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ON WEAK TWINS AND UP-AND-DOWN SUB-PERMUTATIONS

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Abstract

Two permutations $(x_1, \dots, x_w)$ and $(y_1, \dots, y_w)$ are *weakly similar* if $x_i < x_{i+1}$ if and only if $y_i < y_{i+1}$ for all $1 \leqslant i \leqslant w$. Let π be a permutation of the set $[n] = \{1, 2, \dots, n\}$ and let $wt(\pi)$ denote the largest integer w such that π contains a pair of *disjoint* weakly similar sub-permutations (called *weak twins*) of length w. Finally, let $wt(n)$ denote the minimum of $wt(\pi)$ over all permutations π of $[n]$. Clearly, $wt(n) \leq n/2$. In this paper we show that $\frac{n}{12} \leq wt(n) \leq \frac{n}{2} - \Omega(n^{1/3})$. We also study a variant of this problem. Let us say that $(\pi(i_1), ..., \pi(i_j))$, $i_1 < \cdots < i_j$, is an *alternating* (or *up-and-down*) sub-permutation of π if $\pi(i_1) > \pi(i_2) < \pi(i_3) > ...$ or $\pi(i_1) < \pi(i_2) > \pi(i_3) < ....$ Let Π_n be a random permutation selected uniformly from all $n!$ permutations of $[n]$. Stanley has shown that the length of a longest alternating permutation in Π_n is asymptotically almost surely (a.a.s.) close to $2n/3$. We study the maximum length $\alpha(n)$ of a pair of disjoint alternating sub-permutations in Π_n and show that there are two constants $1/3 < c_1 < c_2 < 1/2$ such that a.a.s. $c_1 n \leq \alpha(n) \leq c_2 n$. In addition, we show that the alternating shape is the most popular among all permutations of a given length.

– Dedicated to the memory of Ron Graham

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1. Introduction

Looking for twin objects in mathematical structures has a long and rich tradition going back to ancient geometric dissection problems and culminating in the famous Banach-Tarski Paradox (see [18]). From that research we know, for instance, that two very different looking objects, like the Sun and an apple, or the square and the circle, can be split into finitely many pairwise identical pieces. A general problem is to partition a given structure (or structures) into possibly few pairwise similar substructures. A related issue is to find, in a given structure, a pair of twin substructures, as large as possible.

Despite such 'continuous' origins, questions of that sort can be studied in diverse discrete contexts, with various types of similarity specified between the objects. For instance, Chung, Graham, Erdős, Ulam, and Yao [5] studied edge decompositions of pairs of graphs into pairwise isomorphic subgraphs (see also [6], [12]), while Erdős, Pach, and Pyber [8] looked for twins in a single graph (defined as a pair of edge disjoint isomorphic subgraphs). Axenovich, Person, and Puzynina ([2], [3]) investigated twins in words, and Gawron [11], inspired by their work, initiated exploration of twins in permutations (defined as a pair of disjoint *order-isomorphic* sub-permutations).

Let us dwell on this last problem for a while. By a *permutation* we mean any finite sequence of distinct positive integers. Let $t(n)$ be the maximum number k such that every permutation of length n has a pair of twins, each of length k. By a probabilistic argument, Gawron [11] proved that $t(n) = O(n^{2/3})$ and made a conjecture that this is best possible, that is, $t(n) = \Theta(n^{2/3})$. We confirmed this conjecture in [7] (up to a logarithmic factor) for a *random* permutation. A refinement of our result (getting rid of the logarithmic factor) was then obtained by Bukh and Rudenko [4]. In the deterministic case, the $t(n) \geqslant \Omega(\sqrt{n})$ follows immediately from the famous result of Erdős and Szekeres [9] on monotone subsequences in permutations. Currently, the best lower bound $t(n) = \Omega(n^{3/5})$ is due to Bukh and Rudenko [4].

In this paper we consider a weaker type of similarity of permutations than order-isomorphism in which we only look at the relations between neighboring elements. We say that two permutations $(x_1, \ldots, x_w)$ and $(y_1, \ldots, y_w)$ are *weakly similar* if $x_i < x_{i+1}$ if and only if $y_i < y_{i+1}$ for all $1 \leqslant i \leqslant w$.

This notion can be equivalently defined in terms of shapes. For our purposes, *the shape* of a permutation $\pi = (x_1, \ldots, x_w)$ is defined as a binary sequence $s(\pi) = (s_1, \ldots, s_{w-1})$ with elements from the set $\{+, -\}$, where $s_i = +$ if and only if $x_i < x_{i+1}$, $i = 1, \ldots, w-1$. For instance, $s(6, 1, 4, 3, 7, 9, 8, 2, 5) = (-, +, -, +, +, -, -, +)$. Then, permutations $\pi_x = (x_1, \ldots, x_w)$ and $\pi_y = (y_1, \ldots, y_w)$ are weakly similar if $s(\pi_x) = s(\pi_y)$.

Let $[n] = \{1, 2, \ldots, n\}$ and let π be a permutation of $[n]$, called also an n-*permutation*. Two weakly similar disjoint sub-permutations of π are called *weak* *twins* and the *length of the twins* is defined as the number of elements in just *one* of the sub-permutations. For example, in permutation

$$(6, \boxed{1}, \boxed{4}, 3, \boxed{7}, 9, \boxed{8}, \boxed{2}, \boxed{5}),$$

the blue $(1, 4, 2)$ and red $(7, 8, 5)$ subsequences form weak twins of length 3 (with a common shape $(+, -)$).

