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SUM FORMULA FOR KLOOSTERMAN SUMS AND FOURTH MOMENT OF THE DEDEKIND ZETA-FUNCTION OVER THE GAUSSIAN NUMBER FIELD

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Abstract: We prove the Kloosterman–Spectral sum formula for $\mathrm{PSL}_2(\mathbb{Z}[i])\backslash\mathrm{PSL}_2(\mathbb{C})$, and apply it to derive an explicit spectral expansion for the fourth power moment of the Dedekind zeta-function of the Gaussian number field.
Keywords: automorphic forms, Dedekind zeta-function, fourth moment, Gaussian number field, Kloosterman sums, spectral theory, sum formula.

1. Introduction

Our principal aim in the present article is to establish an explicit formula for the fourth moment of the Dedekind zeta-function ζ_{F} of the Gaussian number field $\mathrm{F} = \mathbb{Q}(i)$:

$$\mathcal{Z}_2(g, \mathrm{F}) = \int_{-\infty}^{\infty} |\zeta_{\mathrm{F}}(\tfrac{1}{2} + it)|^4 g(t)dt, \tag{1.1}$$

where the weight function g is assumed, for the sake of simplicity, to be entire and of rapid decay in any fixed horizontal strip. The basic implement to be utilized is a sum formula for Kloosterman sums over F

$$S_{\mathrm{F}}(m, n;\, c) = \sum_{\substack{d \bmod c \\ (d,c)=1}} \exp\Big(2\pi i \mathrm{Re}\left((md + n\tilde{d})/c\right)\Big) \tag{1.2}$$

with $c, d, m, n \in \mathbb{Z}[i]$ and $d\tilde{d} \equiv 1 \bmod c$. It is an extension of the Kloosterman–Spectral sum formula for $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathrm{PSL}_2(\mathbb{R})$. By a Kloosterman–Spectral sum formula we mean any method of expressing sums of Kloosterman sums in terms of spectral bilinear forms in Fourier coefficients of automorphic forms, and by a Spectral–Kloosterman sum formula any procedure in the opposite direction. As is to be detailed shortly, the existing version of the Spectral–Kloosterman sum formula for $\mathrm{PSL}_2(\mathbb{C})$ concerns only K-trivial automorphic forms over $\mathrm{PSL}_2(\mathbb{C})$, and

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is unsuitable to handle sums of S_F explicitly. We extend the Spectral–Kloosterman sum formula to all K-types and invert the Bessel transformation occurring in it, obtaining the Kloosterman–Spectral sum formula for $\mathrm{PSL}_2(\mathbb{Z}[i])\backslash\mathrm{PSL}_2(\mathbb{C})$. We stress that the Bessel inversion has so far been obtained only for $\mathrm{PSL}_2(\mathbb{R})$ and infinitesimally isomorphic groups.

The explicit formula in Theorem 14.1 expresses $\mathcal{Z}_2(g, \mathrm{F})$ as a sum of a term $M_\mathrm{F}(g)$, an integral transform of g, and a term based on the spectral decomposition of $L^2(\mathrm{PSL}_2(\mathbb{Z}[i])\backslash\mathrm{PSL}_2(\mathbb{C}))$. It is a generalization to F of Theorem 4.2 of [25] that gives for the fourth moment $\mathcal{Z}_2(g, \mathbb{Q})$ of the Riemann zeta-function an explicit formula based on spectral data for $L^2(\mathrm{PSL}_2(\mathbb{Z})\backslash\mathrm{PSL}_2(\mathbb{R}))$; see the final section. In [6], we have generalized that theorem to real quadratic number fields with class number one. In all the three cases, the fourth moment $\mathcal{Z}_2(g, \cdot)$ is linked to spectral data via Kloosterman sums. Thus one may assert that they are built on an essentially common basis. Indeed, the structural similarity among these spectral expansions of the moments is remarkable.

There are, however, notable differences among them as well. In the rational case, the $M_\mathbb{Q}(g)$ is the main term, overshadowing the other explicitly spectral terms. For the quadratic cases, the same does not hold. We shall briefly discuss this peculiar fact for the present case in the final section.

In the case of the Riemann zeta function, the relevant Kloosterman-Spectral sum formula is the one due to Kuznetsov [18], [19]. There a sum of rational Kloosterman sums is expressed in terms of a spectral bilinear form in Fourier coefficients of automorphic forms over the upper half-plane. The test functions on both sides of this equality are related by an integral transformation, given by a Bessel kernel. In applications of the sum formula, it is important to have control over this integral transformation, and, in particular, to be able to invert it. Kuznetsov did this in the works quoted above in an ingenious way. The sum formula for the upper half-plane has been discussed at various places. A self-contained treatment along classical lines can be found in the first two chapters of [25]. For a spectral formulation of the sum formula, the version in [1] has the advantage to stress that the spectral data are tied not to automorphic forms in the upper half-plane but, in fact, to irreducible subspaces of the right regular representation of $\mathrm{PSL}_2(\mathbb{R})$ in $L^2(\mathrm{PSL}_2(\mathbb{Z})\backslash\mathrm{PSL}_2(\mathbb{R}))$.

In the case of [6], we could appeal to [4], which treats totally real number fields. There the Bessel transform in the sum formula on $(\mathrm{PSL}_2(\mathbb{R}))^d$ had to be handled, but that is not essentially more difficult than the Bessel transform for $\mathrm{PSL}_2(\mathbb{R})$.

For our present group $\mathrm{PSL}_2(\mathbb{C})$ such a reduction to smaller groups does not hold; in fact it is the first step of an induction leading to any algebraic number field, and we have to start from scratch. It is true that Miatello and Wallach have given in [23] a wide generalization of the sum formula, to Lie groups of real rank one. They have, however, good reasons to restrict themselves to irreducible representations with a K-trivial vector which are comparable to automorphic forms of weight zero. The relevant integral transform is described by a power series expansion of its kernel function; hence its behavior is only known near the origin. Moreover, the restriction to K-trivial representations makes it unlikely that the same kernel function can be used to describe the inverse transformation. However, we need, for the purpose of the present paper, a sum formula that relates sums of the form

$$\sum_{c\in\mathbb{Z}[i]\setminus\{0\}} S_{\mathrm{F}}(m,n;c)f(c) \tag{1.3}$$

to spectral data for rather arbitrary test functions f on $\mathbb{C}\setminus\{0\}$. That is, we are given a sum of Kloosterman sums to begin with, but not spectral expressions as in [23]. This requires a good control of the relevant integral kernel in much the same manner as Kuznetsov's theory allows us to do for $\mathrm{PSL}_2(\mathbb{R})$. We achieve this by deriving a Spectral–Kloosterman sum formula for $\mathrm{PSL}_2(\mathbb{Z}[i])\backslash\mathrm{PSL}_2(\mathbb{C})$ for each K-type, and combine it into one complete Spectral–Kloosterman sum formula. We invert the Bessel transform in this sum formula, and arrive at the Kloosterman–Spectral sum formula. The major part of the present article is devoted to the development of these Spectral–Kloosterman sum formulas. As is mentioned above, $\mathrm{PSL}_2(\mathbb{C})$ is the first Lie group other than $\mathrm{PSL}_2(\mathbb{R})$, for which this programme has been carried out completely. The results are stated Theorems 10.1 and 13.1. The former theorem can be used to get information on spectral data; and it is the basis of the latter, in which sums of the type (1.3) are spectrally decomposed. The integral transform in these formulae has a product of two Bessel functions as its kernel; see (6.21) and (7.21). Theorem 11.1 gives the inversion of the integral transformation. The integral representation in Theorem 12.1 allows us to bound the kernel function in (7.21) in a practical way for applications, especially to treat $\mathcal{Z}_2(g,\mathrm{F})$.

Once the sum formula in Theorem 13.1 has become available, we can proceed with the study of $\mathcal{Z}_2(g,\mathrm{F})$. The general approach is the same as in the rational case (Chapter 4 of [25]), but the computations are by far more involved, as can be expected. Here it should be made explicit that we shall be concerned solely with establishing an explicit formula for $\mathcal{Z}_2(g,\mathrm{F})$. The asymptotical study of the formula is entirely left for future works. Thus the present article does not contain anything corresponding to Chapter 5 of [25], except the discussion mentioned above as to be made in the final section.

For F the step from the fourth moment to sums of S_{F} is carried out in Section 2, and follows the same lines as in the rational case. In this respect, the real quadratic case in [6] is harder. There we had to deal with infinitely many units. As far as we see, it is essential for our method to assume that the class number is one, for both real and imaginary quadratic fields. We could thus try to deal with any imaginary quadratic number field with class number one, but have exploited, in the present article, arithmetical simplifications offered by the specialization to $\mathrm{F}=\mathbb{Q}(i)$. On the other hand, the derivation of the sum formula as given in Theorems 10.1 and 13.1 could be carried out for any imaginary quadratic number field. If the class number is larger than one, the contribution of the continuous spectrum is only more complicated. The discrete subgroup can be any congruence subgroup, provided we have a Weil type bound of the corresponding generalized Kloosterman sums; actually, any non-trivial estimate suffices. Without such a bound, we would run into additional technical difficulties, that would require the method of [22] for their resolution.

In the present paper we bring together two subjects that have been regarded virtually independent of each other. We have presented the ingredients pertaining automorphic representations of the Lie group $\mathrm{PSL}_2(\mathbb{C})$ in a rather detailed way, in the hope to make them more widely accessible.

Remark. Main results of the present article have been announced in our note [5].

Convention. Notations become available at their first appearances and will continue to be effective throughout the sequel. This applies to those in the above as well. We stress two points especially: (i) The terms left/right invariance/irreducibility are, respectively, abbreviations for the invariance/irreducibility of the relevant function space with respect to the left/right translations by the elements of the group under consideration. (ii) There are mainly two kinds of summation variables, rational and Gaussian integers. The distinction between them will easily be made from the context. Also, group elements, operators, and spaces appear as variables. They are explicitly indicated if there is any danger of confusion.

2. A sum of Kloosterman sums

The aim of this section is to reduce $\mathcal{Z}_2(g,\mathrm{F})$ to a sum of S_F with variable arguments and modulus, indicating the core of the problem that we are going to resolve. We shall partly follow a discussion developed in Section 4.3 of [25].

Thus, let g be as in (1.1). Closely related to $\mathcal{Z}_2(g,\mathrm{F})$ is the function

$$\mathfrak{I}(z_1,z_2,z_3,z_4;g)=\int_{-\infty}^{\infty}\zeta_\mathrm{F}(z_1+it)\zeta_\mathrm{F}(z_2+it)\zeta_\mathrm{F}(z_3-it)\zeta_\mathrm{F}(z_4-it)g(t)dt, \quad (2.1)$$

where all $\mathrm{Re}\, z_j$ are larger than 1. Shifting the contour upward appropriately, this can be continued meromorphically to the whole of $\mathbb{C}^4$. It is regular in a neighbourhood of the point $\mathrm{p}_{\frac12}=(\frac12,\frac12,\frac12,\frac12)$, and

$$\mathcal{Z}_2(g,\mathrm{F})=\mathfrak{I}(\mathrm{p}_{\frac12};g)+a_0g(\tfrac12 i)+b_0g(-\tfrac12 i)+a_1g'(\tfrac12 i)+b_1g'(-\tfrac12 i) \quad (2.2)$$

with certain absolute constants a_0, a_1, b_0, b_1 which could be made explicit. On the other hand, expanding the integrand and integrating term by term, we get

$$\mathfrak{I}(z_1,z_2,z_3,z_4;g)=\frac{1}{16}\sum_{kl\neq0}\frac{\sigma_{z_1-z_2}(k)\sigma_{z_3-z_4}(l)}{|k|^{2z_1}|l|^{2z_3}}\hat g(2\log|l/k|) \quad (2.3)$$

with $k,l\in\mathbb{Z}[i]$. Here

$$\sigma_\nu(n)=\sigma_\nu(n,0),\quad \sigma_\nu(n,p)=\frac14\sum_{d|n}(d/|d|)^{4p}|d|^{2\nu} \quad (2.4)$$

with $p \in \mathbb{Z}$ and the divisibility inside $\mathbb{Z}[i]$, and

$$\hat{g}(x) = \int_{-\infty}^{\infty} g(t)e^{ixt}dt. \tag{2.5}$$

Classifying the summands according as $k = l$ and $k \neq l$, we have, in the region of absolute convergence,

$$\begin{aligned}\mathfrak{I}(z_1, z_2, z_3, z_4; g, \mathrm{F}) = &\frac{\zeta_{\mathrm{F}}(z_1+z_3)\zeta_{\mathrm{F}}(z_1+z_4)\zeta_{\mathrm{F}}(z_2+z_3)\zeta_{\mathrm{F}}(z_2+z_4)}{4\zeta_{\mathrm{F}}(z_1+z_2+z_3+z_4)}\hat{g}(0) \\ &+ \frac{1}{16}\sum_{m\neq 0} |m|^{-2z_1-2z_3} B_m(z_1-z_2, z_3-z_4; g^*(\cdot\,; z_1, z_3)),\end{aligned} \tag{2.6}$$

where

$$B_m(\alpha, \beta; h) = \sum_{n(n+m)\neq 0} \sigma_\alpha(n)\sigma_\beta(n+m)h(n/m), \tag{2.7}$$

and

$$g^*(u; \gamma, \delta) = \frac{\hat{g}(2\log|1+1/u|)}{|u|^{2\gamma}|1+u|^{2\delta}}. \tag{2.8}$$

The first term on the right of (2.6) is due to (14.21).

In order to exploit the relation (2.2), experience in the rational case suggests that we should continue analytically the identity (2.6) to a neighbourhood of the point $\mathrm{p}_{\frac{1}{2}}$, and that such a continuation should be accomplished via spectrally decomposing the function $B_m(\alpha, \beta; g^*(\cdot; \gamma, \delta))$ with a Kloosterman–Spectral sum formula. We shall see, in Section 14, that this is indeed the case. The long process to reach there begins with the following fact on the complex Mellin transform of g^*:

Lemma 2.1. *We put, for $q \in \mathbb{Z}$, $s \in \mathbb{C}$,*

$$\tilde{g}_q(s; \gamma, \delta) = \frac{1}{2\pi}\int_{\mathbb{C}^\times} g^*(u; \gamma, \delta)(u/|u|)^{-q}|u|^{2s}d^\times u, \tag{2.9}$$

where $\mathbb{C}^\times = \mathbb{C}\backslash\{0\}$, and $d^\times u = |u|^{-2}d_+u$ with the Lebesgue measure d_+u on $\mathbb{C}$. Then $\tilde{g}_q(s; \gamma, \delta)$ is regular in the domain

$$\mathrm{Re}\,(s - \gamma - \delta) < 0 \tag{2.10}$$

as a function of three complex variables. More precisely, all of its singularities are in the set $\{\gamma + \delta + \frac{1}{2}|q| + l : \mathbb{Z} \ni l \geq 0\}$, as is implied by the representation

$$\begin{aligned}\tilde{g}_q(s; \gamma, \delta) =&\frac{1}{2}(-1)^q \frac{\Gamma(\gamma+\delta-s+\frac{1}{2}|q|)}{\Gamma(1+s-\gamma-\delta+\frac{1}{2}|q|)} \\ &\times \int_{-\infty}^{\infty} \frac{\Gamma(1-\delta+it)}{\Gamma(\delta-it)} \frac{\Gamma(s-\gamma-it+\frac{1}{2}|q|)}{\Gamma(1+\gamma-s+it+\frac{1}{2}|q|)} g(t)dt,\end{aligned} \tag{2.11}$$

where the contour separates the poles of $\Gamma(1-\delta+it)$ *and those of* $\Gamma(s-\gamma-it+\frac{1}{2}|q|)$ *to the left and the right, respectively; and* s, γ, δ *are assumed to be such that the contour can be drawn. Moreover, if* γ, δ, *and* $\mathrm{Re}\, s$ *are bounded, then we have, regardless of* (2.10),

$$\tilde{g}_q(s;\gamma,\delta) \ll (1+|q|+|s|)^{-A} \tag{2.12}$$

with any fixed $A>0$, *as* $|q|+|s|$ *tends to infinity.*

Proof. The first assertion follows from the observations that $g^*(u;\gamma,\delta) \ll |u|^{-2\mathrm{Re}\,(\gamma+\delta)}$ as $u\to\infty$, and that $g^*(u;\gamma,\delta)$ is of rapid decay as $u\to 0, -1$, which is a consequence of respective upward and downward shifts of the contour in (2.5). To prove the second assertion we assume, temporarily, that

$$\mathrm{Re}\,\gamma < \mathrm{Re}\, s < \mathrm{Re}\,(\gamma+\delta) < \mathrm{Re}\,\gamma + 1, \tag{2.13}$$

which is of course contained in (2.10). Moving to polar coordinates, we have

$$\tilde{g}_q(s;\gamma,\delta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(t)\int_0^\infty r^{2(s-\gamma-it)-1}\int_{-\pi}^{\pi}\frac{e^{i|q|\theta}}{|1+re^{i\theta}|^{2(\delta-it)}}d\theta\, dr\, dt. \tag{2.14}$$

