-(8),

\[ |H^c \setminus F_0| \geq Q(j - s + t, j, t) + |A_j| = (s - t)[j(s + t) + q] + |A_j| := P(j, t) \]

for some \( j = s - t + 1, \ldots, s \). (Again, we suppress the dependence of \( P \) on \( s \).)

Using (6) we can check that \( P(j, t) \) is a quadratic function of \( j \) with a negative coefficient

\[ j^2, \]

and so

\[ P(j, t) \geq \min\{P(s - t + 1, t), P(s, t)\}. \]

Our plan is to express both \( f(t) := P(s, t) \) and \( g(t) := P(s - t + 1, t) \) as polynomials (of degree 3) in \( t \) and show that their minima over \( t, 2 \leq t \leq s - 1 \), are still at least as large as

the R-H-S of (4). After collecting all terms we obtain

\[ f(t) = -\frac{1}{3}t^3 - \frac{1}{2}(5s - 1)t^2 + \left(3s^2 + \frac{3}{2}s + \frac{5}{6}\right)t + 3s^3 - 5s^2 - 2s + 1, \]

and

\[ g(t) = -\frac{4}{3}t^3 - 2(s - 1)t^2 + \left(4s^2 - \frac{2}{3}\right)t + 3s^3 - 6s^2 - s + 2. \]

Since the second derivatives with respect to \( t \) satisfy \( f''(t) = -2t + 1 - 5s < 0 \) and \( g''(t) = -8t - 4s + 4 < 0 \), both functions are concave and the minima are attained at \( t = 2 \) or
\( t = s - 1 \). It remains to compare \( f(2), f(s - 1), g(2), \) and \( g(s - 1) \) against \( W(s) \), the R-H-S of (4). We have

\[
\begin{align*}
f(2) &= 3s^3 + s^2 - 9s + 2, \\
f(s - 1) &= \frac{1}{6} (19s^3 - 43s + 6), \\
g(2) &= 3s^3 + 2s^2 - 9s - 2, \\
g(s - 1) &= \frac{1}{6} (22s^3 - 70s + 36).
\end{align*}
\]

It can be easily checked that

\[
\min\{f(2), f(s - 1), g(2), g(s - 1)\} - W(s) \geq 0
\]

for every \( s \geq 2 \). This completes the proof of (4) and, therefore, also the proof of Theorem 1.1.

References