Let $wt(\pi)$ denote the largest integer w such that π contains weak twins of length w. Further, let $wt(n)$ denote the minimum of $wt(\pi)$ over all n-permutations π. In other words, $wt(n)$ is the largest integer w such that every n-permutation contains weak twins of length w. Our aim is to estimate this function which, unlike its stronger version $t(n)$, turns out to be linear in n.

Theorem 1. *For n large enough,*

$$\frac{n}{12} \leq wt(n) \leq \frac{n}{2} - \Omega(n^{1/3}). \tag{1}$$

Turning to our second result, note that given a sequence $s^{(n)}$ of length $n-1$, it is quite nontrivial to determine the number $N(s^{(n)})$ of n-element permutations with the shape $s^{(n)}$. Of course, there is just one permutation with a given monotone shape, $(+, \dots, +)$ and $(-, \dots, -)$. But already for the alternating shapes, $a_+^{(n)} = (+, -, +, \dots)$ and $a_-^{(n)} = (-, +, -, \dots)$, this is so called André's problem [1], which was solved asymptotically in the 19th century and exactly, in terms of a finite sum of Stirling numbers, only in the 21th century [15] (see also [16]).

The asymptotic formula of André says that, setting $A_n := N(a_+^{(n)}) = N(a_-^{(n)})$,

$$A_n \sim 2(2/\pi)^{n+1} n!.$$

In other words, the probability that a random n-permutation Π_n is alternating (either way) is only $\sim 4(2/\pi)^{n+1}$. On the other hand, by the result of Stanley [17], we know that a.a.s. a random n-permutation contains an alternating subsequence of length at least $\sim 2n/3$, yielding alternating twins of length at least $\sim n/3$ (just split in half a longest alternating sub-permutation in Π_n). In Theorem 2 we show, however, that a.a.s. one can get substantially longer alternating twins; on the other hand, they are much shorter than $n/2$, the absolute upper bound.

To state this result, let $\alpha(\pi)$ be the largest integer w such that π contains weak twins of length w with an alternating shape, $a_+^{(w)}$ or $a_-^{(w)}$. We will call them *alternating twins*. Further, set $\alpha_n := \alpha(\Pi_n)$, where Π_n is a random n-permutation.

Theorem 2. *A.a.s.*

$$\left(\frac{1}{3} + \frac{1}{60} + o(1)\right) n \leq \alpha_n \leq \left(\frac{1}{2} - \frac{1}{120} + o(1)\right) n. \tag{2}$$

We end this paper by proving that, in fact, permutations with alternating shapes are the most popular ones. This result, not directly related to our main theorems, may be of independent interest.

Proposition 1. *For every n and every shape $s^{(n)}$ of length $n-1$, we have $N(s^{(n)}) \leq A_n$.*

The proof of Proposition 1 can be found in Section 3.

Note. We believe that Ron Graham would like the topic of this paper. Not only was he among those who planted the idea of twins into the combinatorial soil, but he also wrote several papers devoted to permutations, both with and without connections to juggling (see, for example, `http://www.math.ucsd.edu/~ronspubs/` for the entire collection of Ron's publications).

2. Proofs of Theorems 1 and 2

2.1. Extremal Points

In our proofs a decisive role is played by local extremes. We call the element i *maximal* in π if $i = 1$ and $\pi(1) > \pi(2)$, or $i = n$ and $\pi(n-1) < \pi(n)$, or $1 < i < n$ and $\pi(i-1) < \pi(i) > \pi(i+1)$. By swapping all signs $<$ and $>$ around, we obtain the notion of *a minimal* point i in π. Maximal and minimal points alternate and are jointly referred to as *extremal*. The points 1 and n are always extremal. Clearly, all extremal points of π form an alternating sequence in π. In fact, as shown by Bóna (see [16], and [13] for a proof), it is the longest one.

Let $E = \{j_1, \ldots, j_k\}$ be the set of all extremal points in π. These points divide the whole range $[n]$ into monotone segments

$$\pi_i = (\pi(j_i), \pi(j_i+1), \ldots, \pi(j_{i+1})), \quad i = 1, \ldots, k-1, \tag{3}$$

which, however, share their endpoints. For a true partition, we define

$$\bar{\pi}_i = (\pi(j_i), \pi(j_i+1), \ldots, \pi(j_{i+1}-1)), \quad i = 1, \ldots, k-2,$$

and leave the last one unchanged, that is, $\bar{\pi}_{k-1} = \pi_{k-1}$.

2.2. Weak Twins

Proof of Theorem 1, lower bound. As the extremal points themselves form an alternating sub-sequence E of π, by splitting it evenly, we obtain a pair of weak twins of length $\lfloor k/2 \rfloor$. Thus, we may assume that $k-1 \leq n/6$, since otherwise $\lfloor k/2 \rfloor \geq k/2 - 1/2 \geq n/12$ and we are done.

Let $Q_1, \dots, Q_\ell$ be those segments among $\bar{\pi}_1, \dots, \bar{\pi}_{k-1}$ which contain at least 4 elements each. It is easy to check that

$$|Q_1| + \cdots + |Q_\ell| \geq \frac{1}{2}n.$$

Indeed, otherwise we would have

$$n = \sum_{i=1}^{k-1} |\bar{\pi}_i| < 3(k-1) + \frac{1}{2}n \leq n,$$

a contradiction. All we need now is the following proposition.