This triple integral is absolutely convergent. Note that we need to deal with the part corresponding to $|r-1|<\varepsilon$, $|\theta\pm\pi|<\varepsilon$ with a small $\varepsilon>0$ separately. The innermost integral is equal to

$$\begin{aligned}&\frac{1}{\Gamma(\delta-it)}\int_0^\infty y^{\delta-it-1}e^{-y(1+r^2)}\int_{-\pi}^{\pi}\exp(i|q|\theta-2ry\cos\theta)d\theta\, dy \\ &= \frac{2\pi(-1)^q}{\Gamma(\delta-it)}\int_0^\infty y^{\delta-it-1}e^{-y(1+r^2)}I_{|q|}(2ry)dy.\end{aligned} \tag{2.15}$$

Thus we have

$$\begin{aligned}\tilde{g}_q(s;\gamma,\delta) &= (-1)^q\int_{-\infty}^{\infty}\frac{2^{-\delta+it}g(t)}{\Gamma(\delta-it)}\int_0^\infty r^{2(s-\gamma)-\delta-it-1} \\ &\quad\times\int_0^\infty y^{\delta-it-1}e^{-\frac{1}{2}y(r+1/r)}I_{|q|}(y)dy\, dr\, dt \\ &= (-1)^q\int_{-\infty}^{\infty}\frac{2^{1-\delta+it}g(t)}{\Gamma(\delta-it)} \\ &\quad\times\int_0^\infty y^{\delta-it-1}K_{2(s-\gamma)-\delta-it}(y)I_{|q|}(y)dy\, dt,\end{aligned} \tag{2.16}$$

where the necessary absolute convergence follows from asymptotic expansions of these Bessel functions. The last integral can be evaluated as a limiting case of formula (1) on p. 410 of [35] coupled with the relation $i^{-|q|}J_{|q|}(iy) = I_{|q|}(y)$. It is equal to

$$\begin{aligned}&\frac{\Gamma(s-\gamma-it+\frac{1}{2}|q|)\Gamma(\gamma+\delta-s+\frac{1}{2}|q|)}{2^{2-\delta+it}\Gamma(|q|+1)} \\ &\qquad\times {}_2F_1(s-\gamma-it+\tfrac{1}{2}|q|, \gamma+\delta-s+\tfrac{1}{2}|q|; |q|+1; 1) \\ &= 2^{\delta-it-2}\frac{\Gamma(1-\delta+it)\Gamma(s-\gamma-it+\frac{1}{2}|q|)\Gamma(\gamma+\delta-s+\frac{1}{2}|q|)}{\Gamma(1+\gamma-s+it+\frac{1}{2}|q|)\Gamma(1-\gamma-\delta+s+\frac{1}{2}|q|)},\end{aligned} \tag{2.17}$$

where we have used Gauss' formula for the value of the hypergeometric function at the point 1. Thus we get (2.11) with the contour $\mathrm{Im}\, t = 0$, provided (2.13). Having obtained this, we use analytic continuation to have the representation (2.11) for those s, γ, δ with which the above separation of poles is possible. Then, to find the location of singularities, we move the contour in (2.11) to $\mathrm{Im}\, t = L$ with a sufficiently large $L > 0$. We may encounter poles at $t = i(1-\delta) + il$ with integers $l \geq 0$, but the relevant residual contributions are easily seen to be entire over $\mathbb{C}^3$, whence the above assertion. As to the bound (2.12), we note that, when $|s| + |q|$ tends to infinity while γ, δ, and $\mathrm{Re}\, s$ are bounded, the contour in (2.11) can be drawn. Then we need only to push the contour down appropriately. The new integral is readily estimated by Stirling's formula. We may encounter poles if q is bounded, but then resulting residues do not disturb (2.12) because of the assumption on g. This ends the proof. ■

Now, returning to (2.7), we note the Ramanujan identity over F: We have, for any $n \in \mathbb{Z}[i]$, $p \in 2\mathbb{Z}$,

$$\sum_{c\neq 0}(c/|c|)^{2p} S_{\mathrm{F}}(n,0;c)|c|^{-2s} = \frac{4}{\zeta_{\mathrm{F}}(s,p/2)} \cdot \begin{cases} \sigma_{1-s}(n,p/2) & \text{if } n \neq 0,\ \mathrm{Re}\, s > 1, \\ \zeta_{\mathrm{F}}(s-1,p/2) & \text{if } n = 0,\ \mathrm{Re}\, s > 2, \end{cases} \tag{2.18}$$

where

$$\zeta_{\mathrm{F}}(s,p) = \frac{1}{4}\sum_{n\neq 0}(n/|n|)^{4p}|n|^{-2s} \tag{2.19}$$

is the Hecke L-function of F associated with the Grössencharakter $(n/|n|)^{4p}$. Applying this with $p = 0$ to the factor $\sigma_\beta(n+m)$ in (2.7), we see that if

$$1 + \max(0, \mathrm{Re}\,\alpha) < \mathrm{Re}\,(\gamma+\delta), \quad \mathrm{Re}\,\beta < -1, \tag{2.20}$$

then we have the absolutely convergent expression

$$B_m(\alpha,\beta; g^*(\cdot\,;\gamma,\delta)) = \frac{1}{4}\zeta_{\mathrm{F}}(1-\beta)\sum_{c\neq 0}|c|^{2(\beta-1)} \times \sum_{\substack{d \bmod c \\ (d,c)=1}} \exp(2\pi i \mathrm{Re}\,(dm/c)) D_m(\alpha, d/c; g^*(\cdot\,;\gamma,\delta)) \tag{2.21}$$

with

$$D_m(\alpha, d/c; g^*(\cdot\,;\gamma,\delta)) = \sum_{n\neq 0}\sigma_\alpha(n)\exp(2\pi i\mathrm{Re}\,(dn/c)) g^*(n/m;\gamma,\delta). \tag{2.22}$$

Note that we have used $g^*(-1;\gamma,\delta) = 0$ in (2.21). We expand $g^*(u;\gamma,\delta)$ into a Fourier series, and apply Mellin inversion to each Fourier coefficient, so that in view of the last lemma we have, for any $u \in \mathbb{C}^\times$ and $\tau < \mathrm{Re}\,(\gamma+\delta)$,

$$g^*(u;\gamma,\delta) = \frac{1}{\pi i}\sum_{q\in\mathbb{Z}}(u/|u|)^q \int_{(\tau)} \tilde{g}_q(s;\gamma,\delta)|u|^{-2s}ds, \tag{2.23}$$

where (τ) is the vertical line $\mathrm{Re}\, s = \tau$. Thus we have

$$D_m(\alpha, d/c; g^*(\cdot\,;\gamma,\delta)) = \frac{1}{\pi i}\sum_{q\in\mathbf{Z}}(m/|m|)^{-q}\int_{(\tau)} X_q(s,\alpha;d/c)|m|^{2s}\tilde{g}_q(s;\gamma,\delta)ds, \tag{2.24}$$

provided $1+\max(0,\mathrm{Re}\,\alpha) < \tau < \mathrm{Re}\,(\gamma+\delta)$, where

$$X_q(s,\alpha;d/c) = \sum_{n\neq 0}\sigma_\alpha(n)\exp(2\pi i\mathrm{Re}\,(dn/c))(n/|n|)^q|n|^{-2s} \tag{2.25}$$

with $(d,c)=1$.

Then we invoke (see [27]): If $q\neq 0$, the function $X_q(s,\alpha;d/c)$ of s is regular for any α. If $q=0$ and $\alpha\neq 0$, it is regular except for the simple poles at $s=1$ and $s=1+\alpha$ with the residues $\pi|c|^{2(\alpha-1)}\zeta_{\mathrm{F}}(1-\alpha)$ and $\pi|c|^{-2(\alpha+1)}\zeta_{\mathrm{F}}(1+\alpha)$, respectively. Moreover, we have, for any combination of parameters,

$$X_q(s,\alpha;d/c) = (c/|c|)^{2q}(\pi/|c|)^{4s-2\alpha-2} \times\frac{\Gamma(1-s+\frac12|q|)\Gamma(1+\alpha-s+\frac12|q|)}{\Gamma(s+\frac12|q|)\Gamma(s-\alpha+\frac12|q|)}X_{-q}(1-s,-\alpha;\tilde{d}/c) \tag{2.26}$$

with $\tilde{d}d\equiv 1 \bmod c$. By the convexity argument of Phragmén and Lindelöf we deduce from this that X_q is of polynomial growth with respect to all involved parameters as far as s remains in an arbitrary but fixed vertical strip.

The last lemma allows us to shift the contour in (2.24) to the left as we like. The functional equation (2.26) yields

Lemma 2.2. *If*

$$1+\max(0,\mathrm{Re}\,\alpha) < \mathrm{Re}\,(\gamma+\delta),\quad |\mathrm{Re}\,\alpha|+\mathrm{Re}\,\beta < -2, \tag{2.27}$$

then we have, for any non-zero $m\in\mathbb{Z}[i]$,

$$B_m(\alpha,\beta;g^*(\cdot\,;\gamma,\delta)) = [B_m^{(0)}+B_m^{(1)}](\alpha,\beta;g^*(\cdot\,;\gamma,\delta)). \tag{2.28}$$

Here

$$\begin{aligned} &B_m^{(0)}(\alpha,\beta;g^*(\cdot\,;\gamma,\delta)) \\ &= 2\pi|m|^2\sigma_{\alpha+\beta-1}(m)\frac{\zeta_{\mathrm{F}}(1-\alpha)\zeta_{\mathrm{F}}(1-\beta)}{\zeta_{\mathrm{F}}(2-\alpha-\beta)}\tilde{g}_0(1;\gamma,\delta) \\ &\quad + 2\pi|m|^{2(\alpha+1)}\sigma_{\beta-\alpha-1}(m)\frac{\zeta_{\mathrm{F}}(1+\alpha)\zeta_{\mathrm{F}}(1-\beta)}{\zeta_{\mathrm{F}}(2+\alpha-\beta)}\tilde{g}_0(1+\alpha;\gamma,\delta)\end{aligned} \tag{2.29}$$

and

$$\begin{aligned}&B_m^{(1)}(\alpha,\beta;g^*(\cdot\,;\gamma,\delta)) \\ &= -\frac14 i\pi^{2\beta-1}\zeta_{\mathrm{F}}(1-\beta)|m|^{\alpha+\beta+1}\sum_{n\neq 0}\sigma_{-\alpha}(n)|n|^{\alpha+\beta-1}S_{m,n}(\alpha,\beta,\gamma,\delta;g),\end{aligned} \tag{2.30}$$

where

$$S_{m,n}(\alpha,\beta,\gamma,\delta;g)=\sum_{c\neq 0}\frac{1}{|c|^2}S_{\mathrm F}(m,n\,;c)\,[g]\Big(\frac{2\pi}{c}\sqrt{mn};\alpha,\beta,\gamma,\delta\Big); \tag{2.31}$$

$$[g](u;\alpha,\beta,\gamma,\delta)=(|u|/2)^{-2(1+\alpha+\beta)}$$
$$\times\sum_{q\in\mathbb Z}(u/|u|)^{-2q}\int_{(\eta)}\frac{\Gamma(1-s+\frac12|q|)\Gamma(1+\alpha-s+\frac12|q|)}{\Gamma(s+\frac12|q|)\Gamma(s-\alpha+\frac12|q|)}\tilde g_q(s;\gamma,\delta)(|u|/2)^{4s}ds \tag{2.32}$$

with $\eta<1+\min(0,\mathrm{Re}\,\alpha)$. *All members on the left sides of* (2.28)-(2.32) *are regular functions of the four complex variables in the domain* (2.27).

Proof. This is analogous to the rational case, which is developed in Section 4.3 of [25]. The first condition in (2.27) comes from (2.20). The second condition there allows us to shift the contour in (2.24) to (ξ) with $1+\frac12\mathrm{Re}\,(\alpha+\beta)<\xi<\min(0,\mathrm{Re}\,\alpha)$. Then the right side of (2.29), which depends on (2.18), is the contribution of the poles at $s=1,1+\alpha$ which occur only when $q=0$. With this choice of the contour, in place of (η) in (2.32), the absolute convergence throughout (2.30)-(2.32) follows solely from (2.12). This gives the regularity assertion. To finish the proof we move the contour from (ξ) to (η) as is specified above. ■

Note that $[g](u;\alpha,\beta,\gamma,\delta)$ is even with respect to u so that the choice of square root makes no difference in (2.31). A relatively closed expression for the transform $g\mapsto[g]$ is available, though it is not much relevant to our present purpose; see Remark at the end of Section 12. The use of the Weil bound (8.14) for $S_{\mathrm F}$ gives the refinement of the second condition in (2.27) to $|\mathrm{Re}\,\alpha|+\mathrm{Re}\,\beta<-\frac32$. It should, however, be observed that the domain (2.27) in $\mathbb C^4$, even with this refinement, does not contain the critical point $(0,0,\frac12,\frac12)$ that corresponds to $\mathrm p_{\frac12}$ in (2.2). In other words, the estimate (8.14) of individual Kloosterman sums does not suffice; we need a massive cancellation among Kloosterman sums. We shall demonstrate, in Section 14, that the Kloosterman–Spectral sum formula in Theorem 13.1 serves this purpose. It can be regarded as a device to separate the variables m, n in (2.31). Taking the result of the separation into (2.30), a sum of products of Hecke series emerges. The fact that these functions are *entire* and of polynomial growth gives rise to the desired analytic, or more precisely, meromorphic continuation of (2.6) to $\mathbb C^4$.

Remark. The right side of (2.7) is called the complex binary additive divisor sum; for its rational counterpart see [24]. The dissection leading to (2.6) is crucial in our argument. We stress that it is not precisely the analogue of the Atkinson dissection in the rational case (see Section 4.2 of [25]). Observe that on the right side of (2.6) three times the first term is hidden in the second term; and thus the latter does not represent the *non-diagonal* part of the sum (2.3) in the traditional sense. A direct extension of Atkinson's device might be to classify the summands in (2.3) according either to the norms of k and l or to the ideals generated by them. Then it would, however, be difficult to relate the result of dissection with any Kloosterman sums or like. In our argument we exploit a geometric feature specific to the ring $\mathbb{Z}[i]$; that is, our dissection is based on the *lattice*, rather than arithmetic, structure of $\mathbb{Z}[i]$. If one tries to consider the fourth moment of the Dedekind zeta-function of any imaginary quadratic number field with class number larger than one, the dissection argument will become an issue. We note that our argument has, nevertheless, a certain generality as well; it extends to

$$\int_{-\infty}^{\infty} |\zeta_{\mathrm{F}}(\tfrac{1}{2}+it,a)\zeta_{\mathrm{F}}(\tfrac{1}{2}+it,b)|^2 g(t)dt \tag{2.33}$$

for any $a,\, b \in \mathbb{Z}$. See also Remark at the end of Section 14.

3. The group $\mathrm{PSL}_2(\mathbb{C})$

The $S_{m,n}$ in (2.31) is a sum of Kloosterman sums. The spectral decomposition of it requires a considerable dose of the representation theory of the Lie group $\mathrm{PSL}_2(\mathbb{C})$.

