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Powers of Hamiltonian cycles in randomly augmented graphs

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Abstract

We study the existence of powers of Hamiltonian cycles in graphs with large minimum degree to which some additional edges have been added in a random manner. It follows from the theorems of Dirac and of Komlós, Sarközy, and Szemerédi that for every $k \geq 1$ and sufficiently large n already the minimum degree $\delta(G) \geq \frac{k}{k+1}n$ for an n-vertex graph G alone suffices to ensure the existence of a kth power of a Hamiltonian cycle. Here we show that under essentially the same degree assumption the addition of just $O(n)$ random edges ensures the presence of the $(k+1)$st power of a Hamiltonian cycle with probability close to one.

KEYWORDS

random graphs; minimum degree conditions; powers of Hamiltonian cycles

1 | INTRODUCTION

All graphs we consider are finite and for simplicity we assume that the vertex set V of any given graph is the set $\{1, \dots, |V|\}$. We recall that for $k \in \mathbb{N}$ the kth power H^k of a graph H is defined to be a graph on the same vertex set, where edges in H^k signify that its vertices have distance at most k in H. Consequently, H^0 is the empty graph on the same vertex set and $H^1 = H$.

For integers $n \geq k+2$ and $k \geq 1$ we consider the set of graphs $\mathcal{P}_n^k$ consisting of all n-vertex graphs G that contain the kth power of a Hamiltonian cycle and we set $\mathcal{P}^k = \bigcup_{n \geq k+2} \mathcal{P}_n^k$. Clearly, $\mathcal{P}_n^k$ is a monotone graph property for fixed n and k, as powers of a Hamiltonian cycle cannot disappear by adding edges to a graph without adding new vertices.

We investigate the probabilities that a given n-vertex graph G with high minimum degree augmented by a binomial random graph $G(n,p)$ spans a kth power of a Hamiltonian cycle, that is, we are interested in $\mathbb{P}(G \cup G(n,p) \in \mathcal{P}_n^k)$. More formally, for $\alpha \in [0,1)$ and $p\colon \mathbb{N} \to [0,1]$ we say (α, p)

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ensures $\mathcal{P}^k$ if

$$\lim_{n\to\infty}\min_{G}\mathbb{P}\Big(G\cup G(n,p(n))\in \mathcal{P}^k_n\Big)=1\,,$$

where the minimum is taken over all n-vertex graphs G with $\delta(G)\geq \alpha n$. We are interested in the "minimal" pairs (α,p) that ensure $\mathcal{P}^k$.

For example, when $p=0$, then this reduces to the classical theorem of Dirac [8] on Hamiltonian cycles for $k=1$ and for $k\geq 2$ to the Pósa-Seymour conjecture [10, 24] and its resolution (for large n) by Komlós, Sarközy, and Szemerédi [16]. These beautiful results then assert that $\big(\frac{k}{k+1},0\big)$ ensures $\mathcal{P}^k$ for every $k\geq 1$.

For the other extreme case, when $\alpha=0$, we arrive at the threshold problem for the existence of powers of Hamiltonian cycles in $G(n,p)$. This was asymptotically solved by Posá [21] for $k=1$ (see also [1, 6] for sharper results). For $k=2$ the threshold is only known up to a factor of poly$(\log n)$ due to Nenadov and Škorić [19]. For $k\geq 3$ the threshold is given by a result of Riordan [22], which was observed by Kühn and Osthus [18]. Writing $\hat{p}_k(n)$ for the threshold for $\mathcal{P}^k$ then these results can be summarized by

$$\hat{p}_1(n)\sim\frac{\ln n}{n}\,,\qquad \Big(\frac{\mathrm{e}}{n}\Big)^{\frac12}\leq \hat{p}_2(n)=O\Big(\frac{(\ln n)^4}{\sqrt{n}}\Big)\,,\qquad \text{and}\qquad \hat{p}_k(n)\sim\Big(\frac{\mathrm{e}}{n}\Big)^{\frac1k}\ \text{ for } k\geq 3\,,$$

where ln stands for the natural logarithm $\log_{\mathrm{e}}$.

We study for $\alpha>\frac{k}{k+1}$ the asymptotics of the smallest function $p=p(n)$ such that (α,p) ensures $\mathcal{P}^{k+1}$. Recall that for $\alpha>\frac{k}{k+1}$ the Komlós-Sarközy-Szemerédi theorem asserts that $(\alpha,0)$ ensures $\mathcal{P}^k$ already. We show that under the same minimum degree assumption for n-vertex graphs G the addition of $O(n)$ random edges suffices to ensure $\mathcal{P}^{k+1}$, which is asymptotically best possible (see discussion below).

Theorem 1.1 *For every integer $k\in\mathbb{N}$ and every $\alpha\in\mathbb{R}$ with $\frac{k}{k+1}<\alpha<1$ there is some constant $C=C(k,\alpha)$ such that for $p=p(n)\geq C/n$ the pair (α,p) ensures $\mathcal{P}^{k+1}$.*

For $k=0$ Theorem 1.1 was already obtained by Bohman, Frieze, and Martin [5]. For larger k only suboptimal upper bounds for $p(n)$ were established so far. The best known bound of the form $\hat{p}_{k+1}(n)/n^{\delta}$ for some $\delta>0$ and $k\geq 2$ was given by Bedenknecht, Han, Kohayakawa, and Mota [3] for $k\geq 2$ (see also [4]).

The following construction shows that Theorem 1.1 is optimal in the sense that for every $\alpha>\frac{k}{k+1}$ there are n-vertex graphs G with $\delta(G)/n\geq\alpha>\frac{k}{k+1}$ that require at least $\Omega(n)$ additional random edges to ensure a $(k+1)$st power of a Hamiltonian cycle.

Let $(k+1)\mid n$ and consider a vertex partition $[n]=V_1\mathbin{\dot\cup}\cdots\mathbin{\dot\cup}V_{k+1}$ with each part of size $n/(k+1)$. Moreover, for every $i=1,\ldots,k+1$ fix some subset $W_i\subseteq V_i$ of size $|W_i|=\lceil\varepsilon n\rceil$ for some arbitrarily small $\varepsilon>0$.

Let G be the n-vertex graph consisting of the union of the complete $(k+1)$-partite graph with vertex partition $V_1\mathbin{\dot\cup}\cdots\mathbin{\dot\cup}V_{k+1}$ and $k+1$ complete bipartite graphs with vertex classes W_i and $V_i\smallsetminus W_i$ for $i=1,\ldots,k+1$. Clearly, $\delta(G)\geq(\frac{k}{k+1}+\varepsilon)n$. However, any copy of C_n^{k+1}, the $(k+1)$st power of a Hamiltonian cycle, contains $\lfloor n/(k+2)\rfloor$ vertex-disjoint copies of K_{k+2} and each of these cliques would require at least one edge contained in some set V_i. Consequently, every such clique has at least one vertex in $\bigcup_{i=1}^{k+1}W_i$ and, hence, G contains at most $\big|\bigcup_{i=1}^{k+1}W_i\big|=(k+1)\lceil\varepsilon n\rceil$ vertex disjoint K_{k+2}'s. This implies that for $\varepsilon\ll(k+1)^{-2}$ one needs to add at least a matching of size $\Omega(n)$ to G before it may have a chance to contain a copy of C_n^{k+1}.

In view of the optimality of Theorem 1.1, the next open problem might be to find the asymptotics of the minimal p such that (α, p) ensures $\mathcal{P}^{k+1}$ for $\alpha = \frac{k}{k+1}$ or even smaller values of α. This problem was also considered by Böttcher, Montgomery, Parczyk, and Person in [4].

2 | METHOD OF ABSORPTION

The proof of Theorem 1.1 is based on the *absorption method*, which has been introduced about a decade ago in [23]. Since then, it has turned out to be an extremely versatile technique for solving a variety of combinatorial problems concerning the existence of spanning substructures in graphs and hypergraphs obeying minimum degree conditions.

A nice feature of this method is that it often makes it possible to split the problem at hand into several subproblems, which may turn out to be more manageable. In the present case, we may reduce Theorem 1.1 to the Propositions 2.1-2.4 formulated later in this section.

Before stating the first of these propositions, we fix some terminology concerning powers of paths. A $(k+1)$-*path* is defined as the $(k+1)$st power of a path. The ordered sets of the first and last $k+1$ vertices are called the *end-sets* of the $(k+1)$-path, which must span $(k+1)$-cliques. If K and K' are the ordered cliques induced by the end-sets of a $(k+1)$-path P, we say that P *connects* K and K' and the vertices of P not contained in $V(K) \cup V(K')$ are its *internal* vertices.

We may now state the so-called Connecting Lemma, which is proved in Section 4. Roughly speaking, it asserts that in the graphs we need to deal with, one may connect any two disjoint $(k+1)$-cliques by means of a "short" $(k+1)$-path. Moreover, we want to declare some small proportion of the vertex set to be "unavailable" for such a connection (e.g., because we already have something else in mind that we want to do with those vertices), then the desired connection does still exist.