Proposition 2. *One can find weak twins in $Q_1 \cup \cdots \cup Q_\ell$ of length at least*

$$\frac{1}{2}\sum_{i=1}^{\ell} |Q_i| - \ell.$$

Before proving the proposition, let us finish the proof of the lower bound in (1). By Proposition 2, there is in π a pair of weak twins of length

$$\frac{1}{2}\sum_{i=1}^{\ell} |Q_i| - \ell \geq \frac{1}{4}n - (k-1) \geq \frac{1}{4}n - \frac{1}{6}n = \frac{n}{12}.$$

□

Proof of Proposition 2. We begin with the following observation. We say that weak twins (A, B), where $A = (\pi(i_1), \dots, \pi(i_k))$, $i_1 < \cdots < i_k$, and $B = (\pi(j_1), \dots, \pi(j_k))$, $j_1 < \cdots < j_k$, are *aligned upward*, respectively, *downward* if the two right-most elements of A and the two right-most elements of B interwind and form a monotone sub-sequence, that is, $j_{k-1} < i_{k-1} < j_k < i_k$ and $\pi(j_{k-1}) < \pi(i_{k-1}) < \pi(j_k) < \pi(i_k)$, or, respectively, $\pi(j_{k-1}) > \pi(i_{k-1}) > \pi(j_k) > \pi(i_k)$.

Claim 1. Let (A, B) be aligned weak twins in π and let $Q = (\pi(m_1), \dots, \pi(m_s))$, $s \geq 4$, be a monotone sub-sequence of π completely to the right of (A, B), that is, $m_1 > i_k$. Then one can extend (A, B) to a new pair of aligned weak twins (A', B') which contains all elements of A, B and Q except for at most 2 elements. The lost elements are either all from Q (the first or the last or both) or one from Q (the last one) and one from A (the last one).

Proposition 2 follows quickly from the above claim. Indeed, by its repeated applications, beginning with selecting a pair of aligned weak twins (A_1, B_1) within Q_1 (here we lose one element in the case when $|Q_1|$ is odd), we recursively construct the desired object losing along the way at most $1 + 2(\ell - 1) < 2\ell$ elements. □

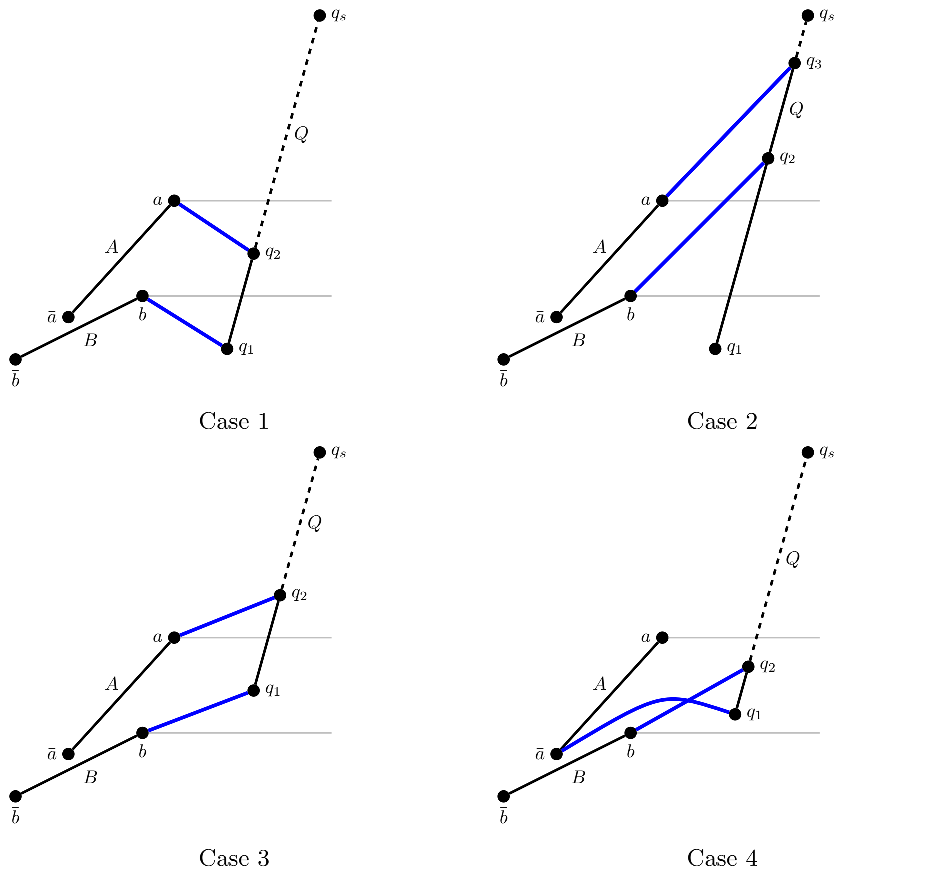


Figure 1: Extending twins in the proof of Claim 1 with increasing Q.

Proof of Claim 1. Without loss of generality, assume that the weak twins (A, B) are aligned upward. However, with respect to Q, we have to consider both cases of its monotonicity. We first assume that Q is increasing.

We are going to examine 4 cases of how the two bottom values in Q position themselves with respect to the two top ones in (A, B) (see Figure 1). Set $a = \pi(i_k)$, $\bar{a} = \pi(i_{k-1})$, $b = \pi(j_k)$, $\bar{b} = \pi(j_{k-1})$, and $q_i = \pi(m_j)$, $j = 1, 2, \ldots, s$. Recall that $\bar{b} < \bar{a} < b < a$.

`Case 1:` $q_1 < b, q_2 < a$. We extend A and B as follows:

$$A' = A, q_2, q_4, \ldots, \quad B' = B, q_1, q_3, \ldots.$$

If s is odd, the point q_s is not used (we say it is lost). Note that due to the order of q_1, q_2, q_3, q_4, the new pair (A', B') is indeed aligned.