With this aim in mind, we shall work with functions on $\mathrm{SL}_2(\mathbb{C})$ which are left-invariant over $\mathrm{SL}_2(\mathbb{Z}[i])$. Since it is implied that they are even, i.e., $f(-\mathrm{g}) = f(\mathrm{g})$, we are actually dealing with

$$G = \mathrm{PSL}_2(\mathbb{C}), \quad \Gamma = \mathrm{PSL}_2(\mathbb{Z}[i]). \tag{3.1}$$

Denoting by $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ the projective image of the elements $\pm\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of $\mathrm{SL}_2(\mathbb{C})$, we put

$$\mathrm{n}[z] = \begin{bmatrix} 1 & z \\ & 1 \end{bmatrix}, \quad \mathrm{h}[u] = \begin{bmatrix} u & \\ & 1/u \end{bmatrix}, \quad \mathrm{k}[\alpha,\beta] = \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix} \tag{3.2}$$

for $z, u, \alpha, \beta \in \mathbb{C}$, $u \neq 0$, $|\alpha|^2 + |\beta|^2 = 1$; and also

$$N = \{\mathrm{n}[z] \colon z \in \mathbb{C}\},\ A = \{\mathrm{a}[r] \colon r > 0\},\ K = \mathrm{PSU}(2) = \{\mathrm{k}[\alpha,\beta] \colon \alpha,\beta \in \mathbb{C}\} \tag{3.3}$$

with $\mathrm{a}[r] = \mathrm{h}[\sqrt{r}]$. We have, for the Euler angles $\varphi, \theta, \psi \in \mathbb{R}$,

$$\mathrm{k}[\alpha,\beta] = \mathrm{h}[e^{i\varphi/2}]\mathrm{k}[\cos(\tfrac{1}{2}\theta), i\sin(\tfrac{1}{2}\theta)]\mathrm{h}[e^{i\psi/2}]. \tag{3.4}$$

We have the Iwasawa decomposition

$$G = NAK, \tag{3.5}$$

which we write, e.g., $\mathrm{g} = \mathrm{nak} = \mathrm{n}[z]\mathrm{a}[r]\mathrm{k}[\alpha,\beta]$, and understand as a coordinate system on G. With it, Haar measures on respective groups are given by

$$d\mathrm{n} = d_+z, \quad d\mathrm{a} = \frac{1}{r}dr, \quad d\mathrm{k} = \frac{1}{8\pi^2}\sin\theta\, d\varphi\, d\theta\, d\psi, \tag{3.6}$$

and

$$dg = \frac{1}{r^2} d\dot{n}\, da\, dk. \tag{3.7}$$

We have, in particular,

$$\int_K dk = 1, \qquad \int_{\Gamma\backslash G} dg = \frac{2}{\pi^2}\zeta_F(2), \tag{3.8}$$

where, with an obvious abuse of notation,

$$\Gamma\backslash G = \Gamma\backslash G/K \cdot K \cong \Gamma\backslash\mathbb{H}^3 \cdot K. \tag{3.9}$$

Here $\mathbb{H}^3$ is the hyperbolic upper half-space, and $\Gamma\backslash\mathbb{H}^3$ is represented by the set

$$\left\{(z,r) \in \mathbb{H}^3 : |\mathrm{Re}\, z| \le \tfrac{1}{2},\ 0 \le \mathrm{Im}\, z \le \tfrac{1}{2},\ |z|^2 + r^2 \ge 1\right\}, \tag{3.10}$$

which is the classical fundamental domain of the Picard group.

Next, the Lie algebra $\mathfrak{g}$ of G has a basis consisting of the six elements

$$\begin{aligned} \mathbf{H}_1 &= \tfrac{1}{2}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \mathbf{V}_1 = \tfrac{1}{2}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \mathbf{W}_1 = \tfrac{1}{2}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \\ \mathbf{H}_2 &= \tfrac{1}{2}\begin{pmatrix} i & \\ & -i \end{pmatrix}, \mathbf{V}_2 = \tfrac{1}{2}\begin{pmatrix} & i \\ -i & \end{pmatrix}, \mathbf{W}_2 = \tfrac{1}{2}\begin{pmatrix} & i \\ i & \end{pmatrix}. \end{aligned} \tag{3.11}$$

These generate the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. We identify them with right differentiations on G; that is, e.g.,

$$(\mathbf{H}_2 f)(g) = \lim_{\mathbb{R}\ni t\to 0} \frac{d}{dt} f(g \exp(t\mathbf{H}_2)) = (\partial_\psi f)(g). \tag{3.12}$$

Then $\mathcal{U}(\mathfrak{g})$ is the set of all left-invariant differential operators on G. The center $\mathcal{Z}(\mathfrak{g})$ of $\mathcal{U}(\mathfrak{g})$ is the polynomial ring $\mathbb{C}[\Omega_+, \Omega_-]$ with the two Casimir elements

$$\Omega_\pm = \frac{1}{8}\left((\mathbf{H}_1 \mp i\mathbf{H}_2)^2 + (\mathbf{V}_1 \mp i\mathbf{W}_2)^2 - (\mathbf{W}_1 \mp i\mathbf{V}_2)^2\right), \tag{3.13}$$

where the factor i means the complexification of respective elements. In terms of the Iwasawa coordinates we have

$$\begin{aligned} \Omega_+ = &\frac{1}{2}r^2\partial_z\partial_{\bar z} + \frac{1}{2}re^{i\varphi}\cot\theta\,\partial_z\partial_\varphi - \frac{1}{2}ire^{i\varphi}\partial_z\partial_\theta - \frac{re^{i\varphi}}{2\sin\theta}\partial_z\partial_\psi \\ &+ \frac{1}{8}r^2\partial_r^2 - \frac{1}{4}ir\partial_r\partial_\varphi - \frac{1}{8}\partial_\varphi^2 - \frac{1}{8}r\partial_r + \frac{1}{4}i\partial_\varphi; \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} \Omega_- = &\frac{1}{2}r^2\partial_z\partial_{\bar z} + \frac{1}{2}re^{-i\varphi}\cot\theta\,\partial_{\bar z}\partial_\varphi + \frac{1}{2}ire^{-i\varphi}\partial_{\bar z}\partial_\theta - \frac{re^{-i\varphi}}{2\sin\theta}\partial_{\bar z}\partial_\psi \\ &+ \frac{1}{8}r^2\partial_r^2 + \frac{1}{4}ir\partial_r\partial_\varphi - \frac{1}{8}\partial_\varphi^2 - \frac{1}{8}r\partial_r - \frac{1}{4}i\partial_\varphi. \end{aligned} \tag{3.15}$$

Restricting ourselves to the maximal compact subgroup K, we note that its Lie algebra $\mathfrak{k}$, and thus its universal enveloping algebra $\mathcal{U}(\mathfrak{k})$ are generated by $\mathbf{H}_2$, $\mathbf{W}_1$, and $\mathbf{W}_2$. The center $\mathcal{Z}(\mathfrak{k})$ of $\mathcal{U}(\mathfrak{k})$ is the polynomial ring $\mathbb{C}[\Omega_\mathfrak{k}]$ with

$$\Omega_\mathfrak{k} = \frac{1}{2}(\mathbf{H}_2^2 + \mathbf{W}_1^2 + \mathbf{W}_2^2). \tag{3.16}$$

In terms of the Iwasawa coordinates we have

$$\Omega_\mathfrak{k} = \frac{1}{2\sin^2\theta}\left(\partial_\varphi^2 + \sin^2\theta\,\partial_\theta^2 + \partial_\psi^2 - 2\cos\theta\,\partial_\varphi\partial_\psi + \sin\theta\cos\theta\,\partial_\theta\right). \tag{3.17}$$

Let $L^2(K)$ be the Hilbert space of all functions on K which are square-integrable over K with respect to the Haar measure dk. To describe the structure of $L^2(K)$, and hence the unitary representations of the compact group K, we put, for $|q| \le l$,

$$(\alpha x - \bar\beta)^{l-q}(\beta x + \bar\alpha)^{l+q} = \sum_{p=-l}^{l} \Phi_{p,q}^l(\mathrm{k}[\alpha,\beta])x^{l-p}. \tag{3.18}$$

We have

$$\Phi_{-p,-q}^l = (-1)^{p+q}\overline{\Phi_{p,q}^l}. \tag{3.19}$$

Also we have, with (3.4),

$$\Phi_{p,q}^l(\mathrm{k}[\alpha,\beta]) = e^{-ip\varphi - iq\psi}\Phi_{p,q}^l(\mathrm{k}[\cos(\tfrac{1}{2}\theta), i\sin(\tfrac{1}{2}\theta)]), \tag{3.20}$$

from which the relations

$$\Omega_\mathfrak{k}\Phi_{p,q}^l = -\tfrac{1}{2}(l^2+l)\Phi_{p,q}^l, \quad \mathbf{H}_2\Phi_{p,q}^l = -iq\Phi_{p,q}^l \tag{3.21}$$

follow with the convention $\Phi_{p,q}^l \equiv 0$ if $|p|$, $|q| \le l$ is violated. The set

$$\{\Phi_{p,q}^l : l \ge 0,\ |p|, |q| \le l\} \tag{3.22}$$

is an orthogonal basis of $L^2(K)$ with norms

$$\|\Phi_{p,q}^l\|_K = \left[\int_K |\Phi_{p,q}^l(\mathrm{k})|^2 dk\right]^{\frac{1}{2}} = \frac{1}{\sqrt{l+\frac{1}{2}}}\binom{2l}{l-p}^{\frac{1}{2}}\binom{2l}{l-q}^{-\frac{1}{2}}. \tag{3.23}$$

The matrices $\Phi_l = (\Phi_{p,q}^l)$ realize all unitary representations of K; in particular, we have

$$\Phi_l(\mathrm{k}_1\mathrm{k}_2) = \Phi_l(\mathrm{k}_1)\Phi_l(\mathrm{k}_2), \quad \mathrm{k}_1,\ \mathrm{k}_2 \in K. \tag{3.24}$$

We arrange (3.22) as

$$L^2(K) = \overline{\bigoplus_{\substack{l,q\\|q|\le l}} L^2(K;\, l,\, q)}, \quad L^2(K;\, l,\, q) = \bigoplus_{|p|\le l}\mathbb{C}\Phi_{p,q}^l. \tag{3.25}$$

We have

$$L^2(K;\, l,\, q) = \{f \in L^2(K) : \Omega_{\mathfrak{k}} f = -\tfrac{1}{2}(l^2+l)f,\, \mathbf{H}_2 f = -iqf\}\,. \tag{3.26}$$

We call $L^2(K;\, l,\, q)$ the subspace of $L^2(K)$ of K-type (l,q).

More generally, in a space in which K acts, we shall say an element has K-type (l,q) if it is a simultaneous eigenvector of $\Omega_{\mathfrak{k}}$ and $\mathbf{H}_2$ with eigenvalues $-\frac{1}{2}(l^2+l)$ and $-iq$, respectively. This concept corresponds to the weight in the theory of modular forms on the upper half plane.

We shall be concerned with representation spaces of $\mathfrak{g}$ for which we can use the principal series representations as a model space. The space of K-finite vectors in the principal series is

$$H(\nu,p) = \{\,\text{finite linear combinations of } \varphi_{l,q}(\nu,p)\,\} \tag{3.27}$$

with

$$\varphi_{l,q}(\nu,p)(\mathrm{na}[r]\mathrm{k}) = r^{1+\nu}\Phi^l_{p,q}(\mathrm{k}) \quad (\nu \in \mathbb{C}). \tag{3.28}$$

Formulas (3.14)–(3.15) and (3.20) imply that $H(\nu,p)$ is a simultaneous eigenspace of $\Omega_\pm$:

$$\Omega_\pm \varphi_{l,q}(\nu,p) = \frac{1}{8}((\nu \mp p)^2 - 1)\varphi_{l,q}(\nu,p). \tag{3.29}$$

The space $H(\nu,p)$ is not G-invariant, but known to be $\mathfrak{g}$-invariant and irreducible for values of (ν,p) that are of interest for our purpose. Restricting functions in $H(\nu,p)$ to K, we have a scalar product on $H(\nu,p)$. If $\mathrm{Re}\,\nu = 0$, the group G acts unitarily in the resulting Hilbert space – unitary principal series. In Section 8 we encounter these unitary representations as models for irreducible subspaces of $L^2(\Gamma\backslash G)$ to be defined there. The irreducibility can in fact be confirmed by computing the actions of the six elements in (3.11) over each $\varphi_{l,q}(\nu,p)$, though we skip it. In what follows we shall see that in most cases $H(\nu,p)$ is the space we are actually dealing with via maps commuting with the action of $\mathfrak{g}$ or equivalently of $\mathcal{U}(\mathfrak{g})$.

Remark. The general theory as well as specific treatments of unitary representations of Lie groups can be found in [17], [32], [33], and [34]. The fundamental region (3.10) is due to Picard [29], and its volume, given in (3.8), to Humbert [13]. Formulas (3.14), (3.15), and (3.17) are obtained by first interpreting (3.13) and (3.16) in terms of the local coordinates $\mathrm{g} = \mathrm{n}[z_1]\mathrm{h}[z_2]\left[\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right]\mathrm{n}[z_3]$ with $(z_1,z_2,z_3) \in \mathbb{C}^3$, over the big cell of the Bruhat decomposition of G, and by changing variables according to the Iwasawa coordinates. Our choice of basis elements (3.22) is somewhat different from common practice as is indicated by (3.23). This is for the sake of convenience for our later discussion. The maps which commute with the action of $\mathfrak{g}$ are usually called intertwining operators.

4. Automorphic forms

Let $C^\infty(\Gamma\backslash G)$ be the space of all smooth left Γ-invariant or Γ-automorphic functions on G. We consider subspaces composed of simultaneous eigenfunctions of $\Omega_\pm$, $\Omega_{\mathfrak{k}}$, and $\mathbf{H}_2$: Let χ be a character on $\mathcal{Z}(\mathfrak{g})$. We put

$$A_{l,q}(\chi) = \Big\{ f \in C^\infty(\Gamma\backslash G) : \Omega_\pm f = \chi(\Omega_\pm) f, \text{ and of } K\text{-type } (l,q) \Big\}. \tag{4.1}$$

Elements of $A_{l,q}(\chi)$ are called left Γ-automorphic forms on G of K-type (l,q) with character χ. Obviously they are counterparts of $\mathrm{PSL}_2(\mathbb{Z})$-automorphic forms on $\mathrm{PSL}_2(\mathbb{R})$.

As being eigenvalues of differential operators, $\chi(\Omega_\pm)$ cannot be arbitrary:

Lemma 4.1. *If $A_{l,q}(\chi) \neq \{0\}$ then $\chi = \chi_{\nu,p}$. Here $\chi_{\nu,p}$ is the character of $\mathcal{Z}(\mathfrak{g})$ defined by*

$$\chi_{\nu,p}(\Omega_\pm) = \frac{1}{8}((\nu \mp p)^2 - 1) \tag{4.2}$$

with certain $\nu \in \mathbb{C}$ and $p \in \mathbb{Z}$, $|p| \le l$, which are uniquely determined modulo $(\nu,p) \mapsto (-\nu,-p)$.