Proposition 2.1 (Connecting Lemma) *For every integer $k \geq 0$ and every $\varepsilon > 0$ there exists some $C > 1$ such that for every n-vertex graph G with $\delta(G) \geq (\frac{k}{k+1} + \varepsilon)n$ and $p = p(n) \geq C/n$ a.a.s. $H = G \cup G(n,p)$ has the following property.*[1]

For every subset $Z \subseteq V$ of size at most $\varepsilon n/2$ and every pair of disjoint, ordered $(k+1)$-cliques K, K', there exists a $(k+1)$-path connecting K and K' with exactly $(k+1)2^{k+1}$ internal vertices from $V \setminus Z$.

As the proof of Theorem 1.1 progresses, the number of vertices we do not want to use for connections anymore gets out of control. Therefore one puts a small set R of vertices aside at the beginning, which is called the *reservoir* and has the property that, actually, we can always connect any two given $(k+1)$-cliques through the reservoir. Of course, in order to use the reservoir multiple times, we shall need again a version, where a small part of the reservoir is "unavailable" at any particular moment. A precise version of the Reservoir Lemma, which is proved in Section 5, reads as follows.

Proposition 2.2 (Reservoir Lemma) *For every integer $k \geq 0$ and every $\varepsilon > 0$, $\gamma \in (0,1)$ there exists $C > 1$ such that for every n-vertex graph G with $\delta(G) \geq (\frac{k}{k+1} + \varepsilon)n$ there exists a set of vertices $R \subseteq V$ of size $\gamma^2 n$ such that for $p = p(n) \geq C/n$ a.a.s. $H = G \cup G(n,p)$ has the following property.*

For every $S \subseteq R$ with $|S| \leq \varepsilon|R|/4$ and for every pair of disjoint, ordered $(k+1)$-cliques K, K' in $G - R$, there exists a $(k+1)$-path connecting K and K' with exactly $(k+1)2^{k+1}$ internal vertices from $R \setminus S$.

[1]As usual *a.a.s.* abbreviates *asymptotically almost surely* and means that the statement holds with *probability tending to 1 as $n \to \infty$*. Strictly speaking, we should therefore consider arbitrary sequences $(G_n)_{n\in\mathbb{N}}$ of n-vertex graphs with $\delta(G_n) \geq (\frac{k}{k+1} + \varepsilon)n$. However, for a less baroque presentation we chose this "simplification" here and in the propositions below.

The next result (proved in Section 6) plays a central rôle and, in fact, this kind of statement gave the absorption method its name. It promises the existence of a very special, so-called *absorbing* $(k+1)$*-path* A, which can "absorb" any small set of vertices. Thus the problem of constructing the $(k+1)$st power of a Hamiltonian cycle gets reduced to the much easier problem of finding the $(k+1)$st power of an almost spanning cycle containing A. Let us remark at this point that the Absorbing Lemma gets utilized after the Reservoir Lemma and the set R appearing below takes this fact into account.

Proposition 2.3 (Absorbing Lemma) *For every integer $k \geq 0$ and every $\varepsilon > 0$ there exist $\gamma \in (0, \varepsilon/4^{k+2})$ and $C > 1$ such that for every n-vertex graph G with $\delta(G) \geq (\frac{k}{k+1} + \varepsilon)n$ and every $p = p(n) \geq C/n$ a.a.s. $H = G \cup G(n,p)$ has the following property.*

For every set of vertices $R \subseteq V$ of size $\gamma^2 n$ the graph $H - R$ contains a $(k+1)$-path A with at most $\gamma n/2$ vertices such that for every $U \subseteq V$ with $|U| \leq 2\gamma^2 n$ the graph $H[V(A) \cup U]$ contains a spanning $(k+1)$-path having the same end-sets as A.

The last ingredient of our argument is a statement to the effect that essentially the whole graph under consideration can be covered by "not too many" $(k+1)$-paths. Such paths can be connected together with the absorbing path A obtained earlier by means of "relatively few" connections to be made through the reservoir, thus producing the desired $(k+1)$st power of an almost spanning cycle. We shall prove this Covering Lemma in Section 7.

Proposition 2.4 (Covering Lemma) *For every integer $k \geq 0$ and every $\varepsilon > 0$, $\gamma \in (0, \varepsilon/2]$ there exists $C > 1$ such that for every n-vertex graph G with $\delta(G) \geq (\frac{k}{k+1} + \varepsilon)n$ and $p = p(n) \geq C/n$ a.a.s. $H = G \cup G(n,p)$ has the following property:*

For every subset $Q \subseteq V$ of size at most γn there exists a family of $\gamma^3 n$ vertex disjoint $(k+1)$-paths in $H - Q$ that cover all but at most $\gamma^2 n$ vertices from $V \setminus Q$.

We conclude the present section with a proof of our main result assuming the four propositions stated above. In fact, we shall not make a direct reference to Proposition 2.1 in the proof below, but it will be employed in the proof of Proposition 2.3 in Section 6.

Proof of Theorem 1.1 Let $k \in \mathbb{N}$ and $\alpha \in \left(\frac{k}{k+1}, 1\right)$ be given and set $\varepsilon = \alpha - \frac{k}{k+1}$. Plugging k and ε into Proposition 2.3 we get $\gamma \in (0, \varepsilon/4^{k+2})$ and $C_3 > 1$. Next we appeal with k, ε, and γ to Propositions 2.2 and 2.4, thus getting two further constants $C_2 > 1$ and $C_4 > 1$. We claim that $C = \max\{C_2, C_3, C_4\}$ is as desired.

So let an n-vertex graph G with $\delta(G) \geq (\frac{k}{k+1} + \varepsilon)n$ as well as some $p \geq C/n$ be given. We need to check that a.a.s. the graph $H = G \cup G(n,p)$ contains the $(k+1)$st power of a Hamiltonian cycle C_n^{k+1}. For this purpose it suffices to prove that every graph $H = G \cup G(n,p)$ exemplifying the conclusion of Proposition 2.x for each $x \in \{2, 3, 4\}$ contains a copy of C_n^{k+1}.

Use Proposition 2.2 for obtaining a reservoir set $R \subseteq V$ of size $\gamma^2 n$. By Proposition 2.3 there exists an absorbing $(k+1)$-path $A \subseteq H - R$. Since $|R| + |V(A)| \leq (\gamma^2 + \gamma/2)n < \gamma n$, we can apply Proposition 2.4 to $Q = R \cup V(A)$ and obtain a collection $\mathcal{P}$ of at most $\gamma^3 n$ vertex-disjoint $(k+1)$-paths covering the whole graph $H - Q$ except for a small set of vertices $U_\star \subseteq V \setminus Q$ with $|U_\star| \leq \gamma^2 n$. Now we want to create the $(k+1)$st power of a cycle $C \subseteq H$

- containing A and each $(k+1)$-path in $\mathcal{P}$ as a subpath,
- such that between any two "consecutive" such subpaths of C there are always exactly $(k+1)2^{k+1}$ vertices from R.

For building C we intend to make $|\mathcal{P}|+1$ successive applications of Proposition 2.2. In each such application we let K and K' be the end-sets of $(k+1)$-paths we wish to connect and we let $S \subseteq R$ be the set of all vertices that we obtained as internal vertices in previous applications of Proposition 2.2. When arriving at the last step of this process closing the cycle C, the set S of vertices we need to exclude has size

$$|S| = (k+1)2^{k+1} \cdot |\mathcal{P}| \leq 4^{k+1}\gamma^3 n \leq \frac{\varepsilon}{4}|R|\,,$$

which justifies the applications of Proposition 2.2.

Now the complement $U = V \setminus V(C)$ satisfies $|U| = |U_\star| + |R \setminus V(C)| \leq 2\gamma^2 n$, whence by Proposition 2.3 there exists a $(k+1)$-path A_U with $V(A_U) = V(A) \cup U$ having the same end-sets as A. Therefore, we can replace A by A_U in C and obtain the desired $(k+1)$st power of a Hamiltonian cycle $C_n^{k+1} \subseteq H$. ■

3 | PRELIMINARIES

In the proofs of the propositions stated in Section 2 we make use of the high minimum degree condition of the given graph G and combine it with properties of $G(n,p)$. We prepare for this by collecting a few observations for such graphs G in Section 3.1 and for the random graph in Section 3.2.

3.1 | Neighborhoods in graphs of large minimum degree

We recall the following standard notation. For a set V and an integer $j \in \mathbb{N}$ we write $V^{(j)}$ for the set of all j-element subsets of V. Given a graph $G = (V,E)$ we write $N_G(u)$ for the neighborhood of a vertex $u \in V$. More generally, for a subset $U \subseteq V$ we set

$$N_G(U) = \bigcap_{u \in U} N(u)$$

for the *joint neighborhood* of U. For simplicity we may suppress G in the subscript and for sets $\{u_1, \dots, u_r\}$ we may write $N(u_1, \dots, u_r)$ instead of $N(\{u_1, \dots, u_r\})$.