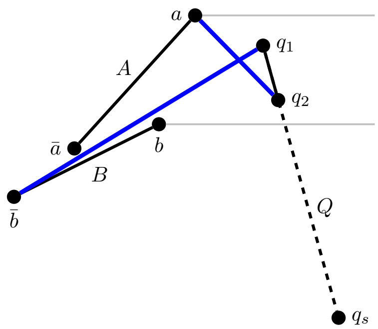


Figure 2: Extending twins in the proof of Claim 1 with decreasing Q.

`Case 2:` $q_1 < b < a < q_2$. Here we set

$$A' = A, q_3, q_5, \ldots, \quad B' = B, q_2, q_4, \ldots.$$

We definitely lose q_1 and, if s is even, we also lose q_s. For $s = 4$ or $s = 5$, the last 4 points of (A', B') are thus b, a, q_2, q_3, which are aligned upward. If $s \geq 6$, then (A', B') is aligned as well.

`Case 3:` $q_1 > b, q_2 > a$. This case is very similar to Case 1, so we omit the details.

`Case 4:` $b < q_1 < q_2 < a$. This is the only case when we lose a point of (A, B). Let A^- denote the sub-sequence A without the last element, a. We set

$$A' = A^-, q_1, q_3, \ldots, \quad B' = B, q_2, q_4, \ldots.$$

Besides a, we may also lose q_s, provided s is even. Observe that for $s = 4$, b, q_1, q_2, q_3 are aligned upward. This exhaust the case when Q is increasing.

For decreasing Q, there are also 4 cases to examine. However, three of them, namely, (i) $a > q_1, b > q_2$, (ii) $q_1 > a > b > q_2$, and (iii) $q_1 > a, q_2 > b$ are very similar to those for increasing Q, so we leave them for the reader. The only somewhat different case is when (iv) $a > q_1 > q_2 > b$ (see Figure 2). Then, denoting by B^- the sub-sequence B without its last element, b, we set

$$A' = A, q_2, q_4, \ldots, \quad B' = B^-, q_1, q_3, \ldots.$$

Besides b, we may also lose q_s, provided s is even. Finally, observe that for $s = 4$, a, q_1, q_2, q_3 are aligned downward, though with the roles of A' and B' switched (which does not really matter to us; formally we should swap A' and B' around). □

Proof of Theorem 1, upper bound. We are going to construct a permutation π on $[n]$, n large enough, with no weak twins longer than $n/2 - cn^{1/3}$ for some $c > 0$.

This permutation will consist of $k' \leq k := \lceil n^{1/3} \rceil$ consecutive increasing segments $P_1, \ldots, P_{k'}$ with $\max P_{i+1} < \min P_i$, of diminishing lengths which have to be chosen carefully. For $i = 1, \ldots, k$, set

$$x_i = 2k^2 - 2(i-1)k - 1.$$

Note that for all $i = 1, \ldots, k$, $x_i > 0$ and x_i is an odd integer. Moreover,

$$\sum_{i=1}^{k} x_i = 2k^3 - 2k\binom{k}{2} - k = k^3 + k^2 - k > n.$$

Let $k' = \min\{j : \sum_{i=1}^{j} x_i \geq n\}$. Then we set $|P_i| = x_i$, $i = 1, \ldots, k'-1$, and $|P_{k'}| = n - \sum_{i=1}^{k'-1} x_i$. Since $\sum_{i=1}^{k} x_i = n + O(k^2)$, with a big margin we have, say, $k' \geq 0.99k$. Also, what is crucial here, for all $i = 1, \ldots, k'-1$, we have $x_i - x_{i+1} \geq 2k$, in fact, with equality except for $i = k'-1$.

So, we define $\pi = (P_1, \ldots, P_{k'})$ in the following manner. We set

$$\pi(1) = n - x_1 + 1, \pi(2) = n - x_1 + 2, \ldots, \pi(x_1) = n \quad \text{and} \quad A_1 = (\pi(1), \ldots, \pi(x_1)).$$

Then we dip down and set

$$\pi(x_1 + 1) = n - x_1 - x_2 + 1, \pi(x_1 + 2) = n - x_1 - x_2 + 2, \ldots, \pi(x_1 + x_2) = n - x_1$$

and

$$A_2 = (\pi(x_1 + 1), \ldots, \pi(x_1 + x_2)),$$

and so on, and so forth.

We now state a proposition from which the desired bound follows quickly.

Proposition 3. *Let (A, B) be weak twins in the permutation π defined above of length $|A| = |B| \geq n/2 - k/3$. Then, for all $1 \leq i < k'$, we have $|A \cap P_i| = |B \cap P_i|$.*

Before proving the proposition, let us finish the proof of the upper bound in Theorem 1. Suppose there is in π a pair of weak twins of length at least $n/2 - k/2$. Since for all $1 \leq i < k'$, $|P_i|$ is odd, in view of Proposition 3, at least one point of each such P_i is missing from (A, B). Hence,

$$|A| = |B| \leq n/2 - (k'-1)/2 \leq n/2 - (0.99k - 1)/2 < n/2 - k/3,$$

a contradiction. □

Proof of Proposition 3. We proceed by (strong) induction on $i = 1, \ldots, k'-1$. Let us start with the base case $i = 1$. Since at most $2k/3$ points of π are not in $A \cup B$, while $|P_1| > 2k/3$, without loss of generality, $A \cap P_1 \neq \emptyset$. It suffices to prove that also $B \cap P_1 \neq \emptyset$, since then, due to the fact that the rest of π lies totally below P_1,

A and B must have the same number of elements in P_1. Suppose to the contrary that $B \cap P_1 = \emptyset$. But then

$$|A \cap P_1| \geq |P_1| - \frac{2}{3}k > |P_2| > |P_3| > \cdots ,$$

so A begins with a longer increasing segment than B does, a contradiction with the notion of weak twins.

