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A NOTE ON UNIQUE REPRESENTATION BASES FOR THE INTEGERS

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Abstract: We prove that each subset A of $\mathbf{Z}$ which fulfills certain Sidon-type condition can be extended to a unique representation basis for $\mathbf{Z}$.
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For a set $A \subseteq \mathbb{Z}$, $h \in \mathbb{N}$, and $n \in \mathbb{Z}$, by $r_{A,h}(n)$ we denote the number of ways n can be represented as a sum $a_1 + a_2 + \cdots + a_h$, where $a_1, a_2, \ldots, a_h \in A$, $a_1 \leq a_2 \leq \cdots \leq a_h$. A set A is an *additive h-basis* for $\mathbb{Z}$ if $r_{A,h}(n) \geq 1$ for all $n \in \mathbb{Z}$, and it is a *unique representation h-basis* if for every $n \in \mathbb{Z}$ we have $r_{A,h}(n) = 1$. Nathanson [2] observed that a unique representation basis for $\mathbb{Z}$ can be constructed by a simple greedy procedure. Thus, there is a striking difference between additive bases for $\mathbb{Z}$ and additive bases for the set of natural numbers $\mathbb{N}$; in the latter case the famous conjecture of Erdős and Turán states that for no additive h-basis for $\mathbb{N}$ the number of representation $r_{A,h}(n)$ can be bounded by an absolute constant. The main reason which makes this problem much easier for integers is that, unlike in the case of the natural numbers where for each additive h-basis B we must have $B \cap [1,n] \geq n^{1/h}$, the additive bases for $\mathbb{Z}$ can be arbitrarily sparse. Since a unique representation basis constructed by Nathanson contained $\Omega(\log n)$ elements a with $|a| \leq n$, he asked if there exist dense unique representation bases for $\mathbb{Z}$; recently, he answered this question in the affirmative [3]. In this note we give another proof of his result and show that, in fact, each subset $A \subseteq \mathbb{Z}$ for which certain Sidon-type property $\mathcal{S}_h$ holds, is contained in a unique representation h-base for $\mathbb{Z}$.

Definition. Let $\gamma : \mathbb{N} \to \mathbb{N}$ be a non-decreasing function on natural numbers, and $h \in \mathbb{N}$, $h \geq 2$. A set $A \subseteq \mathbb{Z}$ has property $\mathcal{S}_h$ (with respect to γ) if for for every $a_1, a_2, \ldots, a_{2h} \in A$, such that $|a_1| \geq |a_2| \geq \ldots \geq |a_{2h}|$, and every choice of

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signs $\varepsilon_i = \pm 1$, $i = 1, 2, \ldots, 2h$,

$$\Big|\sum_{i=1}^{2h} \varepsilon_i a_i\Big| \geq \gamma(|a_1|),$$

unless $a_1, \ldots, a_{2h}$, $\varepsilon_1, \ldots, \varepsilon_{2h}$ are such that for some permutation σ of the set $\{1, \ldots, 2h\}$ we have $a_{\sigma(2i-1)} = a_{\sigma(2i)}$ and $\varepsilon_{\sigma(2i-1)} = \varepsilon_{\sigma(2i)}$, for $i = 1, 2, \ldots, h$.

The main result of this note states that each set with property $\mathcal{S}_h$ is contained in some unique representation h-basis with basically the same density. (Here and below, for $A \subseteq \mathbb{Z}$ we set $A(n) = \big|A \cap [-n, n]\big|$.)

Theorem 1. *Let $\gamma : \mathbb{N} \to \mathbb{N}$ be a nondecreasing function such that $\gamma(n) \to \infty$ as $n \to \infty$, and $h \geq 2$. Then for every subset A of $\mathbb{Z}$ with property $\mathcal{S}_h$, there exists $B \subseteq \mathbb{Z}$ such that $A \cup B$ is a unique representation h-basis for $\mathbb{Z}$, and $B(n)/A(n) \to 0$ as $n \to \infty$.*

Proof. First we define recursively a nondescending family of finite sets B_i, $i \geq 0$. Let $B_0 = \emptyset$. For $i \geq 1$, let $n(i)$ denote the element of the minimum absolute value which is not represented as a sum of h elements from $A \cup B_i$. Note that since B_i is finite and A has property $\mathcal{S}_h$ such an element always exists; e.g., if we choose $a \in A$ such that $\gamma(|a|) \geq 2h \max\{|b| : b \in B_i\}$, then $ha - 1$ can not be expressed as a sum of h elements from $A \cup B_i$. Now let $a_2(i), \ldots, a_{h-1}(i)$ be any elements of A, and let $a_1(i)$ be chosen in such a way that

$$\gamma(|a_1(i)|) > 2h\Big(|n(i)| + \sum_{j=2}^{h-1} a_j(i)\Big),$$

and A contains at least $2^{|n(i)|}$ elements of absolute value smaller than $|a_1(i)|$. Now set $b(i) = n(i) - \sum_{j=1}^{h-1} a_j(i)$, $B_{i+1} = B_i \cup \{b(i)\}$, and $B = \bigcup_{i=0}^{\infty} B_i$. Observe that $B(n) \leq 2\log_2 A(n)$, and so $B(n)/A(n) \to 0$. Note also that each integer is represented as the sum of h elements from $A \cup B$.