Lemma 3.1 *For every integer $k \geq 0$ and $\varepsilon > 0$ the following holds for every n-vertex graph $G = (V,E)$ with $\delta(G) \geq (\frac{k}{k+1} + \varepsilon)n$. For every $j \in [k+1]$ and every $J \in V^{(j)}$ we have*

$$|N(J)| \geq \left(\frac{k+1-j}{k+1} + j\varepsilon\right) n\,. \tag{3.1}$$

Furthermore, for $j \in [k]$ the induced subgraph $G[N(J)]$ satisfies

$$\delta(G[N(J)]) \geq \left(\frac{k-j}{k-j+1} + \varepsilon\right) |N(J)| \tag{3.2}$$

for every $J \in V^{(j)}$.

Proof First observe that De Morgan's law and Boole's inequality imply

$$n - |N(J)| = |V \setminus N(J)| = \left|\bigcup_{u\in J} \left(V \setminus N(u)\right)\right| \leq jn - \sum_{u\in J} |N(u)| \,.$$

Therefore,

$$\begin{aligned} |N(J)| &\geq \sum_{u\in J} |N(u)| - (j-1)n \geq j\delta(G) - (j-1)n \\ &\geq \left(\frac{jk}{k+1} + j\varepsilon\right)n - (j-1)n = \left(\frac{jk-(j-1)(k+1)}{k+1} + j\varepsilon\right)n \,, \end{aligned}$$

which yields (3.1).

Proceeding with (3.2) we note that every $v \in N(J)$ satisfies

$$|N(v) \cap N(J)| \geq \delta(G) - \left(n - |N(J)|\right) \geq \left(1 - \frac{n-\delta(G)}{|N(J)|}\right)|N(J)| \,.$$

Owing to the lower bound on $\delta(G)$ and that (3.1) implies $|N(J)| \geq \frac{k+1-j}{k+1}n$ we deduce

$$\frac{\delta(G[N(J)])}{|N(J)|} \geq 1 - \frac{n-\delta(G)}{|N(J)|} \geq 1 - \frac{1-\frac{k}{k+1}-\varepsilon}{\frac{k+1-j}{k+1}} \geq 1 - \frac{1}{k+1-j} + \varepsilon = \frac{k-j}{k+1-j} + \varepsilon \,,$$

as desired. ∎

3.2 | Janson's inequalities

We shall use the following variant of Janson's inequality [13] (see also [14]).

Theorem 3.2 (Janson's inequality) *Let $\varrho > 0$ and $C > 1$ be constants. Let $F = (V_F, E_F)$ be a forest and let $\mathcal{F}$ be a family of copies of F in K_n with $|\mathcal{F}| \geq \varrho n^{|V_F|}$.*

There exists some constant c_F only depending on F such that for $p \geq C/n$ the probability that $G(n,p)$ contains no copy of F from $\mathcal{F}$ is at most $2^{-c_F\varrho^2 pn^2}$.

The following further customized version of Janson's inequality will be utilized in our proof in Sections 4 and 5. Roughly speaking this version will guarantee that $G(n,p)$ provides the missing edges of a $(k+1)$-path connecting two $(k+1)$-cliques K and K' provided the deterministic graph G guarantees many short k-paths between K and K'.

Corollary 3.3 *For all integers $k,\ \ell \geq 0$ with $(k+1) \mid \ell$ and $\varrho > 0$ there exists $C > 0$ such that for every n-vertex graph G and $p \geq C/n$ the graph $H = G \cup G(n,p)$ satisfies with probability at least $1 - 4^{-n}$ the following property:*

If for a pair of ordered, disjoint $(k+1)$-cliques $K,\ K'$ in G there is a family $\mathcal{P}$ of at least $\varrho n^{\ell+2k+2}$ k-paths $P = x_1 \dots x_{k+1} y_1 \dots y_\ell x'_1 \dots x'_{k+1}$ in G such that $Kx_1 \dots x_{k+1}$ and $x'_1 \dots x'_{k+1}K'$ form $(k+1)$-paths, then there is at least one k-path $P \in \mathcal{P}$ such that KPK' forms a $(k+1)$-path in H.

Proof Let F denote the linear forest on $\ell + 2k + 2$ vertices consisting of $k+1$ disjoint paths on $2 + \ell/(k+1)$ vertices each. For each $P \in \mathcal{P}$ there is a copy F_P of F such that the union $P \cup F_P$ forms a $(k+1)$-path connecting K with K' (see Figure 1).

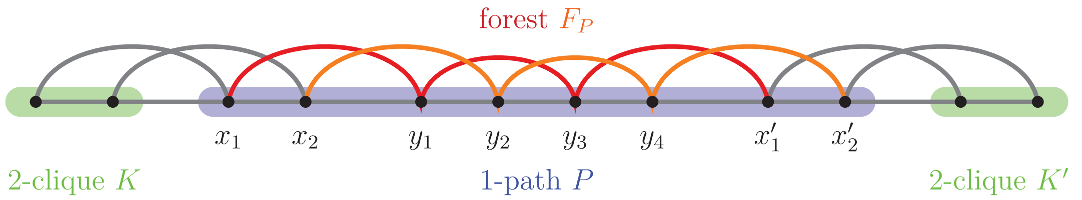

FIGURE 1 For $k = 1$ and $\ell = 4$ completing a 1-path P to a 2-path with a linear forest F_P consisting of two 3-edge paths [Colour figure can be viewed at wileyonlinelibrary.com]

We estimate the probability that at least one of them is a subgraph of $G(n, p)$. Setting $\mathcal{F} = \{F_P : P \in \mathcal{P}\}$ we have $|\mathcal{F}| = |\mathcal{P}| \geq \varrho n^{\ell+2k+2}$, Theorem 3.2 shows that for $C \geq 2c_F^{-1}\varrho^{-2}$ this leads to

$$\mathbb{P}\big(F_P \nsubseteq G(n,p) \text{ for all } F_P \in \mathcal{F}\big) \leq 4^{-n}\,,$$

and the corollary is proved. ∎

4 | PROOF OF THE CONNECTING LEMMA

In this section we establish Proposition 2.1. For that we first prove a deterministic lemma (see Lemma 4.1), which guarantees many short k-paths between every pair of disjoint k-cliques in large graphs G with sufficiently high minimum degree. Similar results appeared before in [12, 17]. We shall employ this result in the proof of Proposition 2.1, where at least one of these k-paths will be “thickened” to a $(k+1)$-path by an application of Janson's inequality in the form of Corollary 3.3.

In Lemma 4.1 it will be convenient to consider *k-walks*, which are defined like k-path, without the restriction that all vertices must be distinct. However, since we consider only graphs without loops, any k consecutive vertices in a k-walk must be distinct. As in the case of k-paths we say a walk connects the ordered k-cliques forming the ends of the walk and internal vertices are counted with their multiplicities (outside the ends).

Lemma 4.1 *For every integer $k \geq 1$ and $\varepsilon > 0$ there exists some $\varrho_k > 0$ such that every n-vertex graph G with $\delta(G) \geq (\frac{k}{k+1} + \varepsilon)n$ satisfies the following for $\ell_k = (k+1)(2^{k+1} - 2)$.*

For all pairs of disjoint, ordered k-cliques K, K' in G the number of k-walks connecting K and K' with ℓ_k internal vertices is at least $\varrho_k n^{\ell_k}$.

Proof We argue by induction on k. For $k = 1$ we have $\ell_1 = 4$ and the statement reduces to showing that any two distinct vertices x and y of an n-vertex graph G with minimum degree $\delta(G) \geq \left(\frac{1}{2} + \varepsilon\right)n$ are connected by $\varrho_1 n^4$ walks with four internal vertices for some $\varrho_1 = \varrho_1(\varepsilon) > 0$. The minimum degree condition implies that there are at least $(1/2 + \varepsilon)^3 n^3$ walks with three edges that start in x. Moreover, by (3.1) for $j = 2$ the end-vertex of each such walk has at least $2\varepsilon n$ joint neighbors with y, which gives rise to at least $2\varepsilon(1/2 + \varepsilon)^3 n^4$ different x-y-walks in G with four internal vertices. This establishes the induction start for $\varrho_1 = \varepsilon/4$.