For the induction step, which is similar to the base step, assume that $|A \cap P_j| = |B \cap B_j|$, for $j = 1, \dots, i \leq k' - 2$. If $|A \cap P_{i+1}| = |B \cap B_{i+1}| = 0$, then we are done. Without loss of generality, assume that $|A \cap P_{i+1}| > 0$. As before, it suffices to show that also $|B \cap P_{i+1}| > 0$. Suppose otherwise. Then, since at most $2k/3$ points of π are not in $A \cup B$, we have

$$|A \cap P_{i+1}| \geq |P_{i+1}| - \frac{2}{3}k > |P_{i+2}| > \cdots .$$

This means, however, that A and B will differ in the length of the first increasing segment commencing to the right of the point $\sum_{j=1}^{i} x_j$. This yields a contradiction with (A, B) being weak twins and completes the proof. □

2.3. Alternating Weak Twins

Recall that the extremal points of π form an alternating sub-sequence. In the proof of the lower bound in Theorem 2, we are going to use this fact and then reiterate it for the sub-permutation π' obtained from π by removing all the extremal points of π. As a crucial tool we invoke the Azuma-Hoeffding inequality for random permutations (see, for example, Lemma 11 in [10] or Section 3.2 in [14]).

Theorem 3. *Let $h(\pi)$ be a function of n-permutations such that if permutation π_2 is obtained from permutation π_1 by swapping two elements, then $|h(\pi_1) - h(\pi_2)| \leq 1$. Then, for every $\eta > 0$,*

$$\mathbb{P}(|h(\Pi_n) - \mathbb{E}[h(\Pi_n)]| \geq \eta) \leq 2\exp(-\eta^2/(2n)).$$

Proof of Theorem 2, lower bound. We are going to show that extremal points are evenly distributed in both 'halves' of Π_n. For mere convenience, we assume that n is even.

Let X_1 and X_2 be the numbers of extremal points in Π_n among, respectively, $\{1, \dots, n/2\}$ and $\{n/2 + 1, \dots, n\}$. Note that the probability that a given point i, $2 \leq i \leq n - 1$, is extremal is $2 \times \frac{1}{3} = \frac{2}{3}$. Thus,

$$\mathbb{E}(X_1) = \mathbb{E}(X_2) = 1 + \left(\frac{n}{2} - 1\right) \times \frac{2}{3} = \frac{n+1}{3}.$$

Now we apply Theorem 3 to show that this expectation is highly concentrated about its mean. To verify the Lipschitz assumption, note that if π_2 is obtained

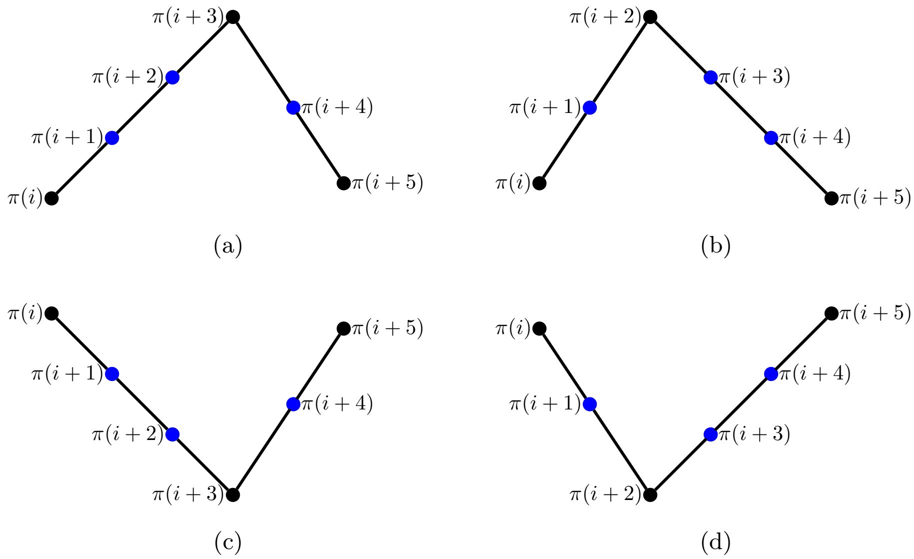


Figure 3: Lucky sixes. The blue points appear in π' as consecutive ones.

from a permutation π_1 by swapping any two of its elements, then trivially $|X_j(\pi_1) - X_j(\pi_2)| \leq 6$, $j = 1, 2$. (A detailed analysis shows that 6 can be replaced by 4 which is optimal.) Consequently, Theorem 3 applied with $h(\pi) = X_j(\pi)/6$ and $\eta = n^{3/5}$ implies

$$\mathbb{P}(|X_j(\Pi_n) - \mathbb{E}[X_j(\Pi_n)]| \geq n^{3/5}) = o(1)$$

implying that a.a.s. $X_j = (1 + o(1))\frac{n}{3}$, $j = 1, 2$.

It is quite hard to characterize the extremal points of π'. Unable to do so, we instead identify a 6-point configuration in π which contains an extremal point of π'. A 6-tuple $\{i, i+1, i+2, i+3, i+4, i+5\}$, $1 \leq i \leq n-5$, is called *a lucky six* if $\pi(i) < \pi(i+1) < \pi(i+2) < \pi(i+3) > \pi(i+4) > \pi(i+5)$ and $\pi(i+2) > \pi(i+4)$, or when all signs $<$ and $>$ are swapped. It should be clear that in a lucky six $i+3$ is an extremal point of π and, most importantly, $i+2$ is an extremal point in π'. Of course, the same property is enjoyed by the symmetrical structures where $\pi(i) < \pi(i+1) < \pi(i+2) > \pi(i+3) > \pi(i+4) > \pi(i+5)$ and $\pi(i+1) < \pi(i+3)$ (and, again, with signs $<$ and $>$ swapped). So, we also call them *lucky sixes*. See Figure 3 for all 4 types of lucky sixes.