Now assume that there exist $n \in \mathbb{Z}$, and $c_1, c_2, \ldots, c_{2h} \in A \cup B$, such that $n = c_1 + \cdots + c_h = c_{h+1} + \cdots + c_{2h}$ for some $\{c_1, \ldots, c_h\} \neq \{c_{h+1}, \ldots, c_{2h}\}$. Then, for each $i = 1, 2, \ldots, 2h$, either $c_i = a_i$ for some $a_i \in A$, or $c_i \in B$ and so, there is $a_i \in A$, such that

$$|c_i + a_i| < \frac{1}{2h}\gamma(|a_i|).$$

Hence, for some $a_1, a_2, \ldots, a_{2h} \in A$, $|a_1| \geq |a_2|, \ldots, |a_{2h}|$, and appropriately chosen signs $\varepsilon_i = \pm 1$, $i = 1, 2, , \ldots, 2h$, we have

$$\Big|\sum_{i=1}^{2h} \varepsilon_i a_i\Big| < |(c_1 + \cdots + c_h) - (c_{h+1} + \cdots + c_{2h})| + \frac{1}{2h}\sum_{i=1}^{2h} \gamma(|a_i|) \leq \gamma(|a_1|),$$

contradicting the fact that A has property $\mathcal{S}_h$. Consequently, $A \cup B$ is a unique representation h-basis. ∎

Theorem 1 immediately implies the existence of dense unique representation h-bases for $\mathbb{Z}$.

Theorem 2. *For every nondecreasing function $g(n)$, such that $g(n) \to \infty$ but $g(n)/n^{1/(2h-1)} \to 0$ as $n \to \infty$, there exists a unique representation h-basis $\hat{A}$ for $\mathbb{Z}$ such that $\hat{A}(n)/g(n) \to 1$ as $n \to \infty$.*

Proof. Let $g(n) \to \infty$ but $g(n)/n^{1/(2h-1)} \to 0$. We shall greedily construct $A \subseteq \mathbb{N}$ such that $A(n)/g(n) \to 1$, and A has property $\mathcal{S}_h$ with respect to $\gamma(n) = n^{1/(2h-1)}/g(n)$. To this end let us define recursively a nondescending family of sets A_n, $n \geq 0$, in the following way. Let $A_0 = \emptyset$. If $|A_n| < g(n)$, then we add to A_n the smallest element $t_n \notin A_n$, such that $t_n \geq n$ and the resulting set $A_{n+1} = A_n \cup \{t_n\}$ has the property $\mathcal{S}_h$; if $|A_n| \geq g(n)$, we put $A_{n+1} = A_n$. Observe that since $g(n)/n^{1/(2h-1)} \to 0$ and $\gamma(n) = n^{1/3}/g(n)$ for every sufficiently large n we have $t_n = (1 + o(1))n$; consequently, for $A = \bigcup_{n \geq 0} A_n$ we have $A(n)/g(n) \to 1$.

Thus, the assertion follows from Theorem 1. ■

Theorem 1 shows that studying densities of unique representation 2-bases of $\mathbb{Z}$ is closely related to studying densities of Sidon sets in $\mathbb{N}$. Indeed, if a subset of $\mathbb{Z}$ is a unique representation 2-basis, then both its positive and negative parts are Sidon sets in $\mathbb{N}$ and $-\mathbb{N}$ respectively. On the other hand, if a subset of $\mathbb{N}$ has the 'strong Sidon' property $\mathcal{S}_2$, then it can be supplemented by a thin set to a unique representation 2-basis for $\mathbb{Z}$. Unfortunately, we do not know much about dense Sidon sets in $\mathbb{N}$: an ingenious construction of Ruzsa [4] gives a Sidon set A in $\mathbb{N}$ with $A(n) = n^{\sqrt{2}-1+o(1)}$; still we cannot prove that there are no Sidon sets $S \subseteq \mathbb{N}$ with $S(n) = n^{1/2+o(1)}$.

We remark also that a well known argument Erdős used to bound densities of Sidon sets in $\mathbb{N}$ (see [1] Theorem II.8) can be mimicked to prove that for each unique representation 2-basis A for $\mathbb{Z}$ we have $\liminf_{n\to\infty} A(n)\sqrt{\log n/n} = O(1)$.

Finally, we observe that Theorems 1 and 2 can be easily generalized in a similar way as in [3]. Thus, for instance, using basically the same argument as in the proof of Theorem 1, one can show the following result.

Theorem 1*. *Let $h \geq 2$, A be subset of $\mathbb{Z}$ with property $\mathcal{S}_h$, and $f : \mathbb{Z} \to \mathbb{N} \cup \{\infty\}$. Then, there exists $B \subseteq \mathbb{Z}$ such that $B(n)/A(n) \to 0$ as $n \to \infty$, and $r_{A\cup B,h}(n) = f(n)$ for every $n \in \mathbb{Z}$.*

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