For the inductive step we assume that the lemma holds for $k - 1$ in place of $k \geq 2$ and we consider a given n-vertex graph $G = (V, E)$ with $\delta(G) \geq (\frac{k}{k+1} + \varepsilon)n$. Given $\varepsilon > 0$ we will use some auxiliary constants ξ, ξ', ξ'', and ξ''' before we define ϱ_k. Moreover, given ϱ_{k-1} by the inductive assumption applied with ε, we shall work under the following hierarchy of constants

$$k^{-1}, \varepsilon \gg \varrho_{k-1}, \xi \gg \xi' \gg \xi'' \gg \xi''' \gg \varrho_k .$$

First we observe that for any $u, w \in V(G)$ the case $j = 2$ of (3.1) and (3.2) yields

$$|N(u,w)| \geq \frac{k-1}{k+1}n \quad \text{and} \quad e(N(u,w)) \geq \left(\frac{k-2}{k-1} + \varepsilon\right)\frac{|N(u,w)|^2}{2} .$$

Hence, it follows from Turán's theorem that $G[N(u,w)]$ induces a copy of K_k and owing to the so-called *supersaturation phenomenon* (see, e.g., [11]) the induced subgraph $G[N(u,w)]$ contains $\Omega(|N(u,w)|^k) = \Omega(n^k)$ copies of K_k. Consequently, there exists $\xi = \xi(k,\varepsilon) > 0$ such that

$$\left|\left\{K_k \subseteq G[N(u,w)]\right\}\right| \geq \xi n^k , \tag{4.1}$$

that is, there are at least ξn^k copies of K_k contained in $G[N(u,w)]$ for any vertices $u, w \in V$.

We consider two disjoint, ordered k-cliques K and K'. As a preliminary step we first extend K' in a greedy manner by k vertices. (This seems like an unnecessary step but it is needed to fulfill a certain divisibility condition at the end of this proof.) The total number of these extensions is, by k applications of (3.1) with $j = k$, at least

$$\left(\left(\frac{1}{k+1} + k\varepsilon\right)n\right)^k \geq \left(\frac{n}{k+1}\right)^k , \tag{4.2}$$

as we do not require that all these vertices are distinct from those in K or K'. Let $\mathcal{L}'$ be the set of all ordered k-tuples obtained this way. By construction for every $L' \in \mathcal{L}'$ we have that $L'K'$ induces a k-walk connecting L' and K' without internal vertices.

Next we connect K with every $L' \in \mathcal{L}'$ by a k-walk. Again we infer from (3.1) that we have $|N(V(K))| \geq \frac{n}{k+1}$ and $|N(V(L'))| \geq \frac{n}{k+1}$. It, therefore, follows from (4.1) that

$$\sum_{u \in N(V(K))} \sum_{w \in N(V(L'))} \left|\left\{M : M \cong K_k \text{ and } M \subseteq G[N(u,w)]\right\}\right| \geq \frac{\xi n^{k+2}}{(k+1)^2} .$$

By a double counting argument, this implies that there are at least $\xi' n^k$ k-cliques M in G for which

$$\left|\left\{(u,w) \in N(V(K)) \times N(V(L')) : M \subseteq G[N(u,w)]\right\}\right| \geq \xi' n^2.$$

For fixed such M we let U_K denote the set of those vertices $u \in N(V(K))$ that belong to at least one such pair and let $W_{L'} \subseteq N(V(L'))$ be defined in the same way. Clearly, we have $|U_K|, |W_{L'}| \geq \xi' n$.

We connect K and L' by k-walks through M. For the k-walk connecting K and M we shall use the properties of $u \in U_K$ and, analogously, we rely on the properties of $w \in W_{L'}$ for the k-walk connecting M and L'. Recall that for every $u \in U_K$ we have $K \subseteq G[N(u)]$, $M \subseteq G[N(u)]$ and an application of (3.2) with $j = 1$ gives

$$\delta(G[N(u)]) \geq \left(\frac{k-1}{k} + \varepsilon\right)|N(u)| .$$

Thus, by the inductive assumption, there are at least $\varrho_{k-1} n^{\ell_{k-1}}$ $(k-1)$-walks connecting the last $k-1$ vertices of K and the first $k-1$ vertices of M and each such walk has ℓ_{k-1} internal vertices. Let $K_+ \subseteq K$

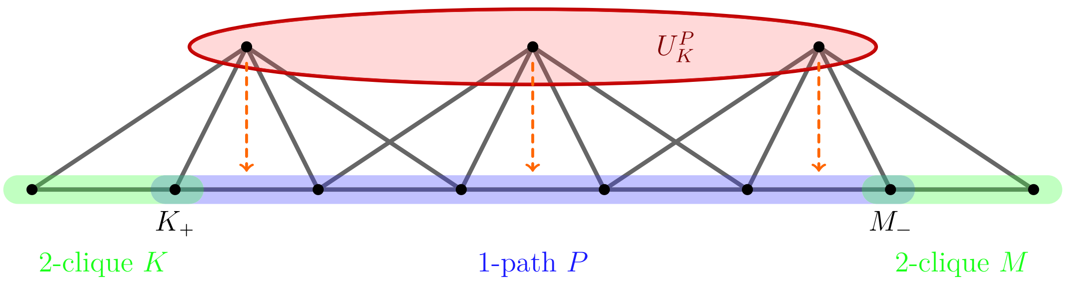

FIGURE 2 Building a 2-path Q for $k = 1$ that connects K and M by adding vertices from U_K^P to a 1-path P connecting K_+ and M_- at the indicated places [Colour figure can be viewed at wileyonlinelibrary.com]

be the ordered $(k-1)$-clique spanned by the last $k-1$ vertices of K and let $M_- \subseteq M$ be the ordered $(k-1)$-clique spanned by the first $k-1$ vertices of M. Repeating this argument for every vertex $u \in U_K$ we obtain at least

$$|U_K| \cdot \varrho_{k-1} n^{\ell_{k-1}} \geq \xi' \varrho_{k-1} n^{1+\ell_{k-1}} = \xi'' n^{1+\ell_{k-1}}$$

pairs (u, P) where $u \in U_K$ and P is a $(k-1)$-walk connecting K_+ and M_- in $G[N(u)]$. As there are no more than $n^{\ell_{k-1}}$ such walks in G, there are at least $\frac{1}{2}\xi'' n$ vertices $u \in U_K$ for which $G[N(u)]$ contains at least $\frac{1}{2}\xi'' n^{\ell_{k-1}}$ of these walks. Let us fix one such $(k-1)$-walk P and denote by U_K^P the subset of U_K consisting of the vertices u such that $P \subseteq G[N(u)]$. Next we construct a k-walk Q from K to M by inserting

$$\frac{\ell_{k-1}}{k} + 1 = 2^k - 1$$

vertices from U_K^P into P in such a way that there are exactly k internal vertices of the $(k-1)$-walk P between each consecutive pair of the vertices of U_K^P (see Figure 2).

Note that any such k-walk Q created this way is indeed a k-walk connecting K and M including the first vertex of K and the last vertex of M, as every vertex $u \in U_K^P \subseteq U_K$ contains K and M in its neighborhood. Note that this way we ensure the existence of at least

$$\frac{1}{2}\xi'' n^{\ell_{k-1}} \cdot \left(\frac{1}{2}\xi'' n\right)^{2^k-1} = \xi''' n^{\ell_{k-1}+2^k-1}$$

k-walks connecting K and M.

The same argument applied for M and L' (instead of K and M) using the set $W_{L'}$ yields $\xi''' n^{\ell_{k-1}+2^k-1}$ k-walks connecting M and L'. Consequently, for fixed M and L' we obtain

$$\left(\xi''' n^{\ell_{k-1}+2^k-1}\right)^2$$

k-walks connecting K and K' that pass through M and L'. We recall that there are at least $\xi' n^k$ choices for the clique M for fixed $L' \in \mathcal{L}'$ and that $|\mathcal{L}'| \geq \left(\frac{n}{k+1}\right)^k$ (see (4.2)). Therefore, the number of k-walks connecting K and K' is at least

$$\left(\xi'''\right)^2 n^{2\ell_{k-1}+2^{k+1}-2} \cdot \xi' n^k \cdot \left(\frac{n}{k+1}\right)^k \geq \varrho_k n^{2\ell_{k-1}+2^{k+1}-2+2k} = \varrho_k n^{\ell_k},$$

where the last identity follows from $\ell_{k-1} = k(2^k - 2)$, which gives indeed

$$2\ell_{k-1} + 2^{k+1} - 2 + 2k = 2k(2^k - 2 + 1) + 2^{k+1} - 2 = (k+1)(2^{k+1} - 2) = \ell_k .$$

This concludes the inductive step and the proof of Lemma 4.1. ■

It is left to deduce Proposition 2.1 from Lemma 4.1. Roughly speaking, Lemma 4.1 verifies the assumptions of Corollary 3.3, which then guarantees that at least one given k-path will be enriched to a $(k+1)$-path by the random graph $G(n,p)$.

Proof of Proposition 2.1 Let $k \geq 0$ and $\varepsilon > 0$ be given. If $k = 0$, then we set $\varrho_0 = 1$, and for $k \geq 1$, we appeal to Lemma 4.1 applied with k and $\varepsilon/2$ and obtain a constant $\varrho_k > 0$. We then let $C > 1$ be given by Corollary 3.3 applied with

$$k\,, \qquad \ell = (k+1)(2^{k+1} - 2)\,, \qquad \text{and} \qquad \varrho = \frac{1}{2^{\ell+1}}\varrho_k \cdot \left(\frac{\varepsilon}{2}\right)^{2k+2} .$$

Finally, let $G = (V, E)$ be an n-vertex graph with $\delta(G) \geq \left(\frac{k}{k+1} + \varepsilon\right)n$ and $p \geq C/n$.