This assertion is a consequence of a study of Fourier coefficients of automorphic forms, which we are going to develop. Thus, for any $f \in C^\infty(\Gamma\backslash G)$ we have the Fourier expansion

$$f(\mathrm{g}) = \sum_{\omega \in \mathbf{Z}[i]} F_\omega f(\mathrm{g}), \tag{4.3}$$

where

$$F_\omega f(\mathrm{g}) = \int_{\Gamma_N\backslash N} \psi_\omega(\mathrm{n})^{-1} f(\mathrm{ng})\, d\mathrm{n} \tag{4.4}$$

with

$$\Gamma_N = \Gamma \cap N, \quad \psi_\omega(\mathrm{n}[z]) = \exp(2\pi i \mathrm{Re}\,(\omega z)). \tag{4.5}$$

Obviously the operator F_ω commutes with every element of $\mathcal{U}(\mathfrak{g})$, implying that, if $f \in A_{l,q}(\chi)$, then $F_\omega f$ is in the space

$$W_{l,q}(\chi,\omega) = \big\{ h \in C^\infty(G) : h(\mathrm{ng}) = \psi_\omega(\mathrm{n}) h(\mathrm{g}), \text{ of } K\text{-type } (l,q) \text{ with character } \chi \big\}. \tag{4.6}$$

Thus the above lemma is a corollary of

Lemma 4.2. *If $W_{l,q}(\chi,\omega) \neq \{0\}$, then there exist $\nu \in \mathbb{C}$ and $p \in \mathbb{Z}$, $|p| \le l$, such that $\chi = \chi_{\nu,p}$.*

Proof. Let $h \in W_{l,q}(\chi,\omega)$. We note that for any fixed $\mathrm{g} \in G$ the function $h(\mathrm{gk})$ of $\mathrm{k} \in K$ belongs to $L^2(K;\, l,\, q)$. In particular we have

$$h(\mathrm{g}) = \sum_{|p| \le l} h_p(\mathrm{na}) \Phi^l_{p,q}(\mathrm{k}). \tag{4.7}$$

The formulas (3.14)–(3.15) and (3.20) imply that the condition $\Omega_\pm h = \chi(\Omega_\pm)h$ is equivalent to

$$\chi(\Omega_+)h_p = \frac{1}{2}(l-p)r\partial_z h_{p+1} + \frac{1}{8}(r^2\partial_r^2 - (1+2p)r\partial_r + 4r^2\partial_z\partial_{\bar z} + p(p+2))h_p, \quad (4.8)$$

$$\chi(\Omega_-)h_p = -\frac{1}{2}(l+p)r\partial_{\bar z} h_{p-1} + \frac{1}{8}(r^2\partial_r^2 + (2p-1)r\partial_r + 4r^2\partial_z\partial_{\bar z} + p(p-2))h_p, \quad (4.9)$$

where it is supposed that $h_p \equiv 0$ if $|p| > l$. We shall first consider the case $\omega = 0$. Then (4.8)–(4.9) can be written as

$$\begin{aligned} &r^2h_p'' - rh_p' + (p^2+1)h_p = \tfrac{1}{2}(a_+ + a_-)h_p, \quad (4.10)\\ &prh_p' - ph_p = \tfrac{1}{4}(a_- - a_+)h_p \end{aligned}$$

with $\chi(\Omega_+) = \frac{1}{8}(a_+ - 1)$, $\chi(\Omega_-) = \frac{1}{8}(a_- - 1)$. If $p \neq 0$ the second equation has a solution space spanned by $r^{1+\nu}$ with $\nu = (a_- - a_+)/4p$; and the first equation gives $a_\pm = (\nu \mp p)^2$. If $p = 0$, we have $a_+ = a_-$. If $a_+ \neq 0$, then $r^{1+\nu}$ and $r^{1-\nu}$ with $\nu^2 = a_+$ span the solutions of the first equation. If $a_+ = 0$ then r and $r\log r$ are the corresponding solutions; and we have $\nu = 0$. This settles the case $\omega = 0$. We next move to the case $\omega \neq 0$. For each $t \in \mathbb{C}\setminus\{0\}$, let ℓ_t be the left translation

$$\ell_t f(\mathrm{g}) = f(\mathrm{h}[t]\mathrm{g}). \quad (4.11)$$

We have

$$\ell_t W_{l,q}(\chi,\omega) = W_{l,q}(\chi, t^2\omega), \quad (4.12)$$

which reduces the problem to the case $\omega = 1$. Any $h \in W_{\ell,q}(\chi,1)$ has the form

$$h(\mathrm{n}[z]\mathrm{a}[r]\mathrm{k}) = \exp(\pi i(z+\bar z)) \sum_{|m|\le l} h_m(r)\Phi^l_{m,q}(\mathrm{k}). \quad (4.13)$$

Again by (4.8)–(4.9) we have

$$\begin{aligned} &r^2h_m'' - (2m+1)rh_m' + (m^2+2m-4\pi^2r^2 - 8\chi(\Omega_+))h_m \quad (4.14)\\ &\qquad = -4\pi i(l-m)rh_{m+1},\\ &r^2h_m'' + (2m-1)rh_m'(r) + (m^2-2m-4\pi^2r^2 - 8\chi(\Omega_-))h_m \quad (4.15)\\ &\qquad = 4\pi i(l+m)rh_{m-1}. \end{aligned}$$

We write $\chi(\Omega_\pm) = \frac{1}{8}(\mu_\pm^2 - 1)$. We may assume, without loss of generality, that

$$0 \le \mathrm{Re}\,\mu_+ \le \mathrm{Re}\,\mu_- . \quad (4.16)$$

It is immediate that there exist constants $c_\pm$, $d_\pm$ such that

$$h_{\pm l}(r) = c_\pm r^{l+1}K_{\mu_\pm}(2\pi r) + d_\pm r^{l+1}I_{\mu_\pm}(2\pi r). \quad (4.17)$$

We consider first the case $\mu_\pm \notin \mathbb{Z}$. Applying inductively the equation (4.15) to $h_l(r)$ we see that in the expansion of $h_{-l}(r)$ all terms are multiples of either $r^{\mu_+ -l+1+2m}$ or $r^{-\mu_+ -l+1+2n}$ with integers $m,\, n \ge 0$. On the other hand, if $c_- \ne 0$ in (4.17), then $h_{-l}(r)$ has a term equal to a multiple of $r^{-\mu_- +l+1}$. Thus we have either $\mu_+ - l + 1 + 2m = -\mu_- + l + 1$ or $-\mu_+ - l + 1 + 2n = -\mu_- + l + 1$. The first identity gives $\mu_+ + \mu_- = 2(l-m)$, whence $0 \le m \le l$ and $\mu_+ = \nu + (l-m)$, $\mu_- = -\nu + (l-m)$ with a $\nu \in \mathbb{C}$. The second gives $\mu_+ = \mu_- - 2(l-n)$; that is, $\mu_+ = \nu - (l-n)$, $\mu_- = \nu + (l-n)$ with a $\nu \in \mathbb{C}$, where we have $0 \le n \le l$ because of (4.16). This settles the case $c_- \ne 0$. In other case, we should have $h_{-l}(r) = d_- r^{l+1} I_{\mu_-}(2\pi r)$. Applying inductively the equation (4.14) to h_{-l}, we proceed in much the same way, and obtain the assertion of the lemma. Next, if $\mu_- \in \mathbb{Z}$, then the above procedure yields that all terms of $h_l(r)$ are multiples of either $r^{\mu_- -l+1+2m}\log r$ or $r^{\mu_- -l+1+2n}$ with integers $m, n \ge 0$. According as either $c_+ \ne 0$ or $= 0$, we have $\mu_- - l + 1 + 2m = \mu_+ + l + 1$ or $\mu_- - l + 1 + 2n = \mu_+ + l + 1$, respectively. Thus we are again led to the same conclusion. Finally, we observe that $\mu_+ \in \mathbb{Z}$ implies $\mu_- \in \mathbb{Z}$; and we end the proof. ■

It should be remarked that in the above it is proved that if $(\nu, p) \ne (0,0)$, then

$$W_{l,q}(\chi_{\nu,p}, 0) = \mathbb{C}\,\varphi_{l,q}(\nu,p) \oplus \mathbb{C}\,\varphi_{l,q}(-\nu,-p); \tag{4.18}$$

otherwise

$$W_{l,q}(\chi_{0,0}, 0) = \mathbb{C}\,\varphi_{l,q}(0,p) \oplus \mathbb{C}\,\partial_\nu \varphi_{l,q}(\nu,p)|_{\nu=0}\,. \tag{4.19}$$

Thus $\dim W_{l,q}(\chi_{\nu,p},0) = 2$, but for $\omega \ne 0$ we know only $\dim W_{l,q}(\chi_{\nu,p},\omega) \le 2$ at this stage.

If a function f on G satisfies the bound

$$f(\mathrm{na}[r]\mathrm{k}) = O(r^b) \tag{4.20}$$

as $r \uparrow \infty$ with a certain real constant b, then we say that f is of polynomial growth. The dependency of the bound on the set where n and k move around is to be mentioned in our discussion. Automorphic forms with polynomial growth are the most interesting ones. The Fourier terms inherit this growth property, and we put

$$W^{\mathrm{pol}}_{l,q}(\chi_{\nu,p},\omega) = \{h \in W_{l,q}(\chi_{\nu,p},\omega) : \text{ of polynomial growth, uniformly over } K\}. \tag{4.21}$$

In this way we get rid of the I-Bessel term in (4.17), which is of exponential growth. On noting basic properties of the K-Bessel function, we have readily

Lemma 4.3. *Let $\omega \ne 0$. If $W^{\mathrm{pol}}_{l,q}(\chi_{\nu,p},\omega)$ is non-zero, then it has dimension one. Any generator h satisfies*

$$h(\mathrm{na}[r]\mathrm{k}) = O\left(|\omega r|^b e^{-2\pi|\omega| r}\right), \tag{4.22}$$

as $r \uparrow \infty$, with a certain real b.

We shall prove in the next section that we have actually $\dim W_{l,q}^{\mathrm{pol}}(\chi_{\nu,p},\omega)=1$ for any $\omega \neq 0$. Moreover, we shall later show that $\dim W_{l,q}(\chi_{\nu,p},\omega) = 2$ always (see (6.17)).

We next introduce the notion of cusp forms: Let

$$A_{l,q}^{\mathrm{pol}}(\chi_{\nu,p}) = \{f \in A_{l,q}(\chi_{\nu,p}) : \text{ of polynomial growth, uniformly over } N \text{ and } K\}; \tag{4.23}$$

and put

$${}^0A_{l,q}(\chi_{\nu,p}) = \{f \in A_{l,q}^{\mathrm{pol}}(\chi_{\nu,p}) : F_0 f = 0\}. \tag{4.24}$$

This is the space of cusp forms of K-type (l,q) with character $\chi_{\nu,p}$. The description of the Fourier terms in (4.17), and (3.14)–(3.15) imply:

Lemma 4.4. *All cusp forms f are real-analytic and of exponential decay:*

$$f(\mathrm{na}[r]\mathrm{k}) = O(e^{-\pi r}) \tag{4.25}$$

uniformly over N, K as r tends to infinity.

Remark. Automorphic forms can be defined on much more general Lie groups, see, e.g., Harish Chandra's lecture notes [11].

5. Jacquet operator

In order to give explicitly an element in the space $W_{l,q}^{\mathrm{pol}}(\chi_{\nu,p},\omega)$, $\omega \neq 0$, we shall appeal to the Jacquet integral. This device turns up in the computation of the Fourier expansion of Poincaré series.

Thus, let f be a function on $\Gamma_N\backslash G$, with which we generate the Poincaré series

$$P_f(\mathrm{g}) = \frac{1}{2}\sum_{\gamma\in\Gamma_N\backslash\Gamma} f(\gamma \mathrm{g}). \tag{5.1}$$

We shall ignore the convergence issue temporarily. Via the Bruhat decomposition we have

$$P_f(\mathrm{g}) = \frac{1}{2}\left(f(\mathrm{g}) + f(\mathrm{h}[i]\mathrm{g})\right) + \frac{1}{4}\sum_{c\neq 0}\sum_{\substack{d \bmod c\\ (d,c)=1}}\sum_{\omega} f(\mathrm{n}[\tilde{d}/c]\mathrm{h}[1/c]\mathrm{wn}[d/c+\omega]\mathrm{g}), \tag{5.2}$$

where $\mathrm{w} = \left[\begin{smallmatrix} & -1\\ 1 & \end{smallmatrix}\right]$; $d\tilde{d} \equiv 1 \bmod c$. The innermost sum is, by the Poisson sum formula, equal to

$$\sum_{\omega} \exp(2\pi i \mathrm{Re}\,(d\omega/c)) \int_N \psi_\omega(\mathrm{n})^{-1} f(\mathrm{n}[\tilde{d}/c]\mathrm{h}[1/c]\mathrm{wng})dn \tag{5.3}$$

with ψ_ω as in (4.5). If we suppose further that f is such that

$$f(\mathrm{ng}) = \psi_{\omega'}(\mathrm{n})f(\mathrm{g}) \tag{5.4}$$

with an $\omega' \in \mathbb{Z}[i]$, then we have

$$\begin{aligned} P_f(\mathrm{g}) &= \frac{1}{2}(f(\mathrm{g}) + f(\mathrm{h}[i]\mathrm{g})) \\ &\quad + \frac{1}{4}\sum_{\omega}\sum_{c\neq 0} S_{\mathrm{F}}(\omega,\omega';c)\int_N \psi_\omega(\mathrm{n})^{-1} f(\mathrm{h}[1/c]\mathrm{wng})dn. \end{aligned} \tag{5.5}$$

Hence we have

$$F_\omega P_f = \frac{1}{2}(\delta_{\omega,\omega'}f + \delta_{\omega,-\omega'}\ell_i f) + \frac{1}{4}\sum_{c\neq 0} S_{\mathrm{F}}(\omega,\omega';c)\mathcal{A}_\omega \ell_{1/c} f, \tag{5.6}$$

where δ is the Kronecker delta, ℓ is as in (4.11), and

$$\mathcal{A}_\xi f(\mathrm{g}) = \int_N \psi_\xi(\mathrm{n})^{-1} f(\mathrm{wng})dn \tag{5.7}$$

is the Jacquet integral. A property of $\mathcal{A}_\xi$ is

$$\ell_t \mathcal{A}_\xi \ell_t = |t|^4 \mathcal{A}_{t^2\xi} \tag{5.8}$$

for any $t \neq 0$; thus only $\mathcal{A}_0$, $\mathcal{A}_1$ matter actually. Obviously $\mathcal{A}_\xi$ commutes with any element of $\mathcal{U}(\mathfrak{g})$.

Now, let us compute $\mathcal{A}_\omega \varphi_{l,q}(\nu,p)$, which is in $W_{l,q}(\chi_{\nu,p},\omega)$. We remark that

$$\mathrm{wn}[z]\mathrm{a}[r] = \mathrm{n}\left[\frac{-\bar{z}}{r^2+|z|^2}\right]\mathrm{a}\left[\frac{r}{r^2+|z|^2}\right]\mathrm{k}\left[\frac{\bar{z}}{\sqrt{r^2+|z|^2}}, \frac{-r}{\sqrt{r^2+|z|^2}}\right]; \tag{5.9}$$

and thus

$$\begin{aligned} &\mathcal{A}_\omega \varphi_{l,q}(\nu,p;\mathrm{na}[r]\mathrm{k}[\alpha,\beta]) \\ &= \psi_\omega(\mathrm{n})r^{1-\nu}\int_{\mathbf{C}} \frac{\mathrm{e}^{-2\pi i \mathrm{Re}\,(\omega r z)}}{(1+|z|^2)^{\nu+1}}\Phi^l_{p,q}\left(\mathrm{k}\left[\frac{\bar{z}}{\sqrt{1+|z|^2}}, \frac{-1}{\sqrt{1+|z|^2}}\right]\mathrm{k}[\alpha,\beta]\right)d_+z. \end{aligned} \tag{5.10}$$

This shows that for $\mathrm{Re}\,\nu > 0$

$$\mathcal{A}_\omega \varphi_{l,q}(\nu,p) \in W^{\mathrm{pol}}_{l,q}(\chi_{\nu,p}). \tag{5.11}$$

We have, by (3.24),

$$\mathcal{A}_\omega \varphi_{l,q}(\nu,p;\mathrm{na}[r]\mathrm{k}) = \psi_\omega(\mathrm{n})\sum_{|m|\le l} v^l_m(r)\Phi^l_{m,q}(\mathrm{k}), \tag{5.12}$$

where

$$v_m^l(r) = r^{1-\nu} \int_{\mathbf{C}} \frac{e^{-2\pi i \mathrm{Re}\,(\omega r z)}}{(1+|z|^2)^{\nu+1}} \Phi_{p,m}^l \left(\mathrm{k}\left[\frac{\bar{z}}{\sqrt{1+|z|^2}}, \frac{-1}{\sqrt{1+|z|^2}} \right] \right) d_+ z. \tag{5.13}$$

The relation

$$\mathrm{k}[e^{-i\phi}\alpha, \beta] = \mathrm{h}[e^{-i\phi/2}]\mathrm{k}[\alpha,\beta]\mathrm{h}[e^{-i\phi/2}] \tag{5.14}$$

and (3.20) imply that, after the change of variables $z = ue^{i\phi}$, the last integral becomes

$$\int_0^\infty \frac{u}{(1+u^2)^{\nu+1}} \Phi_{p,m}^l \left(\mathrm{k}\left[\frac{u}{\sqrt{1+u^2}}, \frac{-1}{\sqrt{1+u^2}} \right] \right) \times \int_{-\pi}^{\pi} \exp\left((p+m)i\phi - 2\pi i \mathrm{Re}\,(\omega r u e^{i\phi}) \right) d\phi \, du. \tag{5.15}$$

Thus we see that if $\omega = 0$, then

$$v_m^l(r) = 2\pi \delta_{m,-p} r^{1-\nu} \int_0^\infty \frac{u}{(1+u^2)^{\nu+1}} \Phi_{p,-p}^l \left(\mathrm{k}\left[\frac{u}{\sqrt{1+u^2}}, \frac{-1}{\sqrt{1+u^2}} \right] \right) du; \tag{5.16}$$

and if $\omega \neq 0$, then

$$v_m^l(r) = 2\pi r^{1-\nu}(i\omega/|\omega|)^{-p-m} \times \int_0^\infty u \frac{J_{p+m}(2\pi|\omega|ru)}{(1+u^2)^{\nu+1}} \Phi_{p,m}^l \left(\mathrm{k}\left[\frac{u}{\sqrt{1+u^2}}, \frac{-1}{\sqrt{1+u^2}} \right] \right) du. \tag{5.17}$$