Let Y_1 and Y_2 be the numbers of lucky sixes $\{i, i+1, i+2, i+3, i+4, i+5\}$ in Π_n for, respectively, $1 \leq i \leq n/2 - 3$ and $n/2 - 1 \leq i \leq n - 5$. Note that the probability that a given 6-tuple is a lucky six is

$$4 \times \frac{\binom{4}{2}}{6!} = \frac{1}{30}.$$

Indeed, considering, for instance, the number of ways to label by $1, \ldots, 6$, the lucky six in Figure 3(a), there is no question that 6 must be at the top, while 5 to its left. The remaining 4 values can be, however, distributed freely between the two pairs, $i, i+1$ and $i+4, i+5$. This explains $\binom{4}{2}$. Thus,

$$\mathbb{E}(Y_1) = \mathbb{E}(Y_2) \sim \frac{n}{60}.$$

Again, a standard application of the Azuma inequality (Theorem 3) yields that a.a.s. $Y_j = (1+o(1))\frac{n}{60}$, $j = 1, 2$.

Let A_j, $j = 1, 2$, be the alternating sub-sequences in, respectively, $\{1, \ldots, n/2\}$ and $\{n/2+1, \ldots, n\}$, consisting of the extremal points of Π_n. Further, let B_j, $j = 1, 2$, be alternating sub-sequences in, respectively, $\{1, \ldots, n/2\}$ and $\{n/2+1, \ldots, n\}$, consisting of the extremal points of Π'_n. By losing at most one point each, one can concatenate A_j with B_{3-j}, $j = 1, 2$, obtaining the desired pair of alternating twins. Noting that $|A_j \cup B_{3-j}| \sim \frac{n}{3} + \frac{n}{60}$ completes the proof of the lower bound in (2). □

Proof of Theorem 2, upper bound. For the proof of the upper bound we need to consider two kinds of special 5-tuples. A 5-tuple $\{i, i+1, i+2, i+3, i+4\}$ is called *cornered* if either the first or the last four consecutive points form a monotone sub-sequence but all five do not (see Figure 4). A 5-tuple $\{i, i+1, i+2, i+3, i+4\}$ is called *crooked* if the three middle points form a monotone sub-sequence but no four points do (see Figure 5). Given a permutation π, let $e(\pi)$ be the number of extremal points in π, and let $co(\pi)$ and $cr(\pi)$ be, respectively, the number of cornered 5-tuples and the number of crooked 5-tuples in π. The following crucial lemma sets an upper bound on the number of elements in two disjoint alternating sub-sequences of π in terms of the three defined above parameters.

Lemma 1. *Let A and B be two disjoint alternating sub-sequences in a permutation π of $[n]$. Then*

$$|A| + |B| \le e(\pi) + co(\pi) + cr(\pi). \tag{4}$$

Deferring the proof of Lemma 1 for later, we now deduce from it the upper bound in (2). Let L count the cornered 5-tuples in the random permutation Π_n and let Z count the crooked 5-tuples in Π_n. Note that the probability that a given 5-tuple is cornered is $4 \times \binom{4}{3}/5! = \frac{8}{60}$ and so, $\mathbb{E}(L) = \frac{8}{60} \times (n-4)$. Note also the that the probability that a given 5-tuple is crooked is $2 \times \frac{11}{5!} = \frac{11}{60}$ (see Figure 6) and so, $\mathbb{E}(W) = \frac{11}{60} \times (n-4)$. Another application of the Azuma inequality (Theorem 3) yields that a.a.s. $L = (1+o(1))\frac{8n}{60}$, while $Z = (1+o(1))\frac{11n}{60}$. Plugging into (4), we finally obtain that

$$\alpha_n \le \frac{1}{2}(1+o(1))\left(\frac{2}{3} + \frac{8}{60} + \frac{11}{60}\right)n = \left(\frac{1}{2} - \frac{1}{120} + o(1)\right)n.$$

□

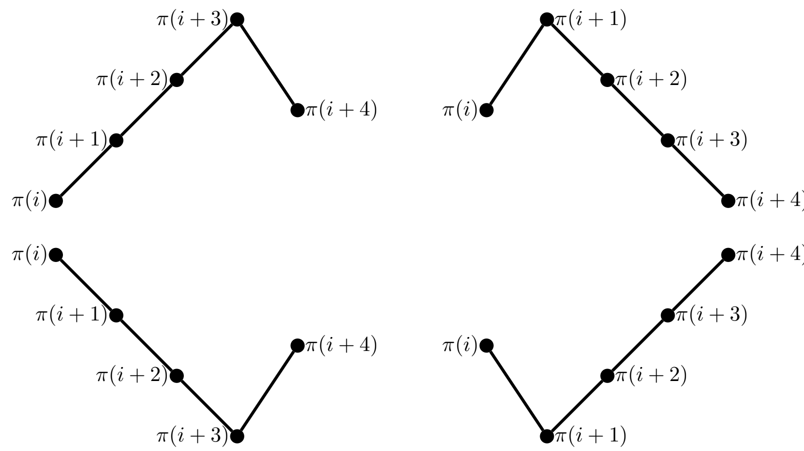


Figure 4: Cornered 5-tuples.