Consider a set $Z \subseteq V$ of size at most $\varepsilon n/2$ and let K and K' be two disjoint, ordered $(k+1)$-cliques in $G - Z$. In order to meet the assumptions of Corollary 3.3 we first show that there are many ways to greedily extend K and K' to $L = x_1 \ldots x_{k+1}$ and $L' = x'_1 \ldots x'_{k+1}$ and then we use Lemma 4.1 to show that there are many possibilities to connect L and L'.

We remedy this by first selecting $(k+1)$-cliques L and L' in $G - Z$ such that KL and $L'K$ form $(k+1)$-walks. In fact, since $\delta(G-Z) \geq \left(\frac{k}{k+1} + \frac{\varepsilon}{2}\right)n$ we infer that $k+1$ applications of (3.1) for $j = k+1$ in $G - Z$ give rise to at least $(\varepsilon n/2)^{k+1}$ such ordered $(k+1)$-cliques L. Similarly, there are at least $(\varepsilon n/2)^{k+1}$ such ordered $(k+1)$-cliques L'. For two such ordered cliques L and L' let L_+ be the last k vertices in L and let L'_- be the first k vertices in L'.

For $k \geq 1$ the graph $G - Z$ satisfies the assumption of Lemma 4.1 with $\varepsilon/2$ instead of ε and, hence, the lemma yields $\varrho_k|V \setminus Z|^\ell$ k-walks connecting L_+ and L'_- in $G - Z$ with $\ell = (k+1)(2^{k+1} - 2)$ internal vertices. For $k = 0$ we have $\ell = 0$ and we note that for the 0-cliques L_+ and L'_- and the empty path might be considered as a 0-path connecting those.

Consequently, for any value of k there are $\varrho_k|V \setminus Z|^\ell$ k-walks connecting L_+ and L'_- for all considered $(k+1)$-cliques L and L'. Going over all such $(k+1)$-cliques L and L' this gives rise to

$$\left(\frac{\varepsilon}{2}n\right)^{2k+2} \cdot \varrho_k\left(\frac{1}{2}n\right)^{\ell}$$

such k-walks $x_1 \ldots x_{k+1} y_1 \ldots y_\ell x'_1 \ldots x'_{k+1}$. Since at most $(2k+\ell)\ell n^{\ell-1}$ of these k-walks may repeat a vertex, that is, walks where the vertices $y_1, \ldots, y_\ell$ are not pairwise different or they are not distinct from K or from K', for sufficiently large n, we may assume that at least half of these k-walks are indeed k-paths disjoint from K and K'. This verifies the assumptions of Corollary 3.3, which with probability at least $1 - 4^{-n}$ yields a desired $(k+1)$-path connecting K and K' in $H = G \cup G(n,p)$

Finally, the union bound over up to at most n^{2k+2} choices for K and K' and at most 2^n choices for Z shows that a.a.s. $H = G \cup G(n,p)$ enjoys the conclusion of Proposition 2.1. ■

5 | PROOF OF THE RESERVOIR LEMMA

Proof of Proposition 2.2 Consider a random subset $R \subseteq V$ with $|R| = \gamma^2 n$ chosen uniformly at random. Since $\delta(G) \geq (\frac{k}{k+1} + \varepsilon)n$, it follows from a version of Chernoff's inequality appropriate for hypergeometric distributions that for each vertex $v \in V$ the bad event that $|N(v) \cap R| < (\frac{k}{k+1} + \frac{\varepsilon}{2})|R|$ holds has probability $e^{-\Omega(n)}$. Thus, by the union bound, the probability that there exists some $v \in V$ for which this bad event occurs is $o(1)$.

This proves, in particular, that there exists some set $R \subseteq V$ with $|R| = \gamma^2 n$ and

$$|N(v) \cap R| \geq \left(\frac{k}{k+1} + \frac{\varepsilon}{2}\right)|R| \quad \text{for every } v \in V. \tag{5.1}$$

For the rest of the proof we fix some such set $R \subseteq V$ having these properties. Notice that (5.1) immediately entails that

$$|N(J) \cap R| \geq \tfrac{1}{2}\varepsilon|R| \quad \text{holds for all } J \in V^{(k+1)}. \tag{5.2}$$

Let us now fix two ordered $(k+1)$-cliques K and K' in $G - R$ as well as a subset $S \subseteq R$ with $|S| \leq \frac{1}{4}\varepsilon|R|$. Consider the bad event $\mathcal{E}$ that there is no $(k+1)$-path in H connecting K with K' having

$$\ell = (k+1)2^{k+1}$$

internal vertices all of which belong to $R \setminus S$. It suffices to prove that

$$\mathbb{P}(\mathcal{E}) \leq 4^{-n}. \tag{5.3}$$

This is because there are at most n^{k+1} possibilities for each of K and K' and at most 2^n possibilities for S, meaning that once (5.3) is established it will follow that the probability that H fails to have the desired property is at most $n^{2k+2}2^n \cdot o(4^{-n}) = o(1)$, as desired.

For the proof of (5.3) we note that due to (5.2) we can greedily extend K to a $(k+1)$-path KL, where L denotes some ordered $(k+1)$-clique in $G[R \setminus S]$. More precisely, since $|S| \leq \frac{1}{4}\varepsilon|R|$ each vertex of such a clique L can be chosen in at least $\frac{1}{4}\varepsilon|R|$ many ways and thus the set $\mathcal{L}$ containing all such cliques L satisfies $|\mathcal{L}| \geq (\frac{1}{4}\varepsilon|R|)^{k+1}$.

Applying the same reasoning to backwards extensions of K' we infer that the set $\mathcal{L}'$ consisting of all ordered $(k+1)$-cliques L' in $G[R \setminus S]$ for which $L'K'$ is a $(k+1)$-path in G has at least the size $|\mathcal{L}'| \geq (\frac{1}{4}\varepsilon|R|)^{k+1}$.

Now let $\mathcal{P}$ be the collection of all k-paths in $G[R \setminus S]$ having ℓ vertices that start with a member of $\mathcal{L}$ and end with a member of $\mathcal{L}'$. To derive a lower bound on $|\mathcal{P}|$ we note that as a consequence of (5.1) the graph $G[R \setminus S]$ satisfies the assumptions of Lemma 4.1 with $\varepsilon/4$ here in place of ε there. Thus for some sufficiently small choice of $\varrho_k > 0$, Lemma 4.1 guarantees that for every $L \in \mathcal{L}$ and $L' \in \mathcal{L}'$ there are at least $\varrho_k|R \setminus S|^{\ell-2k-2}$ k-walks with $\ell - 2k - 2$ internal vertices connecting the last k vertices of L with the first k vertices of L'. Without loss of generality we may assume that $\varrho_k \ll \varepsilon^{2k+2}/2^{\ell}$ and since most of these walks are indeed paths for sufficiently large n, this shows that

$$|\mathcal{P}| \geq \frac{\varrho_k}{2}|\mathcal{L}||\mathcal{L}'||R \setminus S|^{\ell-2k-2} \geq \frac{\varrho_k}{2}\left(\frac{\varepsilon}{4}\right)^{2k+2}\left(1 - \frac{\varepsilon}{4}\right)^{\ell}|R|^{\ell} \geq \varrho_k^2|R|^{\ell}. \tag{5.4}$$

Consequently, we can invoke Corollary 3.3 for ℓ, k, and $\varrho = \varrho_k^2$, which yields (5.3) and thereby Proposition 2.2 is proved. ■

6 | PROOF OF THE ABSORBING LEMMA

The present section is dedicated to the proof of Proposition 2.3. As in many earlier applications of the absorbing method the core idea is to take a random collection of $\Omega(n)$ small configurations called *absorbers*, which are then connected by means of the Connecting Lemma to form the desired path A.

The absorbers we shall use later will simply be $(k+1)$-paths on $2k+2$ vertices. When such a path P appears in the neighborhood of some vertex x, we have the liberty to insert x in the middle of P, thus creating a longer $(k+1)$-path. In other words, the path P can *absorb* x. Now the plan is to construct A so as to contain many disjoint absorbers and to make sure that for every $x \in V$ there will be at least $2\gamma^2 n$ absorbers in A capable of absorbing x.

For standard reasons in the area detailed more fully below, the task of proving Proposition 2.3 gets thus reduced to estimating the number of such absorbers in H. This requires to deal with the interplay of the deterministic part G and the random part $G(n,p)$ of H. It turns out to be convenient to insist that our absorbers are entirely contained in G, except for their "middle edges," which will have to be taken from $G(n,p)$. It thus becomes necessary to argue that $G(n,p)$ is likely to "complete" many x-absorbers for every $x \in V$ and for doing so we exhibit auxiliary graphs B_x with $\Omega(n^2)$ edges and show that a.a.s. $G(n,p)$ intersects each of them in $\Omega(n)$ edges.