The integral in (5.16) is, by the definition of $\Phi_{p,-p}^l$, equal to

$$\begin{aligned} &(-1)^{l-p} \sum_{a=0}^{l-|p|} (-1)^a \binom{l+|p|}{a} \binom{l-|p|}{a} \int_0^\infty \frac{u^{2a+1}}{(1+u^2)^{\nu+l+1}} du \\ &= \frac{1}{2} \frac{\Gamma(l+|p|+1)\Gamma(\nu+|p|)}{\Gamma(\nu+l+1)\Gamma(2|p|+1)} \sum_{a=0}^{l-|p|} (-1)^a \binom{l-|p|}{a} \frac{(\nu+|p|)_a}{(2|p|+1)_a} \\ &= \frac{1}{2} \frac{\Gamma(l+1-\nu)}{\Gamma(l+1+\nu)} \frac{\Gamma(|p|+\nu)}{\Gamma(|p|+1-\nu)}. \end{aligned} \tag{5.18}$$

The last line depends on the identity

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{(\alpha)_j}{(\beta)_j} = \frac{(\beta-\alpha)_k}{(\beta)_k}, \quad (\alpha)_j = \alpha(\alpha+1)\cdots(\alpha+j-1), \tag{5.19}$$

which can be shown by induction. On the other hand, we observe that if $p+m<0$ in the integrand in (5.17) then we may replace (p, m) by $(-p, -m)$ without affecting the value of the integral, since we have (3.19) and $J_{-a} = (-1)^a J_a$ for $a \in \mathbb{Z}$. Thus, according as $\mathrm{sgn}(p+m) = \pm$, the integral is equal to

$$
\begin{aligned}
&(-1)^{l-p} \sum_{a=0}^{\min\{l\mp m, l\mp p\}} (-1)^a \binom{l\mp m}{a}\binom{l\pm m}{l\mp p - a} \\
&\qquad\times \int_0^\infty \frac{u^{|m+p|+1+2a} J_{|m+p|}(2\pi|\omega|ru)}{(1+u^2)^{\nu+l+1}}\,du \\
&= (-1)^{l-p} \sum_{a=0}^{\min\{l\mp m, l\mp p\}} \sum_{b=0}^{a} (-1)^b \binom{l\mp m}{a}\binom{l\pm m}{l\mp p - a}\binom{a}{b} \\
&\qquad\times \int_0^\infty \frac{u^{|m+p|+1} J_{|m+p|}(2\pi|\omega|ru)}{(1+u^2)^{\nu+l+1-b}}\,du.
\end{aligned}
\tag{5.20}
$$

Exchanging the order of summation, the sum over a taken inside is equal to

$$
\begin{aligned}
\xi_p^l(m,b) &= \frac{b!(2l-b)!}{(l-p)!(l+p)!} \\
&\times \binom{l - \frac{1}{2}(|m+p|+|m-p|)}{b}\binom{l-\frac{1}{2}(|m+p|-|m-p|)}{b}.
\end{aligned}
\tag{5.21}
$$

In fact it is

$$
\begin{aligned}
&\sum_{a=b}^{\min\{l\mp m, l\mp p\}} \binom{l\mp m}{a}\binom{l\pm m}{l\mp p - a}\binom{a}{b} \\
&= \frac{(l-m)!(l+m)!}{b!} \sum_{a=b}^{\min\{A,B\}} \frac{1}{(A-a)!(B-a)!(a-b)!(a+c)!}
\end{aligned}
\tag{5.22}
$$

with $A = l\mp m,\ B = l \mp p,\ c = |m+p|$; and on the assumption $A \le B$

$$
\begin{aligned}
\sum_{a=b}^{\min\{A,B\}} &= \frac{1}{(A-b)!(B-b)!(c+b)!}\sum_{d=0}^{A-b}(-1)^d\binom{A-b}{d}\frac{(b-B)_d}{(c+b+1)_d} \\
&= \frac{(A+B+c-b)!}{(A-b)!(B-b)!(A+c)!(B+c)!}
\end{aligned}
\tag{5.23}
$$

because of (5.19). Hence we get (5.21).

Collecting these and invoking the formula

$$
\int_0^\infty \frac{u^{\tau+1}J_\tau(tu)}{(1+u^2)^{\eta+1}}\,du = \frac{(t/2)^\eta}{\Gamma(\eta+1)} K_{\tau-\eta}(t) \quad (t>0),
\tag{5.24}
$$

which holds for $-1 < \mathrm{Re}\,\tau < 2\mathrm{Re}\,\eta + \frac{3}{2}$ (formula (2) on p. 434 of [35]), we obtain

Lemma 5.1. *We have that for* $\omega = 0$

$$\mathcal{A}_0\varphi_{l,q}(\nu,p) = \pi\frac{\Gamma(l+1-\nu)}{\Gamma(l+1+\nu)}\frac{\Gamma(|p|+\nu)}{\Gamma(|p|+1-\nu)}\varphi_{l,q}(-\nu,-p); \tag{5.25}$$

and for $\omega \neq 0$

$$\begin{aligned}&\mathcal{A}_\omega\varphi_{l,q}(\nu,p)(\mathrm{na}[r]\mathrm{k}) \\ &= 2(-1)^{l-p}\pi^\nu|\omega|^{\nu-1}\psi_\omega(\mathrm{n})\sum_{|m|\le l}(i\omega/|\omega|)^{-m-p}\alpha^l_m(\nu,p;|\omega|r)\Phi^l_{m,q}(\mathrm{k})\end{aligned} \tag{5.26}$$

with

$$\begin{aligned}&\alpha^l_m(\nu,p;r) \\ &= \sum_{j=0}^{l-|m+p|/2-|m-p|/2}(-1)^j\xi^l_p(m,j)\frac{(\pi r)^{l+1-j}}{\Gamma(\nu+l+1-j)}K_{\nu+l-|m+p|-j}(2\pi r).\end{aligned} \tag{5.27}$$

We see that with respect to ν the function $\mathcal{A}_0\varphi_{l,q}(\nu,p)$ is meromorphic, and for $\omega \neq 0$ the function $\mathcal{A}_\omega\varphi_{l,q}(\nu,p)$ is entire. Thus, taking into account analytic continuation, we may extend $\mathcal{A}_\omega$ so that $\mathcal{A}_\omega\varphi_{l,q}(\nu,p)$ is given by the right side members of (5.25)–(5.26), as far as they are regular. In this way we define the *Jacquet operator*:

$$\mathcal{A}_\omega:\ H(\nu,p)\to W^{\mathrm{pol}}(\chi_{\nu,p},\omega) \tag{5.28}$$

where the right side is the space spanned by all $W^{\mathrm{pol}}_{l,q}(\chi_{\nu,p},\omega)$, $|p|\le l$, $|q|\le l$. The function $\mathcal{A}_\omega\varphi_{l,q}(\nu,p)$ spans the space $W^{\mathrm{pol}}_{l,q}(\chi_{\nu,p},\omega)$, $\omega\neq 0$, for all values of $\nu\in\mathbb{C}$, $p\in\mathbb{Z}$. In particular, since the space $W^{\mathrm{pol}}_{l,q}(\chi_{-\nu,-p},\omega)$ is identical to $W^{\mathrm{pol}}_{l,q}(\chi_{\nu,p},\omega)$, the function $\mathcal{A}_\omega\varphi_{l,q}(-\nu,-p)$ is a multiple of $\mathcal{A}_\omega\varphi_{l,q}(\nu,p)$. Checking the coefficients of $\Phi^l_{l,q}(\mathrm{k})$ in these functions we find the functional equation

$$\begin{aligned}&(\pi|\omega|)^{-\nu}(i\omega/|\omega|)^p\Gamma(l+1+\nu)\mathcal{A}_\omega\varphi(\nu,p) \\ &\qquad= (\pi|\omega|)^{\nu}(i\omega/|\omega|)^{-p}\Gamma(l+1-\nu)\mathcal{A}_\omega\varphi(-\nu,-p).\end{aligned} \tag{5.29}$$

Note that the term Jacquet operator is limited to its application to the space $H(\nu,p)$, whereas we use the term Jacquet integral wherever it applies. This abuse of terminology should not cause confusion in our later discussion.

Now, the most important example of automorphic forms that are not cuspidal but of polynomial growth is offered by the Eisenstein series of K-type (l,q):

$$e_{l,q}(\nu,p;\mathrm{g}) = \frac{1}{2}\sum_{\gamma\in\Gamma_N\backslash\Gamma}\varphi_{l,q}(\nu,p)(\gamma\mathrm{g})\qquad(\mathrm{Re}\,\nu>1) \tag{5.30}$$

with $p\in 2\mathbb{Z}$. We need this condition on p to have a non-trivial sum; note that (3.20) implies $\varphi_{l,q}(\nu,p;\mathrm{h}[i]\mathrm{g}) = (-1)^p\varphi_{l,q}(\nu,p;\mathrm{g})$. The series converges absolutely in the indicated domain of ν, and is regular there, which is the same as in the K-trivial case. The Fourier expansion of $e_{l,q}(\nu,p)$ can be obtained as an application of the foregoing discussion. Obviously we have

$$\mathcal{A}_\omega\ell_{1/c}\varphi_{l,q}(\nu,p)(\mathrm{a}[r]\mathrm{k}) = (c/|c|)^{2p}|c|^{-2(\nu+1)}\mathcal{A}_\omega\varphi_{l,q}(\nu,p)(\mathrm{a}[r]\mathrm{k}). \tag{5.31}$$

Thus, by (2.18), we have

Lemma 5.2. *The Eisenstein series $e_{l,q}(\nu,p)$, $p\in 2\mathbb{Z}$, is meromorphic over $\mathbb{C}$ with respect to ν. When it is regular, we have the Fourier expansion*

$$e_{l,q}(\nu,p)=\varphi_{l,q}(\nu,p)+\pi^{2\nu}\frac{\Gamma(l+1-\nu)}{\Gamma(l+1+\nu)}\frac{\zeta_{\mathrm{F}}(1-\nu,p/2)}{\zeta_{\mathrm{F}}(1+\nu,p/2)}\varphi_{l,q}(-\nu,-p) \qquad (5.32)$$
$$+\frac{1}{\zeta_{\mathrm{F}}(1+\nu,p/2)}\sum_{\omega\neq 0}\sigma_{-\nu}(\omega,p/2)\mathcal{A}_\omega\varphi_{l,q}(\nu,p).$$

We also have the functional equation

$$\pi^{-\nu}\Gamma(l+1+\nu)\zeta_{\mathrm{F}}(1+\nu,p/2)e_{l,q}(\nu,p) \qquad (5.33)$$
$$=\pi^{\nu}\Gamma(l+1-\nu)\zeta_{\mathrm{F}}(1-\nu,p/2)e_{l,q}(-\nu,-p).$$

Proof. These assertions are consequences of the previous lemma, the identity (5.29), and the functional equation

$$\pi^{-\nu}\Gamma(|p|+\nu)\zeta_{\mathrm{F}}(\nu,p/2)=\pi^{\nu-1}\Gamma(|p|+1-\nu)\zeta_{\mathrm{F}}(1-\nu,p/2). \qquad (5.34)$$

This ends the proof. ■

Note that in the present arithmetical situation we do not need to establish Langlands' analytic continuation [20] of Eisenstein series. We stress also that the above discussion implies that each cusp-form $\psi\in {}^0A_{l,q}(\chi_{\nu,p})$ has the Fourier expansion

$$\psi=\sum_{\omega\neq 0}\mathrm{c}(\omega)\mathcal{A}_\omega\varphi_{l,q}(\nu,p) \quad \text{or} \quad F_\omega\psi=\mathrm{c}(\omega)\mathcal{A}_\omega\varphi_{l,q}(\nu,p) \qquad (5.35)$$

with certain complex numbers $\mathrm{c}(\omega)$. Because of this, instead of considering individual automorphic forms, we study systems that behave under the action of $\mathfrak{g}$ in the same way as the $\varphi_{l,q}(\nu,p)$. Thus, automorphic representations move to the focus of interest; that are linear maps from the model space $H(\nu,p)$ to $C^\infty(\Gamma\backslash G)$ that commute with the action of $\mathfrak{g}$. Specifically we have, for any $\mathbf{X}\in\mathfrak{g}$,

$$F_\omega\mathbf{X}\psi=\mathrm{c}(\omega)\mathcal{A}_\omega\mathbf{X}\varphi_{l,q}(\nu,p). \qquad (5.36)$$

This means that the result of a right differentiation applied to a cusp-form is a sum of a finite linear combination of cusp-forms, since $H(\nu,p)$ is $\mathfrak{g}$-invariant. Moreover, we see that the set of automorphic functions $\mathfrak{g}\psi$ or $\mathcal{U}(\mathfrak{g})\psi$ share the Fourier coefficients $\{\mathrm{c}(\omega)\}$ in the sense expressed by the identity (5.36). In passing, we note that in (5.35) we have

$$\mathrm{c}(-\omega)=(-1)^p\mathrm{c}(\omega). \qquad (5.37)$$

This is because $\mathrm{h}[i]\in\Gamma$ and $\ell_i\mathcal{A}_\omega=\mathcal{A}_{-\omega}\ell_i^{-1}$.

Remark. The operator $\mathcal{A}_\omega$ has been studied by Jacquet [14] for more general groups than $\mathrm{PSL}_2(\mathbb{C})$. For $\mathrm{PSL}_2(\mathbb{R})$, one obtains an expression in terms of Whittaker functions. Basic properties, like (5.34), of Hecke L-functions associated with Grössencharakters can be found in [12].

6. Goodman–Wallach operator

The Jacquet integral has given a solution to the system (4.14)–(4.15), which is at most of polynomial growth in the sense of (4.20); and it has fixed the operator $\mathcal{A}_\omega$. The formula (4.17) suggests, however, the existence of a solution of exponential growth. To construct such a solution we shall employ a method due to Goodman and Wallach [10]. We shall have a map

$$\mathcal{B}_\omega : H(\nu,p) \to W(\chi_{\nu,p},\omega), \tag{6.1}$$

where the right side is spanned by all $W_{l,q}(\chi_{\nu,p},\omega)$, $|p| \le l$, $|q| \le l$.