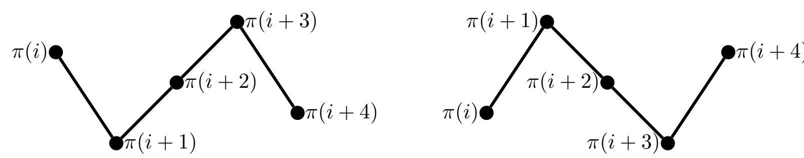


Figure 5: Crooked 5-tuples.

It remains to prove Lemma 1.

Proof of Lemma 1. Let A and B be given as in the lemma. Let E be the set of extremal points in π and F the set of points neighboring the extremal points but not extremal themselves. We are going to construct an injective mapping $\phi : A \cup B \to E \cup F$. Then, noting that $|F| = co(\pi) + cr(\pi)$, completes the proof.

Let $j_1, \ldots, j_k$ be all extremal points in π. These points divide the whole range $[n]$ into monotone segments defined in (3) which we now express in terms of the numbers of their inner points ℓ_i:

$$\pi_i = (\pi(j_i), \pi(j_i + 1), \ldots, \pi(j_i + \ell_i), \pi(j_{i+1}))$$

$i = 1, \ldots, k-1$. Note that ℓ_i can equal 0. Before constructing the desired mapping ϕ, let us examine the distribution of the set $A \cup B$ among the segments π_i. Our first observation is that each segment contains at most two elements of A and at most two elements of B. Moreover, if π_i contains exactly two elements of A, then one of them is a minimal element of A and the other – a maximal element of A, and the

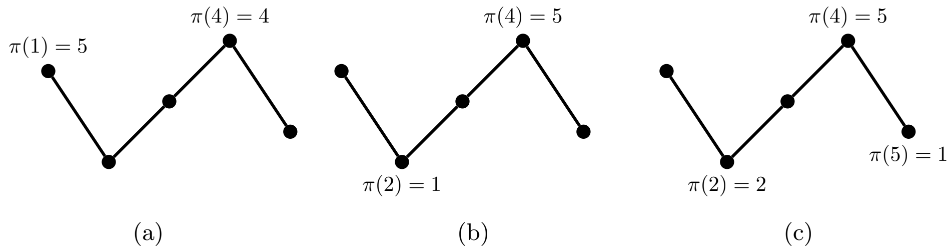


Figure 6: (a) If $\pi(1) = 5$, then $\pi(4) = 4$ and the remaining number of choices is $\binom{3}{2}$. (b) If $\pi(4) = 5$ and $\pi(2) = 1$, then we have 3! choices. (c) Finally, if $\pi(4) = 5$ and $\pi(5) = 1$, then there are 2! remaining choices.

same is true for B. But most crucial is the following property concerning a pair of consecutive segments π_i and π_{i+1}. If j_{i+1} is maximal, respectively, minimal in π, then there is in total at most one maximal, resp., minimal element of A on these segments.

Knowing all this, it is easy to see that the following construction is, indeed, an injection. If π_i is increasing, then to the maximal elements of $A \cup B$ lying on π_i, assign the top-most two elements of π_i, that is, to $\pi(j_i + \ell_i), \pi(j_{i+1})$, in any feasible fashion. While to the minimal elements of $A \cup B$ lying on π_i assign the two down-most elements of π_i, that is, to $\pi(j_i), \pi(j_i + 1)$. If π_i is decreasing, we proceed similarly, but with the pairs $\pi(j_i + \ell_i), \pi(j_{i+1})$ and $\pi(j_i), \pi(j_i + 1)$ swapped. □

3. Proof of Proposition 1

Given a sequence $s = (s_1, \ldots, s_r)$ with $s_i \in \{+, -\}$ and a linearly ordered set S of size $|S| = r + 1$, denote by $\mathcal{N}_S(s)$ the set of all permutations π of S with the shape $s(\pi) = s$. If $S = [r + 1]$, then we abbreviate $\mathcal{N}(s) := \mathcal{N}_{[r+1]}(s)$. Further, let $N_S(s) = |\mathcal{N}_S(s)|$. Observe that $N_S(s)$ does not depend on S, so we skip the subscript $_S$ altogether here.

The complement of a sequence $s = (s_1, \ldots, s_r)$ is naturally defined as the sequence $\bar{s} = (\bar{s}_1, \ldots, \bar{s}_r)$, where $\{s_i, \bar{s}_i\} = \{+, -\}$ for each i. In other words, one replaces each $+$ in s with $-$, and vice versa. It is easy to see that $N(s) = N(\bar{s})$.

Recall that $A_n = N(a_+^{(n)}) = N(a_-^{(n)})$. Our proof of Proposition 1 is by induction on n and, in its final accord, utilizes the following known identity involving the sequence A_n (see, for example, [16]):

$$\sum_{k=0}^{n} \binom{n}{k} A_k A_{n-k} = 2A_{n+1}. \tag{5}$$

What is more, our proof is also inspired by the idea behind the proof of (5), which is to build a permutation of $[n+1]$ beginning with positioning the element $n+1$, and then separately counting the completions to the left and to the right of it. Also, as the R-H-S of (5) is a double of what we want, we are doomed to count in permutations with the complementary shape as well.

Proof of Proposition 1. For $n \leq 3$, the proposition follows by inspection. Fix $n \geq 3$ and assume it is true for all $n' \leq n$. Given a shape $s^{(n+1)} := s = (s_1, \ldots, s_n)$ our goal is to show that $N(s) \leq A_{n+1}$.