Accordingly, the proof of Proposition 2.3 consists of four steps.

- Define for each $x \in V$ a graph B_x on V of size $\Omega(n^2)$ depending only on G.
- State properties $G(n,p)$ is likely to have that will imply the existence of A in a deterministic sense.
- Perform a random selection of $\Omega(n)$ absorbers.
- Connect these absorbers, thus obtaining A.

Proof of Proposition 2.3 We work with a hierarchy

$$k^{-1}, \varepsilon \gg \beta \gg \gamma \gg C^{-1}$$

and we consider an n-vertex graph $G=(V,E)$ with $\delta(G) \geq (\frac{k}{k+1}+\varepsilon)n$.

The graphs B_x

Let P denote the $(k+1)$-path on $2k+2$ vertices $1,\ldots,2k+2$ and let P^- be the graph obtained from P by deleting the middle edge between $k+1$ and $k+2$. Notice that the chromatic number of P^- is (at most) $k+1$, an admissible coloring being the map $\varphi : V(P^-) \longrightarrow [k+1]$ assigning the colors $1,\ldots,k+1,k+1,1,\ldots,k$ in this order to the vertices of P^-, that is, explicitly

$$\varphi(i) = \begin{cases} i & \text{if } 1 \leq i \leq k+1, \\ k+1 & \text{if } i = k+2, \\ i-k-2 & \text{if } k+3 \leq i \leq 2k+2. \end{cases}$$

We claim that for every vertex $x \in V$ there are at least

$$\beta n^{2k+2} \text{ ordered copies of } P^- \text{ in } G[N(x)]\,. \tag{6.1}$$

This is clear for $k=0$, as in this case the graph P^- has two vertices and no edges. If $k>0$ we apply Lemma 3.1 to $J=\{x\}$ and learn that the graph $G[N(x)]$ has order at least $\frac{k}{k+1}n$ and minimum degree at least $(\frac{k-1}{k}+\varepsilon)|N(x)|$. So by the Erdős-Stone theorem there is at least one copy of P^- in $G[N(x)]$ and by supersaturation (see, e.g., [11]) there are indeed at least βn^{2k+2} ordered copies of P^- in $G[N(x)]$, which completes the proof of (6.1).

Let B_x be a graph on V whose edges are the pairs vv' with the property that there are at least βn^{2k} injective graph homomorphism $\varphi : P^- \to G[N(x)]$ with $\varphi(k+1) = v$ and $\varphi(k+2) = v'$. It follows from the discussion above that

$$e(B_x) \geq \beta n^2/2\,. \tag{6.2}$$

Properties of $G(n,p)$

We will now check that the following statements hold a.a.s.

1. $G(n,p)$ has at most Cn edges.
2. There are at most $2C^2n$ ordered pairs (e,e') of intersecting edges in $G(n,p)$.
3. For every $R \subseteq V$ with $|R| \leq \gamma^2 n$ and every $v \in V$ at least $\beta Cn/4$ edges of $B_x - R$ appear in $G(n,p)$.

Notice that (1) is straightforward by Chernoff's inequality. For (2) we remark that the random variable counting such pairs has expected value and variance $O_C(n)$ and, therefore, Chebyshev's inequality applies. Finally, for every R and x as in (3) we have $e(B_x - R) \geq \beta n^2/2 - |R|n \geq \beta n^2/3$ by (6.2) and $\gamma \ll \beta$. Thus the expected value of the number $X_{R,x}$ of edges that $G(n,p)$ and $B_x - R$ have in common is at least $\beta Cn/3$. In view of Chernoff's inequality (see [15, Section 2.1]) and $C \gg \beta^{-1}$ it follows that

$$\mathbb{P}(X_{R,x} < \beta Cn/4) < e^{-\beta Cn/96} < 4^{-n}\,.$$

Taking the union bound over all choices for the pair (R,x) we infer that (3) fails with a probability of at most $n2^n \cdot 4^{-n} = o(1)$.

Having thus proved (1)-(3) to hold a.a.s. we shall henceforth regard $G(n,p)$ as a fixed graph having these properties, for which, moreover, the conclusion of Proposition 2.1 is valid.

As we shall see, these assumptions imply the existence of the desired absorbing path. Let us fix a set $R \subseteq V$ with $|R| \leq \gamma^2 n$ from now on.

Selection of absorbers

An ordered copy $\vec{v} = (v_1,\ldots,v_{2k+2}) \in (V \setminus R)^{2k+2}$ of P^- in $G - R$ with $v_{k+1}v_{k+2} \in E(G(n,p))$ is called an *absorber*. Notice that by (1) there exist at most Cn^{2k+1} absorbers.

In case all vertices of an absorber $\vec{v}$ are in $N_G(x)$ for some vertex $x \in V$ we say that $\vec{v}$ is an *x-absorber*. As explained earlier, the rationale behind this terminology is that if the path A we are about to construct happens to contain an x-absorber $\vec{v} = (v_1,\ldots,v_{2k+2})$, then we may replace this part of A by the $(k+1)$-path $(v_1,\ldots,v_{k+1},x,v_{k+2},\ldots,v_{2k+2})$ whenever we wish to "absorb" x into A. Later we shall refer to this option as the *absorbing property* of $\vec{v}$. We contend that

$$\text{for every } x \in V \text{ there are at least } \beta^2 Cn^{2k+1}/4 \text{ many } x\text{-absorbers.} \tag{6.3}$$

Notice that by (3) this would follow from the fact that for every edge vv' that $B_x - R$ and $G(n,p)$ have in common there are at least $\beta n^{2k}/2$ many x-absorbers having v and v' in their $(k+1)$st and $(k+2)$nd position, respectively. Now for $vv' \in e(B_x)$ there are actually at least βn^{2k} such configurations in V

and at most $2k|R|n^{2k-1}$ of them can fail to be x-absorbers for the reason of containing a vertex from R. Due to $|R| \le \gamma^2 n$ and $\gamma \ll \beta$ at most $\beta n^{2k}/2$ candidates get discarded in this way, and thereby (6.3) is proved.

Now let $\mathcal{F}$ be a random set of absorbers containing each absorber independently and uniformly at random with probability $q = \gamma^{3/2}C^{-1}n^{-2k}$. Since

$$\mathbb{E}[|\mathcal{F}|] \le Cn^{2k+1} \cdot q = \gamma^{3/2}n\,,$$

Markov's inequality entails

$$\mathbb{P}(|\mathcal{F}| \le 3\gamma^{3/2}n) > 2/3\,. \tag{6.4}$$

An ordered pair $(\vec{v}, \vec{w})$ of absorbers is said to be *overlapping* if they have a vertex in common. When two absorbers overlap, then either their middle edges are disjoint or they are not. The first case appears at most $(Cn)^2 \cdot 4k^2n^{4k-1}$ many times by (1), while the second case appears at most $8C^2n \cdot n^{4k}$ times by (2). So altogether there are at most $(4k^2 + 8)C^2n^{4k+1}$ pairs of overlapping absorbers. Hence, the expected number of overlapping pairs $(\vec{v}, \vec{w}) \in \mathcal{F}^2$ is at most $(4k^2+8)\gamma^3 n$, and a further application of Markov's inequality yields

$$\mathbb{P}(\text{there are at most } \gamma^{5/2}n \text{ overlapping pairs in } \mathcal{F}^2) > 2/3\,. \tag{6.5}$$

Since for each $x \in V$ the expected number of x-absorbers in $\mathcal{F}$ is by (6.3) at least $\beta^2\gamma^{3/2}n/4$, Chernoff's inequality implies

$$\mathbb{P}(\text{there are at least } 3\gamma^2 n \text{ many } x\text{-absorbers in } \mathcal{F} \text{ for every } x \in V) > 2/3\,. \tag{6.6}$$

In view of (6.4)-(6.6) there is an instance $\mathcal{F}_\star$ of $\mathcal{F}$ having the three properties whose probabilities were just shown to be larger than $2/3$. Delete from $\mathcal{F}_\star$ all absorbers belonging to an overlapping pair and denote the resulting set of absorbers by $\mathcal{F}_{\star\star}$. Notice that $\mathcal{F}_{\star\star}$ enjoys the following properties

- $|\mathcal{F}_{\star\star}| \le 3\gamma^{3/2}n$,
- no two absorbers in $\mathcal{F}_{\star\star}$ overlap, and
- for each $x \in V$ there are at least $2\gamma^2 n$ many x-absorbers in $\mathcal{F}_{\star\star}$.