Thus, let $\varphi \in H(\nu,p)$ be arbitrary. We shall find an infinite vector $\{a(m,n) : m,n \ge 0\}$, which depends only on ν,p,ω, so that

$$\mathcal{B}_\omega\varphi(\mathrm{g}) = \sum_{m,n\ge 0} a(m,n)\partial_z^m\,\partial_{\bar z}^n\,\varphi(\mathrm{wn}[z]\mathrm{w}^{-1}\mathrm{g})|_{z=0} \tag{6.2}$$

satisfies

$$\mathcal{B}_\omega\varphi(\mathrm{ng}) = \psi_\omega(\mathrm{n})\mathcal{B}_\omega\varphi(\mathrm{g}), \tag{6.3}$$

or

$$\partial_t\mathcal{B}_\omega\varphi(\mathrm{n}[t]\mathrm{g})|_{t=0} = \pi i\omega\mathcal{B}_\omega\varphi(\mathrm{g}), \quad \partial_{\bar t}\mathcal{B}_\omega\varphi(\mathrm{n}[t]\mathrm{g})|_{t=0} = \pi i\bar\omega\mathcal{B}_\omega\varphi(\mathrm{g}). \tag{6.4}$$

We note that

$$\mathrm{wn}[z]\mathrm{w}^{-1}\begin{pmatrix}0 & t\\ 0 & 0\end{pmatrix}(\mathrm{wn}[z]\mathrm{w}^{-1})^{-1} = t\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} + tz\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} - tz^2\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}. \tag{6.5}$$

Considering the exponential of the right side in a vicinity of $t = 0$, we have

$$\begin{aligned}&\partial_t\varphi(\mathrm{wn}[z]\mathrm{w}^{-1}\mathrm{n}[t]\mathrm{g})|_{t=0} \\ &\quad = \partial_t\varphi(\mathrm{h}[e^{zt}]\mathrm{wn}[z]\mathrm{w}^{-1}\mathrm{g})|_{t=0} + \partial_t\varphi(\mathrm{wn}[z^2t+z]\mathrm{w}^{-1}\mathrm{g})|_{t=0} \\ &\quad = (1+\nu-p)z\varphi(\mathrm{wn}[z]\mathrm{w}^{-1}\mathrm{g}) + z^2\partial_z\varphi(\mathrm{wn}[z]\mathrm{w}^{-1}\mathrm{g}),\end{aligned} \tag{6.6}$$

since for $\xi = \xi_1 + i\xi_2$ $(\xi_1,\,\xi_2 \in \mathbb{R})$

$$\varphi(\mathrm{h}[e^\xi]\mathrm{na}[r]\mathrm{k}) = \varphi(\mathrm{a}[e^{2\xi_1}r]\mathrm{h}[e^{i\xi_2}]\mathrm{k}) = e^{2(1+\nu)\xi_1-2p\xi_2 i}\varphi(\mathrm{a}[r]\mathrm{k}). \tag{6.7}$$

The formula (6.6) gives

$$\begin{aligned} &\partial_t\, \partial_z^m\, \partial_{\bar z}^n \varphi(\mathrm{wn}[z]\mathrm{w}^{-1}\mathrm{n}[t]\mathrm{g})|_{(z,t)=(0,0)} \\ &\qquad = m(\nu - p + m)\partial_z^{m-1}\, \partial_{\bar z}^n \varphi(\mathrm{wn}[z]\mathrm{w}^{-1}\mathrm{g})|_{z=0}\,. \end{aligned} \tag{6.8}$$

In just the same way one may show that

$$\begin{aligned} &\partial_{\bar t}\, \partial_z^m\, \partial_{\bar z}^n \varphi(\mathrm{wn}[z]\mathrm{w}^{-1}\mathrm{n}[t]\mathrm{g})|_{(z,t)=(0,0)} \\ &\qquad = n(\nu + p + n)\partial_z^{m}\, \partial_{\bar z}^{n-1} \varphi(\mathrm{wn}[z]\mathrm{w}^{-1}\mathrm{g})|_{z=0}\,. \end{aligned} \tag{6.9}$$

From these and (6.4) we see that the coefficients $a(m,n)$ should satisfy the recurrence relation

$$\begin{aligned} \pi i\omega\, a(m,n) &= (m+1)(\nu - p + m + 1)a(m+1,n), \\ \pi i\bar\omega\, a(m,n) &= (n+1)(\nu + p + n + 1)a(m,n+1). \end{aligned} \tag{6.10}$$

We set the side condition

$$a(0,0) = \{\Gamma(\nu+1+p)\Gamma(\nu+1-p)\}^{-1}. \tag{6.11}$$

Then we are led to

$$a(m,n) = \frac{(\pi i\omega)^m(\pi i\bar\omega)^n}{m!n!\Gamma(\nu+1-p+m)\Gamma(\nu+1+p+n)}\,. \tag{6.12}$$

With this choice of the vector, the sum (6.2) converges absolutely for any element $\varphi \in H(\nu,p)$. Indeed, the analyticity of $z \mapsto \varphi(\mathrm{wn}[z]\mathrm{w}^{-1}g)$ provides us with a necessary bound of the derivatives. We stress that the sum is entire with respect to ν.

Obviously the operator $\mathcal{B}_\omega$ commutes with all elements of $\mathcal{U}(\mathfrak{g})$; and it maps $\varphi_{l,q}(\nu,p)$ into $W_{l,q}(\chi_{\nu,p},\omega)$. Thus there should be an expansion of $\mathcal{B}_\omega\varphi_{l,q}(\nu,p)$ in terms of $\Phi^l_{m,q}$, $|m| \le l$:

Lemma 6.1. *We have, for any* $\omega \neq 0$,

$$\begin{aligned} &\mathcal{B}_\omega\varphi_{l,q}(\nu,p)(\mathrm{na}[r]\mathrm{k}) \\ &\quad = (\pi|\omega|)^{-\nu-1}\psi_\omega(\mathrm{n}) \sum_{|m|\le l} (-i\omega/|\omega|)^{p-m}\beta^l_m(\nu,p;|\omega|r)\Phi^l_{m,q}(\mathrm{k}), \end{aligned} \tag{6.13}$$

where

$$\beta^l_m(\nu,p;r) = \sum_{j=0}^{l-|m+p|/2-|m-p|/2} \xi^l_p(m,j)\frac{(\pi r)^{l+1-j}}{\Gamma(\nu+l+1-j)} I_{\nu+l-|m+p|-j}(2\pi r). \tag{6.14}$$

We have also

$$\begin{aligned}\pi^{-2}(\pi|\omega|)^{-\nu}(-i\omega/|\omega|)^{p}\Gamma(l+1+\nu)\mathcal{A}_\omega\varphi_{l,q}(\nu,p) \\ = -\frac{(\pi|\omega|)^{\nu}}{\sin\pi\nu}(i\omega/|\omega|)^{-p}\Gamma(l+1+\nu)\mathcal{B}_\omega\varphi_{l,q}(\nu,p) \\ +\frac{(\pi|\omega|)^{-\nu}}{\sin\pi\nu}(i\omega/|\omega|)^{p}\Gamma(l+1-\nu)\mathcal{B}_\omega\varphi_{l,q}(-\nu,-p),\end{aligned} \tag{6.15}$$

which is a refinement of (5.29).

Proof. Let us suppose that $\nu \notin \mathbb{Z}$. On the right side of (5.27) we replace the K-Bessel function by its defining expression:

$$K_\xi(u) = \frac{\pi}{2\sin\pi\xi}(I_{-\xi}(u) - I_\xi(u)). \tag{6.16}$$

Then the function $\alpha^l_m(\nu,p;r)$ is a difference of two parts; one is r^ν times a power series in r, and the other $r^{-\nu}$ times another power series. Taking these into the system (4.14)–(4.15), we see that each part satisfies the system. The first part is equal to a multiple of $\beta^l_m(\nu,p;r)$, whence the right side of (6.13) belongs to $W_{l,q}(\chi_{\nu,p},\omega)$. The other part yields another member of $W_{l,q}(\chi_{\nu,p},\omega)$; and these two are linearly independent. Since we have shown $\dim W_{l,q}(\chi_{\nu,p},\omega) \le 2$ already, we find that

$$\dim W_{l,q}(\chi_{\nu,p},\omega) = 2 \tag{6.17}$$

under the present specification. On the other hand, it is easy to see that there is a power series P such that $\mathcal{B}_1\varphi_{l,q}(\nu,p;\mathrm{a}[r]) = a(0,0)r^{1+\nu}P(r)$ with $P(0)=1$. Hence $\mathcal{B}_1\varphi_{l,q}(\nu,p)$ should be a constant multiple of the right side of (6.13) with $\omega=1$. The constant is equal to 1, as can be seen by checking the term with $m=p$. Observing that

$$\ell_t\mathcal{B}_1\ell_t^{-1} = \mathcal{B}_{t^2} \tag{6.18}$$

because of $\mathrm{h}[t^{-1}]\mathrm{wn}[z]\mathrm{w}^{-1}\mathrm{h}[t] = \mathrm{wn}[t^2z]\mathrm{w}^{-1}$, we get (6.13) for general non-zero ω. As to (6.15) we note that $\mathcal{B}_\omega\varphi_{l,q}(\nu,p)$ and $\mathcal{B}_\omega\varphi_{l,q}(-\nu,-p)$ are linearly independent elements of $W^{\mathrm{pol}}_{l,q}(\chi_{\nu,p},\omega)$; and thus $\mathcal{A}_\omega\varphi_{l,q}(\nu,p)$ is a linear combination of them. Computing the coefficients of $\Phi^l_{l,q}(\mathrm{k})$ in these three elements we obtain (6.15). The case $\nu\in\mathbb{Z}$ is settled with analytic continuation, since both sides of (6.13) are entire in ν; and (6.15) is similar. This ends the proof. ∎

Now, we shall show that operators $\mathcal{A}_{\omega_1}$ and $\mathcal{B}_{\omega_2}$ are related in a way which will turn out to be important in our later discussions of the sum formula for Kloosterman sums. We observe that by (6.13) we have $\mathcal{B}_{\omega_2}\varphi_{l,q}(\nu,p;\mathrm{na}[r]\mathrm{k}) = O(r^{\mathrm{Re}\,\nu+1})$ as $r\downarrow 0$ for any $\omega_2\neq 0$. Hence the Jacquet integral $\mathcal{A}_{\omega_1}\mathcal{B}_{\omega_2}\varphi_{l,q}(\nu,p;\mathrm{na}[r]\mathrm{k})$ converges for $\mathrm{Re}\,\nu>0$:

Lemma 6.2. *Let* $\omega_2\neq 0$, $\mathrm{Re}\,\nu>0$. *Then we have that*

$$\mathcal{A}_0\mathcal{B}_{\omega_2}\varphi_{l,q}(\nu,p) = (-1)^p\frac{\sin\pi\nu}{\nu^2-p^2}\frac{\Gamma(l+1-\nu)}{\Gamma(l+1+\nu)}\varphi_{l,q}(-\nu,-p); \tag{6.19}$$

and for $\omega_1 \neq 0$

$$\begin{aligned} &\mathcal{A}_{\omega_1}\mathcal{B}_{\omega_2}\varphi_{l,q}(\nu,p) \\ &= (\pi^2|\omega_1\omega_2|)^{-\nu}(\omega_1\omega_2/|\omega_1\omega_2|)^p \mathfrak{J}_{\nu,p}(2\pi\sqrt{\omega_1\omega_2})\mathcal{A}_{\omega_1}\varphi_{l,q}(\nu,p) \end{aligned} \tag{6.20}$$

with

$$\mathfrak{J}_{\nu,p}(u) = |u/2|^{2\nu}(u/|u|)^{-2p} J^*_{\nu-p}(u) J^*_{\nu+p}(\bar{u}). \tag{6.21}$$

Here $J^*_\nu(x)$ *is the entire function of* x *which is equal to* $J_\nu(x)(x/2)^{-\nu}$ *for* $x > 0$.

Proof. Let $\varphi \in H(\nu,p)$ with $\mathrm{Re}\,\nu > 0$. We may take the integral defining $\mathcal{A}_{\omega_1}\mathcal{B}_{\omega_2}\varphi$ inside the sum for $\mathcal{B}_{\omega_2}$. The (m,n)-th term is equal to

$$\begin{aligned} &a(m,n)\int_{\mathbf{C}} \exp(-2\pi i \mathrm{Re}\,(\omega_1 z_1))\partial_z^m \partial_{\bar z}^n\, \varphi(\mathrm{wn}[z+z_1]\mathrm{g})|_{z=0}\, d_+z_1 \\ &= a(m,n)\int_{\mathbf{C}} \exp(-2\pi i \mathrm{Re}\,(\omega_1 z))\partial_z^m \partial_{\bar z}^n\, \varphi(\mathrm{wn}[z]\mathrm{g})\, d_+z \\ &= a(m,n)(\pi i\omega_1)^m(\pi i\bar\omega_1)^n \mathcal{A}_{\omega_1}\varphi(\mathrm{g}). \end{aligned} \tag{6.22}$$

This and (5.25), (6.12) readily give (6.19)–(6.20). ∎

Remark. The operator $\mathcal{B}_\omega$ is due to Goodman and Wallach [10], but for a more general context than $\mathrm{PSL}_2(\mathbb{C})$. Miatello and Wallach use it to express Fourier coefficients of Poincaré series in terms of their τ-function, which coincides with our $\mathfrak{J}_{\nu,p}$ if specialized to $\mathrm{PSL}_2(\mathbb{C})$; see Proposition 2.7 in [22]. An extension of [10] is given in [21].

7. Lebedev transform

The Lebedev or the K-Bessel transform

$$f \mapsto \int_0^\infty f(r)K_\nu(r)\frac{dr}{r} \tag{7.1}$$

plays a significant rôle in the theory of sum formulas for rational Kloosterman sums. Since the function K_ν appears in the Fourier expansion of the classical Eisenstein series over $\mathbb{H}^2$, it appears natural to anticipate that the corresponding function, i.e., $\mathcal{A}_\omega\varphi_{l,q}(\nu,p)$, in the Fourier expansion of $e_{l,q}(\nu,p)$ should work analogously in the present context. We shall show in later sections that this is indeed the case. Here we shall carry out some preparatory work. In particular we shall prove an extension of the one-sided inversion of the Lebedev transform:

$$\eta(\nu) = \frac{i}{\pi^2}\int_0^\infty K_\nu(r)r^{-1}\int_{(0)} \eta(\xi)K_\xi(r)\xi\sin(\pi\xi)d\xi\, dr, \tag{7.2}$$

where η is to satisfy an appropriate regularity and decay condition.

Thus, let $\omega \neq 0$ and put

$$P_{l,q}(N\backslash G,\omega) = \Big\{ f \in C^\infty(G) : f(\mathrm{ng}) = \psi_\omega(\mathrm{n}) f(\mathrm{g}),\ \text{of } K\text{-type } (l,q), \qquad (7.3)$$
$$f(\mathrm{a}[r]\mathrm{k}) = O(r^{1+\sigma_0}) \text{ as } r \downarrow 0, = O(r^{1-\sigma_\infty}) \text{ as } r \uparrow \infty \Big\},$$

where constants σ_0, $\sigma_\infty > 0$ may depend on each f. We define an extension $\mathcal{L}^\omega_{l,q}$ of the Lebedev transform applied to an $f \in P_{l,q}(N\backslash G,\omega)$ by

$$\mathcal{L}^\omega_{l,q} f(\nu,p) = \Gamma(l+1-\nu)\frac{(\pi|\omega|)^\nu(-i\omega/|\omega|)^{-p}}{\pi^2\|\Phi^l_{p,q}\|_K}\int_{N\backslash G} f(\mathrm{g})\overline{A_\omega \varphi_{l,q}(-\bar\nu,p)(\mathrm{g})}\, d\dot{\mathrm{g}}, \quad (7.4)$$

where $d\dot{\mathrm{g}} = r^{-3}dr\, d\mathrm{k}$ for $\mathrm{g} = \mathrm{a}[r]\mathrm{k} \in N\backslash G$. From (5.27) it follows that the integral converges absolutely for $|\mathrm{Re}\,\nu| < \sigma_0$, and that $\mathcal{L}^\omega_{l,q} f(\nu,p)$ is a regular function there. If $\mathrm{Re}\,\nu < 0$, then we have an integral representation for $A_\omega\varphi_{l,q}(-\bar\nu,p)(\mathrm{g})$, which being inserted into (7.4) yields an absolutely convergent double integral over $N\backslash G \times N$. Hence for $-\sigma_0 < \mathrm{Re}\,\nu < 0$ the last integral is equal to

$$\int_G f(\mathrm{g})\overline{\varphi_{l,q}(-\bar\nu,p;\mathrm{wg})}d\mathrm{g} = \int_{N\backslash G}\int_N f(\mathrm{wng})\overline{\varphi_{l,q}(-\bar\nu,p;\mathrm{g})}d\mathrm{n}\, d\dot{\mathrm{g}} \qquad (7.5)$$
$$= \int_{N\backslash G} A_0 f(\mathrm{g})\overline{\varphi_{l,q}(-\bar\nu,p;\mathrm{g})}d\dot{\mathrm{g}}.$$

We may write

$$A_0 f(\mathrm{na}[r]\mathrm{k}) = \sum_{|m|\le l} u_m(r)\Phi^l_{m,q}(\mathrm{k}). \qquad (7.6)$$

Then, we find that for $-\min(\sigma_0,\sigma_\infty) < \mathrm{Re}\,\nu < \sigma_0$

$$\mathcal{L}^\omega_{l,q} f(\nu,p) = \pi^{-2}(\pi|\omega|)^\nu(-i\omega/|\omega|)^{-p}\|\Phi^l_{p,q}\|_K\Gamma(l+1-\nu)\int_0^\infty u_p(r)r^{-\nu-2}dr. \quad (7.7)$$

Using this relation we shall show that there exists a one-sided inversion of $\mathcal{L}^\omega_{l,q}$:

Theorem 7.1. *Let us assume that the function* η *is defined over the set*

$$\{(\nu,p)\in\mathbb{C}\times\mathbb{Z} : |\mathrm{Re}\,\nu| \le \sigma,\ |p| \le l\} \qquad (7.8)$$

with a fixed $\sigma > 1$, *and satisfies the conditions:*

1. $\eta(\nu,p)$ *is holomorphic on a neighbourhood of the strip* $|\mathrm{Re}\,\nu| \le \sigma$,
2. $\eta(\nu,p) \ll e^{-\pi|\mathrm{Im}\,\nu|/2}(1+|\mathrm{Im}\,\nu|)^{-A}$ *for any* $A > 0$,
3. $\eta(\nu,p) = \eta(-\nu,-p)$.