For each $k = 0, 1, \ldots, n$, let $\mathcal{N}_k(s) = \{\pi \in \mathcal{N}(s) : \pi(k+1) = n+1\}$. As $n+1$ is always a maximum element of π, we have $\mathcal{N}_k(s) \neq \emptyset$ if and only if $s_k = +$ and $s_{k+1} = -$. Thus, setting $K^\wedge = \{k : s_k = + \text{ and } s_{k+1} = -\}$, we have $\mathcal{N}(s) = \bigcup_{k \in K^\wedge} \mathcal{N}_k(s)$, and, as the sets under the union are obviously disjoint, $N(s) = \sum_{k \in K^\wedge} N_k(s)$, where $N_k(s) = |\mathcal{N}_k(s)|$. For a fixed k, let us focus on the number $N_k(s)$. Every permutation in $\mathcal{N}_k(s)$ consists of a 'prefix' u, followed by $n+1$, followed by a suffix v. Introducing 'truncated' shapes $s'_k = (s_1, \ldots, s_{k-1})$ and $s''_k = (s_{k+2}, \ldots, s_n)$, u and v must satisfy $s(u) = s'_k$ and $s(v) = s''_k$. Hence, using also the induction assumption,

$$N_k(s) = \binom{n}{k} N(s'_k) N(s''_k) \leq \binom{n}{k} A_k A_{n-k}.$$

The same is true for the complementary shape $\bar{s}$ as well. Recalling that $N(\bar{s}) = N(s)$ and noticing that the set

$$\{k : \bar{s}_k = + \text{ and } \bar{s}_{k+1} = -\} = \{k : s_k = - \text{ and } s_{k+1} = +\} =: K^\vee$$

is disjoint from $K^\wedge$, we thus conclude that

$$2N(s) \leq \sum_{k \in K^\wedge \cup K^\vee} \binom{n}{k} A_k A_{n-k} \leq \sum_{k=0}^{n} \binom{n}{k} A_k A_{n-k} = 2A_{n+1},$$

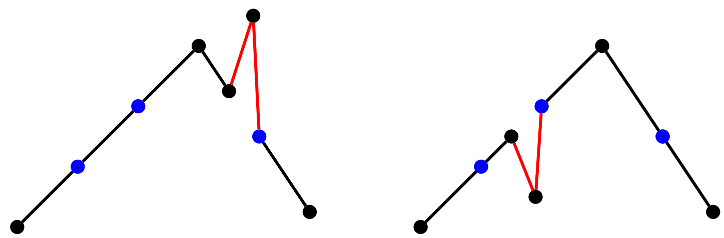
where the last equality is (5).

□

4. Concluding Remarks

We believe that the lower bound in Theorem 1 can be improved and it is plausible to conjecture that $wt(n) \sim \frac{n}{2}$. As a matter of fact, if π happens to be an n-permutation with $e(\pi) = o(n)$, then the construction used in the proof of (1) yields $wt(\pi) \sim \frac{n}{2}$.

It is also not difficult to see that the lower bound on α_n in Theorem 2 can be improved. Let π' be the sub-permutation obtained from π by removing all the extremal points of π. Recall that in the proof of the lower bound (2) we



(a) (b)

Figure 7: (a) 117 choices; (b) 105 choices.

estimated $e(\pi')$ by using the lucky six tuples. But one can also consider more "lucky" structures. This can be done by incorporating zigzags into the lucky six tuples as in Figure 7, for example. This already gives an improvement on the lower bound on α_n:

$$\alpha_n \geq \left(\frac{1}{3} + \frac{1}{60} + \frac{1}{2} \cdot 4 \cdot \frac{117+105}{8!} + o(1)\right) n.$$

Now we can consider longer lucky tuples (of length $10, 12, 14, ...$) and use computer to calculate the corresponding expectations. Computer simulations suggest that

$$\alpha_n \geq (1/3 + 0.1006...)n.$$

We do not know what the exact value of the second term is here, since it is not even clear how to compute the expected value $\mathbb{E}(e(\Pi'_n))$.

Another direction of related studies would be to consider a more general notion of *weak r-twins*, defined as r pairwise disjoint subsequences of a permutation with the same shape. One naturally expects that the analogous function $wt^{(r)}(n)$ should satisfy $wt^{(r)}(n) \sim \frac{n}{r}$.

Finally, let us point at some natural counting problems involving the notion of weak similarity of permutations. For instance, define, for even n, the sequence T_n which counts all n-permutations that are weak twins of length $n/2$, that is, all n-permutations that can be split into two sub-permutations with the same shape. What is the asymptotic growth of the sequence T_n?

A related problem stems from Proposition 1. We proved there that no shape is more represented among all permutations of length n than the alternating ones. It follows from the proof of Proposition 1 that actually the number A_n is strictly bigger than B_n — the largest among the numbers $N(s)$ with s being a non-alternating sequence of length $n-1$. It can be shown that $A_n/B_n \leq 2$. Indeed, by swapping the first two elements, we see that $N(b^{(n)}) = N'(a_+^{(n)})$, where $b^{(n)} = (-,-,+,-,+,\dots)$ is the shape of permutations which begin with a decreasing triple and then alternate, while $N'(a_+^{(n)})$ counts those alternating permutations π of length n for which $\pi(2) > \pi(1) > \pi(3)$. In turn, swapping the first and the third element of an alternating permutation counted by $N(a_+^{(n)}) - N'(a_+^{(n)})$, that is, one for which $\pi(2) > \pi(3) > \pi(1)$, yields that $N'(a_+^{(n)}) \geq \frac{1}{2}A_n$. It would be interesting to compute $\lim_{n\to\infty} A_n/B_n$, if it exists.

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