Building the absorbing path

An iterative application of Proposition 2.1 allows us to connect the members of $\mathcal{F}_{\star\star}$ into a single path $A \subseteq G - R$ with

$$|V(A)| \le (2k+2)|\mathcal{F}_{\star\star}| + (k+1)2^{k+1}(|\mathcal{F}_{\star\star}| - 1) \le \gamma n/2\,.$$

In each of those applications of the Connecting Lemma, we take K and K' to be end-sets of the two $(k+1)$-paths we wish to connect, and we let Z be the union of the other vertices in the path system we currently have with R. Since at every moment the $(k+1)$-paths we are currently dealing with will have at most $\gamma n/2$ vertices in total and $|R| \le \gamma^2 n$, we will have $|Z| \le \gamma n$ in each of our $|\mathcal{F}_{\star\star}| - 1$ applications of Proposition 2.1, as required.

Using the absorbing property of x-absorbers in a greedy manner one sees immediately that the $(k+1)$-path A just constructed has the required property. ■

7 | PROOF OF THE COVERING LEMMA

This section deals with the proof of Proposition 2.4. Roughly speaking, our strategy is as follows. By known results [7, 9] the minimum degree condition imposed on G is more than enough to guarantee that we can cover essentially all vertices of $G' = G - Q$ with vertex-disjoint copies of the graph K^-_{k+2} which arises from a clique of order $k + 2$ by the deletion of a single edge. A standard application of the regularity method for graphs would allow to strengthen this result so as to obtain, for any bounded number m, a covering of an overwhelming proportion of the vertices of G' by vertex-disjoint copies of the m-blow-up $K^-_{k+2}(m)$ of a K^-_{k+2}. Explicitly, this is the graph arising from a K^-_{k+2} upon replacing each of its vertices x by an independent set V_x of size m and each of its edges xy by a complete bipartite graph $K_{m,m}$ joining V_x and V_y. An important point here is that there is a tremendous amount of flexibility in the construction of such an almost-covering of G' by copies of $K^-_{k+2}(m)$.

Now for any $K^-_{k+2}(m)$ in G it may happen that an appropriate path on $2m$ vertices in $G(n,p)$ augments it to a graph containing a spanning $(k+1)$-path in H. Of course, for any particular $K^-_{k+2}(m)$ in G this is an extremely unlikely event having a probability of only $o(1)$. However, owing to the aforementioned flexibility in the construction of an almost $K^-_{k+2}(m)$-covering of G', it becomes asymptotically almost surely possible to ensure that we only take copies $K^-_{k+2}(m)$ for which such a path in $G(n,p)$ is available.

In the two subsequent subsections we provide some of the background alluded to in the two foregoing paragraphs, while the proof of Proposition 2.4 will be given in Section 7.3.

7.1 | K^-_r-factors

For $r \geq 3$ let K^-_r denote the graph obtained from the clique K_r by deleting one edge. A K^-_r-factor of a graph G is a spanning subgraph of G each of whose connected components is isomorphic to K^-_r. It was proved by Enomoto, Kaneko, and Tuza [9] that every sufficiently large connected graph G with $\delta(G) \geq \frac{1}{3}|V(G)|$ whose number of vertices is divisible by 3 contains a K^-_3-factor. For larger values of r the tight minimum degree condition ensuring the existence of a K^-_r-factor was determined by Cooley, Kühn, and Osthus [7]. By combining the results in those two references one obtains the following.

Theorem 7.1 *For every integer $r \geq 3$ there exists an integer n_0 such that every connected graph G with $n \geq n_0$ vertices, $r \mid n$, and*

$$\delta(G) \geq \left(1 - \frac{r-1}{r(r-2)}\right) n$$

contains a K^-_r-factor.

For the application we have in mind the following "imperfect" consequence of this result, where we omit the divisibility assumption on n and allow a bounded number of left-over vertices, will be more convenient.

Corollary 7.2 *For every integer $k \geq 1$ there exists $n_0 \in \mathbb{N}$ such that every graph G with $n \geq n_0$ vertices and*

$$\delta(G) \geq \left(1 - \frac{k+1}{k(k+2)}\right) n$$

contains a collection of vertex disjoint copies of K^-_{k+2} which together cover all but at most $(k+2)^2$ vertices of G.

Proof We check that the number n_0 provided by Theorem 7.1 suffices. Let r be the integer satisfying $0 \le r \le k+1$ and $n \equiv r \pmod{k+2}$. Add $k+2-r > 0$ new vertices to G and connect them to all other vertices (and to each other). The graph thus obtained satisfies the assumptions of Theorem 7.1, and hence it contains a K^-_{k+2}-factor. When returning to G we can "lose" at most $k+2-r$ copies of K^-_{k+2}, wherefore at most $(k+2)^2$ vertices remain uncovered by the "surviving" copies of K^-_{k+2}. ■

7.2 | The graph regularity method

Mainly in order to fix some notation we shall now state a version of Szemerédi's Regularity Lemma from [25]. For two real numbers $\delta > 0$ and $d \in [0,1]$, a graph G and two nonempty disjoint sets $A, B \subseteq V(G)$, we say that the pair (A, B) is (δ, d)-*quasirandom* if for all $X \subseteq A$ and $Y \subseteq B$ the inequality

$$\big|e(X,Y) - d|X||Y|\big| \le \delta|A||B|$$

holds. The pair (A, B) is δ-*quasirandom* if it is (δ, d)-quasirandom for $d = e(A,B)/|A||B|$.

Theorem 7.3 (Szemerédi's Regularity Lemma) *Given $\delta > 0$ and $t_0 \in \mathbb{N}$ there exists an integer T_0 such that every graph $G = (V, E)$ on $n \ge t_0$ vertices admits a partition*

$$V = V_0 \cupdot V_1 \cupdot \cdots \cupdot V_t$$

of its vertex set such that

1. *$t \in [t_0, T_0]$, $|V_0| \le \delta|V|$, and $|V_1| = \cdots = |V_t|$, and*
2. *for every $i \in [t]$ the set $\big\{j \in [t] \setminus \{i\} : (V_i, V_j)$ is not δ-quasirandom$\big\}$ has size at most δt.*

Any partition as in Theorem 7.3 is called δ-*quasirandom* or just *quasirandom*. In the literature one often finds other versions of the Regularity Lemma, where instead of the second condition above one requires that at most δt^2 pairs (V_i, V_j) with distinct $i, j \in [t]$ fail to be δ-quasirandom. Applying such a regularity lemma to appropriate constants $\delta' \ll \delta$ and $t'_0 \gg \max(t_0, \delta^{-1})$, and relocating partition classes involved in many irregular pairs to V_0, one can obtain the version stated here.

Next we state the Counting Lemma accompanying Szemerédi's Regularity Lemma.

Lemma 7.4 (Counting Lemma) *Let F be a graph with vertex set $[f]$ and let G be another graph with a partition $V(G) = V_1 \cupdot \cdots \cupdot V_f$ such that (V_i, V_j) is δ-quasirandom whenever $ij \in F$. Then the number of ordered copies of F in G, that is, the number of f-tuples $(v_1, \ldots, v_f) \in V_1 \times \cdots \times V_f$ such that $v_i v_j \in G$ whenever $ij \in F$, equals*

$$\left(\prod_{ij \in F} d_{ij} \pm e(F)\delta\right) \prod_{i=1}^{f} |V_i|,$$

where $d_{ij} = \frac{e(V_i,V_j)}{|V_i||V_j|}$ is the density of (V_i, V_j), which is set to 0 in case $V_i = \varnothing$ or $V_j = \varnothing$.

7.3 | The covering lemma

We are now ready for the proof of the covering lemma.

Proof of Proposition 2.4 We begin by choosing several constants fitting into the hierarchy

$$k^{-1}, \varepsilon \gg \gamma \gg m^{-1}, \delta, t_0^{-1} \gg T_0^{-1} \gg \tau \gg C^{-1}$$

and we consider an n-vertex graph $G = (V, E)$ with $\delta(G) \geq (\frac{k}{k+1} + \varepsilon)n$.

Next we describe a deterministic property the random graph $G(n, p)$ for $p = C/n$ is likely to have and the remainder will then be dedicated to showing that this property implies the conclusion of our Covering Lemma in a deterministic way.