We put

$$\mathcal{M}^\omega_{l,q}\eta(\mathrm{g}) = \frac{1}{2\pi^3 i}\sum_{|p|\le l}\frac{(-i\omega/|\omega|)^p}{\|\Phi^l_{p,q}\|_K}\int_{(0)}\eta(\nu,p)(\pi|\omega|)^{-\nu}\Gamma(l+1+\nu) \qquad (7.9)$$
$$\times A_\omega\varphi_{l,q}(\nu,p)(\mathrm{g})\nu^{\epsilon(p)}\sin\pi\nu\, d\nu$$

with $\epsilon(0) = 1$ *and* $\epsilon(p) = -1$ *for* $p \neq 0$. *Then* $\mathcal{M}^{\omega}_{l,q}\eta \in P_{l,q}(N\backslash G, \omega)$, *and we have*

$$\mathcal{L}^{\omega}_{l,q}\mathcal{M}^{\omega}_{l,q}\eta(\nu, p) = \frac{2}{\pi}\frac{\nu^{\epsilon(p)+1}}{p^2 - \nu^2} \prod_{1\le j\le l} (j^2 - \nu^2) \cdot \eta(\nu, p). \tag{7.10}$$

Proof. We shall consider the first assertion. To estimate $\mathcal{M}^{\omega}_{l,q}\eta(\mathrm{na}[r]\mathrm{k})$ as $r \uparrow \infty$, we note that the integral formula

$$K_\xi(u) = \frac{\Gamma(\xi + \frac{1}{2})}{2\sqrt{\pi}(u/2)^\xi} \int_0^\infty \frac{e^{iux}}{(1 + x^2)^{\xi+\frac{1}{2}}} dx, \tag{7.11}$$

which holds for $u > 0$, $\mathrm{Re}\,\xi > -\frac{1}{2}$, gives, after a multiple use of partial integration,

$$K_\xi(u) \ll e^{-\pi|\xi|/2}((1 + |\xi|)/u)^{\mathrm{Re}\,\xi+k} \tag{7.12}$$

for each fixed $k \geq 1$, uniformly for $|\mathrm{Re}\,\xi| < \frac{1}{2}k$, $u > 0$. This and (5.26)–(5.27) imply that $\mathcal{M}^{\omega}_{l,q}\eta(\mathrm{na}[r]\mathrm{k})$ is of rapid decay with respect to r as $r \uparrow \infty$. Next, to treat the case where $r \downarrow 0$ we observe that by (7.12) the contour in (7.9) can be shifted to (α) with $0 < \alpha < \sigma$. Then (3.23), (6.15), and the condition 3. give

$$\mathcal{M}^{\omega}_{l,q}\eta(\mathrm{g}) = \frac{i}{2\pi} \sum_{|p|\le l} \frac{(i\omega/|\omega|)^{-p}}{\|\Phi^l_{p,q}\|_K} \left\{ \int_{(\alpha)} + \int_{(-\alpha)} \right\} \eta(\nu, p)(\pi|\omega|)^\nu \tag{7.13}$$
$$\times\, \Gamma(l + 1 + \nu)\mathcal{B}_\omega \varphi_{l,q}(\nu, p)(\mathrm{g})\nu^{\epsilon(p)} d\nu.$$

We may shift the contour $(-\alpha)$ to (α), and have

$$\mathcal{M}^{\omega}_{l,q}\eta(\mathrm{g}) = \frac{i}{\pi} \sum_{|p|\le l} \frac{(i\omega/|\omega|)^{-p}}{\|\Phi^l_{p,q}\|_K} \int_{(\alpha)} \eta(\nu, p)(\pi|\omega|)^\nu \tag{7.14}$$
$$\times\, \Gamma(l + 1 + \nu)\mathcal{B}_\omega \varphi_{l,q}(\nu, p)(\mathrm{g})\nu^{\epsilon(p)} d\nu$$
$$+\, l! \sum_{1\le|p|\le l} \frac{(i\omega/|\omega|)^{-p}}{\|\Phi^l_{p,q}\|_K} \eta(0, p)\mathcal{B}_\omega \varphi_{l,q}(0, p)(\mathrm{g}).$$

The formulas (6.13)–(6.14) imply that as $r \downarrow 0$ the first sum on the right is $O(r^{1+\alpha})$, and also

$$l! \sum_{1\le|p|\le l} \frac{(i\omega/|\omega|)^{-p}}{\|\Phi^l_{p,q}\|_K} \eta(0, p)\mathcal{B}_\omega \varphi_{l,q}(0, p)(\mathrm{g}) \tag{7.15}$$
$$= b(\eta)\psi_\omega(\mathrm{n})r^2\Phi^l_{0,q}(\mathrm{k}) + O(r^3)$$
$$= b(\eta)\mathcal{B}_\omega\varphi_{l,q}(1, 0)(\mathrm{g}) + O(r^3),$$

where $l \geq 1$, and $b(\eta) = -\pi l \cdot l!\eta(0, 1)|\omega|\|\Phi^l_{1,q}\|_K^{-1}$. Collecting these we have indeed $\mathcal{M}^{\omega}_{l,q}\eta \in P_{l,q}(N\backslash G, \omega)$. We note that the last line in (7.15) is to play a rôle in Section 9.

Consequently we may use (7.7) in computing $\mathcal{L}^{\omega}_{l,q}\mathcal{M}^{\omega}_{l,q}\eta(\nu,p)$. We thus apply $\mathcal{A}_0$ to the identity (7.14). It is easy to check the absolute convergence that is necessary to exchange the order of integration, and by (6.19) we have

$$\mathcal{A}_0\mathcal{M}^{\omega}_{l,q}\eta(\mathrm{g}) = \frac{i}{\pi}\sum_{|p|\le l}\frac{(-i\omega/|\omega|)^p\Phi^l_{p,q}(\mathrm{k})}{\|\Phi^l_{p,q}\|_K} \tag{7.16}$$
$$\times\int_{(\alpha)}\eta(-\nu,p)(\pi|\omega|)^{\nu}\Gamma(l+1-\nu)\frac{\nu^{\epsilon(p)}\sin\pi\nu}{\nu^2-p^2}r^{1-\nu}d\nu.$$

This and (7.7) yield, via the Mellin inversion,

$$\mathcal{L}^{\omega}_{l,q}\mathcal{M}^{\omega}_{l,q}\eta(\nu,p) = -2\pi^{-2}\Gamma(l+1+\nu)\Gamma(l+1-\nu)\frac{\nu^{\epsilon(p)}\sin\pi\nu}{\nu^2-p^2}\eta(\nu,p), \tag{7.17}$$

which ends the proof. ■

The above discussion implies in particular that $\mathcal{M}^{\omega}_{l,q}\eta\in L^2(N\backslash G)$. Related to this we shall show a Parseval property of the transform $\mathcal{M}^{\omega}_{l,q}$:

Lemma 7.1. *Let η and θ satisfy the three conditions in the last theorem. Then we have*

$$\int_{N\backslash G}\mathcal{M}^{\omega}_{l,q}\eta(\mathrm{g})\overline{\mathcal{M}^{\omega}_{l,q}\theta(\mathrm{g})}\,d\dot{\mathrm{g}} \tag{7.18}$$
$$=\frac{1}{\pi^2 i}\sum_{|p|\le l}\int_{(0)}\eta(\nu,p)\overline{\theta(\nu,p)}\frac{\nu^{2\epsilon(p)+1}\sin\pi\nu}{p^2-\nu^2}\prod_{j=1}^{l}(j^2-\nu^2)\,d\nu.$$

Proof. We replace $\mathcal{M}^{\omega}_{l,q}\theta(\mathrm{g})$ by its defining expression. The resulting double integral over $N\backslash G\times i\mathbb{R}$ is easily seen to be absolutely convergent; and consequently we have

$$\int_{N\backslash G}\mathcal{M}^{\omega}_{l,q}\eta(\mathrm{g})\overline{\mathcal{M}^{\omega}_{l,q}\theta(\mathrm{g})}\,d\dot{\mathrm{g}}$$
$$=\frac{1}{2\pi i}\sum_{|p|\le l}\int_{(0)}\mathcal{L}^{\omega}_{l,q}\mathcal{M}^{\omega}_{l,q}\eta(\nu,p)\overline{\theta(\nu,p)}\nu^{\epsilon(p)}\sin\pi\nu\,d\nu, \tag{7.19}$$

which gives the assertion. ■

Further, we shall show

Lemma 7.2. *For any non-zero ω_1, ω_2, τ we define the map*

$$\kappa(\omega_1,\omega_2,\tau):\ \eta\mapsto\mathcal{K}_{\nu,p}(2\pi\tau\sqrt{\omega_1\omega_2})\eta, \tag{7.20}$$

where

$$\mathcal{K}_{\nu,p}(\xi)=\frac{1}{\sin\pi\nu}\{\mathcal{J}_{-\nu,-p}(\xi)-\mathcal{J}_{\nu,p}(\xi)\} \tag{7.21}$$

with $\mathcal{J}_{\nu,p}$ defined in (6.21). *Then we have, for η as in the last theorem,*

$$A_{\omega_1}\ell_\tau\mathcal{M}^{\omega_2}_{l,q}\eta(\nu,p)=|\pi\tau|^2\mathcal{M}^{\omega_1}_{l,q}\kappa(\omega_1,\omega_2,\tau)\eta(\nu,p). \tag{7.22}$$

Proof. It is trivial that $\kappa(\omega_1,\omega_2,\tau)\eta$ satisfy the three conditions in the last theorem. Hence the right side of (7.22) is well-defined. To transform the left side we use (7.14). Formally we have

$$\begin{aligned}A_{\omega_1}\ell_\tau\mathcal{M}^{\omega_2}_{l,q}\eta(\nu,p)(g)=&\frac{i}{\pi}|\tau|^2\sum_{|p|\le l}\frac{(i\tau^2\omega_2/|\tau^2\omega_2|)^{-p}}{\|\Phi^l_{p,q}\|_K}\int_{(\alpha)}\eta(\nu,p)(\pi|\tau^2\omega_2|)^{\nu} \qquad (7.23)\\ &\times\Gamma(l+1+\nu)A_{\omega_1}\mathcal{B}_{\tau^2\omega_2}\varphi_{l,q}(\nu,p)(g)\nu^{\epsilon(p)}d\nu\\ &+l!|\tau|^2\sum_{1\le|p|\le l}\frac{(i\tau^2\omega_2/|\tau^2\omega_2|)^{-p}}{\|\Phi^l_{p,q}\|_K}\eta(0,p)A_{\omega_1}\mathcal{B}_{\tau^2\omega_2}\varphi_{l,q}(0,p)(g),\end{aligned}$$

where we have used (6.18). To verify the exchange of the order of integrals implicit in (7.23) we need only to invoke (7.15); note that it also allows us to use (6.20) even for $A_{\omega_1}\mathcal{B}_{\tau^2\omega_2}\varphi_{l,q}(0,p)$, $p\neq 0$. Thus we have, by (6.20),

$$\begin{aligned}A_{\omega_1}\ell_\tau\mathcal{M}^{\omega_2}_{l,q}\eta(\nu,p)(g)=&\frac{i}{\pi}|\tau|^2\sum_{|p|\le l}\frac{(-i\omega_1/|\omega_1|)^{p}}{\|\Phi^l_{p,q}\|_K}\int_{(\alpha)}\eta(\nu,p)(\pi|\omega_1|)^{-\nu} \qquad (7.24)\\ &\times\Gamma(l+1+\nu)\mathcal{J}_{\nu,p}(2\pi\tau\sqrt{\omega_1\omega_2})A_{\omega_1}\varphi_{l,q}(\nu,p)(g)\nu^{\epsilon(p)}d\nu\\ &+l!|\tau|^2\sum_{1\le|p|\le l}\frac{(-i\omega_1/|\omega_1|)^{p}}{\|\Phi^l_{p,q}\|_K}\eta(0,p)\mathcal{J}_{0,p}(2\pi\tau\sqrt{\omega_1\omega_2})A_{\omega_1}\varphi_{l,q}(0,p)(g).\end{aligned}$$

We shift the contour (α) of one half of the last integral to $(-\alpha)$; then the last sum disappears. To the integrand over $(-\alpha)$ we apply the functional equation (5.29). After a rearrangement we get (7.22). ■

Remark. This section is a detailed work-out of the last chapter of [34] in the case of $\mathrm{PSL}_2(\mathbb{C})$. It is in fact the harmonic analysis of the space of N-equivariant functions. The $\mathcal{L}^{\omega}_{l,q}$ could be called a Whittaker transform, but we regard it rather as an extension of the Lebedev transform, paying respect to its origin. For (7.1)–(7.2) see Section 2.6 of [25]. For an interpretation of $\mathcal{K}_{\nu,p}$ see Section 15.

8. The space $L^2(\Gamma\backslash G)$

In the next section we shall treat inner-products of certain Poincaré series, especially their spectral decompositions. Here we shall briefly develop the relevant spectral theory of the space $L^2(\Gamma\backslash G)$ composed of all left Γ-automorphic functions on G which are square integrable over $\Gamma\backslash G$ with respect to the measure induced by dg. To this end we shall employ the unitary representation of G realized over $L^2(\Gamma\backslash G)$ via right translations by elements of G. We shall see that automorphic forms on G, especially the basis elements for the Parseval formula over $L^2(\Gamma\backslash G)$, do not occur singly but are parametrized through maps of the model space $H(\nu,p)$ and live in right-irreducible subspaces of $L^2(\Gamma\backslash G)$ sharing Fourier coefficients, as is indicated at the end of Section 5. This point of view will be essential in describing the sum formula for S_{F}.

We first observe that the constant function and all cusp-forms over G belong to $L^2(\Gamma\backslash G)$, because of (3.8)–(3.10) and (4.25). We have

$$L^2(\Gamma\backslash G) = \mathbb{C} \oplus {}^0L^2(\Gamma\backslash G) \oplus {}^eL^2(\Gamma\backslash G). \tag{8.1}$$

Here ${}^0L^2(\Gamma\backslash G)$ is the subspace spanned by all cusp-forms, and called the cuspidal subspace; the subspace ${}^eL^2(\Gamma\backslash G)$ is the orthogonal complement. The space ${}^0L^2(\Gamma\backslash G)$ is G-invariant, and we have a decomposition

$${}^0L^2(\Gamma\backslash G) = \overline{\bigoplus V} \tag{8.2}$$

into countably many subspaces V irreducible with respect to the action of G. Each V has a dense subspace that is a common eigenspace of the Casimir elements, and Lemma 4.1 implies that we should have

$$\Omega_\pm|_V = \chi_{\nu_V,p_V}(\Omega_\pm)\cdot 1. \tag{8.3}$$

It is known that for the group Γ all V are of unitary principal series type, and we can suppose that

$$\nu_V \in i[0,\infty). \tag{8.4}$$

Moreover, there exists a linear isomorphism

$$T_V : H(\nu_V,p_V) \to V, \tag{8.5}$$

which has a dense image and commutes with the action of $\mathfrak{g}$. This has the following immediate consequences: We have the decomposition

$$V = \overline{\bigoplus_{|p_V|\le l,\,|q|\le l} V_{l,q}}, \quad V_{l,q} = \mathbb{C}\,T_V\varphi_{l,q}(\nu_V,p_V), \tag{8.6}$$

where $V_{l,q}$ is the subspace spanned by all cusp-forms of K-type (l,q) in V; that is, $\dim V_{l,q} = 1$. Besides, the unitary structure of $H(\nu_V,p_V)$ mentioned at the end of Section 3 is transferred by T_V into (8.6), and we have

$$\|T_V\varphi_{l,q}(\nu_V,p_V)\|_{\Gamma\backslash G} = \|\Phi^l_{p_V,q}\|_K \tag{8.7}$$

with the norm $\|\cdot\|_{\Gamma\backslash G}$ corresponding to (8.9). By (5.35) we have the Fourier expansion

$$T_V\varphi_{l,q}(\nu_V,p_V) = \sum_{\omega\ne 0} c_V(\omega)\mathcal{A}_\omega\varphi_{l,q}(\nu_V,p_V). \tag{8.8}$$

The Fourier coefficients $c_V(\omega)$ depends only on V and ω. This is because both T_V and A_ω commute with the action of $\mathfrak{g}$. The vector $\{c_V(\omega)\}$ is fixed by V up-to an arbitrary multiplier of unit absolute value.