For every sequence $\vec{X} = (X_1, \ldots, X_{k+2})$ of disjoint subsets of V we define a family $\mathcal{F}(\vec{X})$ of $2m$-vertex paths with vertex set V as follows. Consider the set of all pairs (Y_1, Y_2) of m-sets with $Y_1 \subseteq X_1$ and $Y_2 \subseteq X_2$ such that there are further m-sets $Y_i \subseteq X_i$ for $i \in [3, k+2]$ such that $Y_1 \cup \cdots \cup Y_{k+2}$ spans a copy of $K_{k+2}^{-}(m)$ in G having all Y_i-$Y_{i'}$ edges for all $1 \leq i < i' \leq k+2$ with $(i, i') \neq (1, 2)$. For each such pair (Y_1, Y_2) choose a spanning path $P(Y_1, Y_2)$ on $Y_1 \cup Y_2$ that alternates between the two classes. The family $\mathcal{F}(\vec{X})$ consists of all these paths taken over all choices of (Y_1, Y_2) as above. Finally, let

$$\mathcal{J} = \left\{ \vec{X} = (X_1, \ldots, X_{k+2}) : \ |\mathcal{F}(\vec{X})| \geq \tau n^{2m} \right\}.$$

By Janson's inequality (see Theorem 3.2), a sufficiently large choice of C guarantees

$$\mathbb{P}\left(P \nsubseteq G(n,p) \text{for all} P \in \mathcal{F}(\vec{X})\right) \leq o(2^{-(k+2)n})$$

for each $\vec{X} \in \mathcal{J}$. Since $|\mathcal{J}| \leq 2^{(k+2)n}$ holds trivially, the union bound informs us that the event $\mathcal{E}$ that for every $\vec{X} \in \mathcal{J}$ there is a path $P \in \mathcal{F}(\vec{X})$ with $P \subseteq G(n, p)$ has probability $1 - o(1)$. Henceforth we assume that $\mathcal{E}$ occurs.

Applying Theorem 7.3 to $G' = G - Q$ we obtain for some $t \in [t_0, T_0]$ a δ-quasirandom partition

$$V \setminus Q = V_0 \cup V_1 \cup \cdots \cup V_t$$

of G'. Let Γ be the *reduced graph* with vertex set $[t]$ defined in such a way that a pair $ij \in [t]^{(2)}$ forms an edge of Γ if and only if the pair (V_i, V_j) is δ-quasirandom with density $d_{ij} = e(V_i, V_j)/|V_i||V_j| \geq \frac{1}{(k+1)^2}$. We contend that

$$\text{if } k > 0, \text{ then } \delta(\Gamma) \geq \left(1 - \frac{k+1}{k(k+2)}\right) t. \tag{7.1}$$

For the proof of this estimate we consider an arbitrary $i \in [t]$ and note that the minimum degree condition imposed on G yields

$$e(V_i, V) \geq \left(\frac{k}{k+1} + \varepsilon\right) |V_i| n.$$

On the other hand, it readily follows from the definitions of a δ-quasirandom partition and Γ that

$$\begin{aligned} e(V_i, V) &\leq e(V_i, Q \cup V_0) + \delta t |V_i|^2 + d_\Gamma(i)|V_i|^2 + (t - d_\Gamma(i))\frac{|V_i|^2}{(k+1)^2} \\ &\leq (\gamma + \delta)|V_i|n + \delta |V_i| n + \frac{1}{(k+1)^2}|V_i|n + \frac{k(k+2)}{(k+1)^2} \cdot \frac{d_\Gamma(i)}{t} \cdot |V_i| n \end{aligned}$$

Provided that $\gamma + 2\delta \le \varepsilon$ the combination of both estimates yields

$$\frac{d_\Gamma(i)}{t} \ge \frac{(k+1)^2}{k(k+2)} \left(\frac{k}{k+1} - \frac{1}{(k+1)^2} \right) = 1 - \frac{k+1}{k(k+2)}$$

and thereby (7.1) is proved.

Now the main work that remains to be done is to show the following statement.

Claim 7.5 *If $K \subseteq V(\Gamma)$ induces a K^-_{k+2} and $V_K = \bigcup_{i \in K} V_i$, then all but at most $\frac{1}{2}\gamma^2|V_K|$ vertices of $H[V_K]$ can be covered by a family of vertex disjoint $(k+1)$-paths each on $(k+2)m$ vertices.*

Assuming for the moment that we already know this, the proof of Proposition 2.4 can be completed as follows. If $k \ge 1$, then by Corollary 7.2 and (7.1) we know that Γ contains an almost perfect K^-_{k+2}-factor $\mathcal{K}$ covering all but at most $(k+2)^2$ vertices of Γ. As a K^-_2 is the empty graph on two vertices, such a factor $\mathcal{K}$ exists for $k = 0$ as well. Applying Claim 7.5 to each K^-_{k+2} in $\mathcal{K}$ we obtain a family of vertex disjoint $(k+1)$-paths in $H - Q$ covering all but at most $\left(\delta + \frac{(k+2)^2}{t_0} + \frac{1}{2}\gamma^2\right)n$ vertices, and by $\delta, t_0^{-1} \ll \gamma$ this is at most $\gamma^2 n$. Moreover, the number of these $(k+1)$-paths can be at most $\frac{n}{(k+2)m}$, which by $\gamma \gg m^{-1}$ is indeed at most $\gamma^3 n$.

It remains to prove Claim 7.5. To this end we may suppose that $V(K) = [k+2]$ and that the (perhaps) missing edge of the K^-_{k+2} is $\{1, 2\}$. Let $\mathcal{P}$ be a maximum collection of vertex-disjoint $(k+1)$-paths with $(k+2)m$ vertices in the $(k+2)$-partite graph $H[V_1, \dots, V_{k+2}]$. For each $i \in [k+2]$ let $X_i \subseteq V_i$ be the set of vertices in V_i which are not used by these paths. Since each path in $\mathcal{P}$ needs to consist of m vertices from each V_i, it follows that $|X_1| = \cdots = |X_{k+2}| = x$ holds for some integer x. Now it suffices to prove $x \le \frac{1}{2}\gamma^2|V_1|$, so assume for the sake of contradiction that this fails.

We intend to derive $|\mathcal{F}(\vec{X})| \ge \tau n^{2m}$ from the alleged largeness of x, which will tell us that $\vec{X} \in \mathscr{J}$. To this end we shall first obtain a lower bound on the number Ω of copies of $K^-_{k+2}(m)$ in $G[X_1, \dots, X_{k+2}]$ having

- m vertices in each X_i and
- all edges between the vertices in X_i and $X_{i'}$ for $1 \le i < i' \le k+2$ with $(i, i') \neq (1, 2)$.

For each $i \in [k+2]$ let $X_i = X_{i,1} \cup \cdots \cup X_{i,m}$ be a partition of X_i into m sets of size x/m. Now for $1 \le i < i' \le k+2$ with $(i, i') \neq (1, 2)$ we have $ii' \in E(\Gamma)$, which indicates that the pair $(V_i, V_{i'})$ is $(\delta, d_{ii'})$-quasirandom in G for some $d_{ii'} \in [\frac{1}{(k+2)^2}, 1]$. For $j, j' \in [m]$ we have $|X_{ij}| \ge \frac{\gamma^2}{2m}|V_i|$ and $|X_{i'j'}| \ge \frac{\gamma^2}{2m}|V_j|$ by our indirect assumption on $x = |X_i| = |X_j|$ and thus the pair $(X_{ij}, X_{i'j'})$ is $(\delta^\star, d_{ii'})$-quasirandom in G, where $\delta^\star = \frac{4m^2\delta}{\gamma^4}$. By Lemma 7.4 applied to $F = K_{k+2}(m)$ and the vertex classes X_{ij} with $i \in [k+2]$ and $j \in [m]$ it follows that

$$\Omega \ge \left(\frac{1}{(k+2)^{2e(K^-_{k+2}(m))}} - e\left(K^-_{k+2}(m)\right)\delta^\star \right) \left(\frac{x}{m}\right)^{m(k+2)},$$

which by $\tau \ll \delta, m^{-1}, T_0^{-1} \ll \gamma \ll k^{-1}$ gives $\Omega \ge \tau n^{m(k+2)}$. In particular, there are at least τn^{2m} pairs of m-sets (Y_1, Y_2) with $Y_1 \subseteq X_1$ and $Y_2 \subseteq X_2$ which can be completed to a copy of $K^-_{k+2}(m)$ in $G[X_1, \dots, X_{k+2}]$ by appropriate further m-sets $Y_i \subseteq X_i$ for $i \in [3, k+2]$. For these reasons, we have indeed $|\mathcal{F}(\vec{X})| \ge \tau n^{2m}$ and $\vec{X} \in \mathscr{J}$.

Thus the occurrence of $\mathcal{E}$ supplies a path $P \in \mathcal{F}(\vec{X})$ with $P \subseteq G(n, p)$. For $i \in [3, k+2]$ let $Y_i \subseteq X_i$ be m-sets witnessing $P \in \mathcal{F}(\vec{X})$. Since $V(P) \cup Y_3 \cup \cdots \cup Y_{k+2}$ spans a $(k+1)$-path in $H[X_1, \dots, X_{k+2}]$, we get a contradiction to the maximality of the collection $\mathcal{P}$ chosen earlier. This concludes the proof of Claim 7.5 and, hence, the proof of Proposition 2.4. ■

Note added in proof

After the present work was submitted, Nenadov and Trujić [20] showed that Theorem 1.1 is true even if one replaces $\mathcal{P}^{k+1}$ by $\mathcal{P}^{2k+1}$. Independently, a more general result was recently obtained by Antoniuk and the current authors [2].

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