We now restrict the decomposition (8.1) to the subspace $L^2(\Gamma\backslash G)_{l,q}$ spanned by all square-integrable left Γ-automorphic functions of K-type (l,q). Then the cuspidal part is well described by the above assertions. What remains is the non-cuspidal part, and it is rendered in terms of Eisenstein series of K-type (l,q), as is embodied in the fundamental

Theorem 8.1. *Let $f_1, f_2 \in L^2(\Gamma\backslash G)_{l,q}$, and denote their inner-product by*

$$\langle f_1, f_2\rangle_{\Gamma\backslash G} = \int_{\Gamma\backslash G} f_1(\mathrm{g})\overline{f_2(\mathrm{g})}d\mathrm{g}. \tag{8.9}$$

Then we have the Parseval identity

$$\begin{aligned}\langle f_1, f_2\rangle_{\Gamma\backslash G} =& \delta_{l,0}\frac{\pi^2}{2\zeta_{\mathrm{F}}(2)}\langle f_1, 1\rangle_{\Gamma\backslash G}\langle 1, f_2\rangle_{\Gamma\backslash G} \\ &+\sum_V \frac{1}{\|\Phi^l_{p_V,q}\|^2_K}\langle f_1, T_V\varphi_{l,q}(\nu_V,p_V)\rangle_{\Gamma\backslash G}\langle T_V\varphi_{l,q}(\nu_V,p_V), f_2\rangle_{\Gamma\backslash G} \\ &+\sum_{\substack{|p|\le l\\ p\in 2\mathbf{Z}}} \frac{1}{2\pi i\|\Phi^l_{p,q}\|^2_K}\int_{(0)} E_{l,q}[\nu,p;f_1]\overline{E_{l,q}[\nu,p;f_2]}d\nu,\end{aligned} \tag{8.10}$$

where the convergence is absolute throughout. Here V runs over a complete orthogonal system of right-irreducible cuspidal subspaces of $L^2(\Gamma\backslash G)$ that intersect the space $L^2(\Gamma\backslash G)_{l,q}$ non-trivially. Also

$$E_{l,q}[\nu,p;f_j] = \int_{\Gamma\backslash G} f_j(\mathrm{g})\overline{e_{l,q}(-\bar\nu,p;\mathrm{g})}d\mathrm{g} \tag{8.11}$$

in the sense of norm convergence.

Proof. This is a special case of a general result due to Langlands [20] (see also [11]). We stress, however, that our particular assertion could be established in a direct way. ∎

Next, we shall take into consideration the action of Hecke operators: We define the Hecke operator labeled with $n \in \mathbb{Z}[i]$ by

$$\mathcal{T}_n : \psi(\mathrm{g}) \mapsto \frac{1}{4|n|}\sum_{d|n}\sum_{b \bmod d}\psi(\mathrm{n}[b/d]\mathrm{h}[\sqrt{n}/d]\mathrm{g}), \tag{8.12}$$

where ψ is to be left Γ-automorphic; the choice of the square root is irrelevant. If $\psi \in {}^0\!A_{l,q}(\chi_{\nu,p})$, then we have, from (5.8) and (5.35),

$$\mathcal{T}_n\psi = \frac{(n/|n|)^p}{4|n|^\nu}\sum_{\omega\ne0}c(\omega)\sum_{d|(\omega,n)}|d|^{2\nu}(d/|d|)^{-2p}\mathcal{A}_{\omega n/d^2}\varphi_{l,q}(\nu,p). \tag{8.13}$$

The commutativity of the algebra $\{\mathcal{T}_n\}$ and the metric property of each $\mathcal{T}_n$ in $L^2(\Gamma\backslash G)$ are analogous to the rational case. Since the right side of (8.12) commutes with the right translation by elements of G, we may assume that every V in (8.2) is an eigenspace of $\mathcal{T}_n$ for all n with the eigenvalue $t_V(n) \in \mathbb{R}$. The Weil bound

$$S_{\mathrm{F}}(\omega_1, \omega_2; c) \ll |(\omega_1, \omega_2, c)||c|\sigma_0(c, 0) \tag{8.14}$$

yields the estimate

$$t_V(n) \ll |n|^{1/2+\varepsilon} \tag{8.15}$$

for any fixed $\varepsilon > 0$, where the implicit constant depends only on ε; see Corollary 10.1 below. The equation (8.13) implies that in (8.8)

$$t_V(n)c_V(\omega) = \frac{1}{4}|n|^{\nu_V}(n/|n|)^{-p_V} \sum_{d|(\omega,n)} |d|^{-2\nu_V}(d/|d|)^{2p_V} c_V(\omega n/d^2). \tag{8.16}$$

In particular we have, for all $n \in \mathbb{Z}[i]$,

$$c_V(n) = c_V(1)|n|^{-\nu_V}(n/|n|)^{p_V} t_V(n), \tag{8.17}$$

where we have used (5.37). This and (8.16) give

$$t_V(m)t_V(n) = \frac{1}{4} \sum_{d|(m,n)} t_V(mn/d^2). \tag{8.18}$$

We have

$$t_V(1) = 1, \quad t_V(-n) = t_V(n), \quad t_V(in) = \epsilon_V t_V(n) \tag{8.19}$$

with $\epsilon_V = \pm 1$.

Further, let

$$H_V(s) = \frac{1}{4} \sum_{n \neq 0} t_V(n)|n|^{-2s} \tag{8.20}$$

be the Hecke series associated with the irreducible subspace V under the convention (8.17); note that when $\epsilon_V = -1$ this vanishes identically. The identity (8.18) implies that in the region of absolute convergence

$$H_V(s_1)H_V(s_2) = \frac{1}{4}\zeta_{\mathrm{F}}(s_1 + s_2) \sum_{n \neq 0} \sigma_{s_1-s_2}(n) t_V(n)|n|^{-2s_1}. \tag{8.21}$$

Properties of $H_V(s)$ as a function of s can be read from

Lemma 8.1. *Let $b \in \mathbb{Z}$, and put*

$$H_V(s, b) = \frac{1}{4} \sum_{n \neq 0} t_V(n)(n/|n|)^b |n|^{-2s}. \tag{8.22}$$

Take $b \in 2\mathbb{Z}$ and $i^b = \epsilon_V$ to have a non-trivial sum. Then $H_V(s,b)$ is entire in s and satisfies the functional equation

$$\begin{aligned}&\pi^{1-2s}\Gamma(s+\tfrac{1}{2}(|p_V+b|+\nu_V))\Gamma(s+\tfrac{1}{2}(|p_V-b|-\nu_V))H_V(s,b) \\ &= (-1)^{\max(|b|,|p_V|)}\pi^{2s-1}\Gamma(1-s+\tfrac{1}{2}(|p_V-b|+\nu_V)) \\ &\qquad \times \Gamma(1-s+\tfrac{1}{2}(|p_V+b|-\nu_V))H_V(1-s,-b).\end{aligned} \tag{8.23}$$

Proof. We consider the integral

$$\begin{aligned}&Y_V(s,b;\mathrm{k}) \\ &= \int_{\mathbf{C}^\times} |u|^{4s-2}(u/|u|)^{-2b}T_V\varphi(\mathrm{h}[u]\mathrm{k})d^\times u \\ &= \int_{|u|\ge 1} \Big(|u|^{4s-2}(u/|u|)^{-2b}T_V\varphi(\mathrm{h}[u]\mathrm{k}) + |u|^{2-4s}(u/|u|)^{2b}T_V\varphi(\mathrm{h}[u]\mathrm{wk})\Big)d^\times u,\end{aligned} \tag{8.24}$$

where $\varphi = \varphi_{l,l}(\nu_V, p_V)$ with $l = \max(|b|, |p_V|)$, and w is as in (5.2). Obviously $Y_V(s,b;\mathrm{k})$ is entire in s, and $Y_V(1-s,-b;\mathrm{wk}) = Y_V(s,b;\mathrm{k})$. In view of (5.8), (8.8), and (8.17), we have, for $\mathrm{Re}\, s$ sufficiently large,

$$Y_V(s,b;\mathrm{k}) = 4c_V(1)H_V(s,b)\int_{\mathbf{C}^\times} |u|^{4s-2}(u/|u|)^{-2b}\mathcal{A}_1\varphi(\mathrm{h}[u]\mathrm{k})d^\times u. \tag{8.25}$$

By (5.26)–(5.27) this integral is equal to

$$\begin{aligned}&2(-1)^{l-p_V}i^{b-p_V}\pi^{1+\nu_V}\Phi^l_{-b,l}(\mathrm{k})\int_0^\infty r^{2s-2}\alpha^l_{-b}(\nu_V,p_V;r)dr \\ &= 2\frac{(-1)^{l-p_V}i^{b-p_V}\pi^{l+2+\nu_V}}{\Gamma(l+1+\nu)}\xi^l_{p_V}(-b,0)\Phi^l_{-b,l}(\mathrm{k})\int_0^\infty r^{2s+l-1}K_{l+\nu-|p_V-b|}(2\pi r)dr.\end{aligned} \tag{8.26}$$

On noting that $\Phi^l_{b,l}(\mathrm{wk}) = (-1)^{l+b}\Phi^l_{-b,l}(\mathrm{k})$, we obtain (8.23).

Remark. For the spectral theory of automorphic forms on semisimple Lie groups see, e.g., [11]. The assertion (8.4) depends on the absence of exceptional eigenvalues for the non-Euclidean Laplacian over $\Gamma\backslash\mathbb{H}^3$ (see Proposition 6.2 in Chapter 7 of [9]). That suffices, as the complementary series occurs only for $p = 0$. The bound (8.14) is a special case of Theorem 10 of [2], which applies to all number fields. It should be stressed that for our purpose it is enough to have any non-trivial exponent in place of $1+\varepsilon$, which is best possible. In the proof of the last lemma we followed [15]; see the proof of Theorem 6.4 there.

As to (8.2) it may be worth mentioning the following *multiplicity one* result: For given $\pm(\nu,p) \in i\mathbb{R} \times \mathbb{Z}$ and $\{t(n) \in \mathbb{C} : n \in \mathbb{Z}[i], n \neq 0\}$, there is at most one irreducible subspace V of ${}^0L^2(\Gamma\backslash G)$ with $(\nu_V, p_V) = \pm(\nu,p)$, and $t_V(n) = t(n)$ for all n. Indeed, the Fourier expansion (5.35) and (8.17) show that an automorphic form of a given K-type is determined by (ν,p) and the $t(n)$ up to a scalar factor. This shows that the decomposition (8.2) is unique if we impose the condition that the spaces V are invariant under all Hecke operators.

9. Preliminary sum formula

We now enter into the discussion of the sum formula for Kloosterman sums S_{F}. The formula will be derived via spectral and geometric computations of an inner-product of two particular Poincaré series. This is analogous to the rational case. However, the choice of these series gives rise to a discussion. A possible way to take is to use an explicit function as a seed to generate the Poincaré series, which extends Selberg's argument for the rational case. This works well if we restrict ourselves to the K-trivial case, but it does not seem to extend easily to the K-non-trivial situation. On the other hand, a method that Miatello–Wallach [22], [23] developed for a far more general situation offers us a flexible way to choose the seed function. Here we shall follow their argument, adopting it to our present specifications.

Thus we shall employ $\mathcal{M}_{l,q}^{\omega}\eta$ to generate a Poincaré series, where η is to satisfy the three conditions given in Theorem 7.1. We notice immediately that (7.15) causes, in general, a convergence problem. This reminds us a similar situation that Hecke encountered in his investigation of holomorphic modular forms of weight 2. He used analytic continuation to overcome the difficulty. In much the same spirit we shall consider, in view of the last line of (7.15), the sum

$$[\mathcal{B}_\omega]\varphi_{l,q}(\nu,p)(\mathrm{g}) = \frac{1}{2}\sum_{\gamma\in\Gamma_N\backslash\Gamma} \mathcal{B}_\omega\varphi_{l,q}(\nu,p)(\gamma\mathrm{g}) \tag{9.1}$$

with non-zero $\omega\in\mathbb{Z}[i]$, though we actually need only the case $p=0$. By (5.30) and (6.13) we see that the sum converges absolutely, for $\mathrm{Re}\,\nu>1$, to a left Γ-automorphic function of K-type (l,q) with character $\chi_{\nu,p}$. The combination of (5.5)–(5.7) and (6.18)–(6.21) yields that

$$\begin{aligned}[\mathcal{B}_\omega]\varphi_{l,q}(\nu,p)(\mathrm{g}) &= \frac{1}{2}(\mathcal{B}_\omega+(-1)^p\mathcal{B}_{-\omega})\varphi_{l,q}(\nu,p)(\mathrm{g}) \\ &\quad + (-1)^p\frac{\sin\pi\nu}{\nu^2-p^2}\frac{\Gamma(l+1-\nu)}{\Gamma(l+1+\nu)}\frac{\sigma_{-\nu}(\omega,p/2)}{\zeta_{\mathrm{F}}(1+\nu,p/2)}\varphi_{l,q}(-\nu,-p)(\mathrm{g}) \\ &\quad + \sum_{\omega'\neq0}\mathcal{J}_{\nu,p}(\omega,\omega')\mathcal{A}_{\omega'}\varphi_{l,q}(\nu,p)(\mathrm{g}),\end{aligned} \tag{9.2}$$

where the second line appears only when $p\in2\mathbb{Z}$; and

$$\mathcal{J}_{\nu,p}(\omega,\omega') = \frac{1}{4\pi^2|\omega\omega'|^\nu}\left(\frac{\omega\omega'}{|\omega\omega'|}\right)^p\sum_{c\neq0}\frac{1}{|c|^2}S_{\mathrm{F}}(\omega,\omega';c)\mathfrak{J}_{\nu,p}\left(\frac{2\pi}{c}\sqrt{\omega\omega'}\right). \tag{9.3}$$

The bound (8.14) implies that for $\mathrm{Re}\,\nu>\frac{1}{2}$ the function $\mathcal{J}_{\nu,p}(\omega,\omega')$ is regular and of polynomial order in ω, ω'. Thus $[\mathcal{B}_\omega]\varphi_{l,q}(\nu,p)(\mathrm{g})$ is regular for $\mathrm{Re}\,\nu>\frac{1}{2}$, and analytically continues to a left Γ-automorphic function of K-type (l, q) with character $\chi_{\nu,p}$. It is, however, of exponential growth with respect to r, $\mathrm{g} = \mathrm{na}[r]\mathrm{k}$; and thus it does not belong to $L^2(\Gamma\backslash G)$. We then appeal to a common practice: we attach a factor $\rho(\gamma\mathrm{g})$ to each summand of (9.1). Here $\rho(\mathrm{g}) = \rho(r)$ with an abuse of notation, and $\rho(r)$ is smooth, being equal to 1 for $r \le r_0$ and to 0 for $r > r_0 + 1$ with $r_0 \ge 2$. If $r > 1$ then this affects actually only two terms in (9.1), which correspond to the cosets represented by $\gamma = 1$ and $\mathrm{h}[i]$. We thus have, instead of (9.1)–(9.2), that

$$[\rho\mathcal{B}_\omega]\varphi_{l,q}(\nu,p)(\mathrm{g}) = \frac{1}{2}\sum_{\gamma\in\Gamma_N\backslash\Gamma} \rho(\gamma\mathrm{g})\mathcal{B}_\omega\varphi_{l,q}(\nu,p)(\gamma\mathrm{g}) \tag{9.4}$$

for $\mathrm{Re}\,\nu > 1$, and that if $r > 1$, $\mathrm{Re}\,\nu > \frac{1}{2}$,

$$\begin{aligned}[\rho\mathcal{B}_\omega]\varphi_{l,q}(\nu,p)(\mathrm{g}) &= [\mathcal{B}_\omega]\varphi_{l,q}(\nu,p)(\mathrm{g}) \\ &\quad + \frac{1}{2}(\rho(r)-1)(\mathcal{B}_\omega + (-1)^p\mathcal{B}_{-\omega})\varphi_{l,q}(\nu,p)(\mathrm{g}).\end{aligned} \tag{9.5}$$

Note that as $r \uparrow \infty$

$$[\rho\mathcal{B}_\omega]\varphi_{l,q}(\nu,p)(\mathrm{g}) \ll r^{1-\mathrm{Re}\,\nu} \tag{9.6}$$

uniformly for $\mathrm{Re}\,\nu > \frac{1}{2}$.

Returning to (7.14)–(7.15) we define $\mathcal{M}_{l,q}^{\omega,*}\eta$ by

$$\mathcal{M}_{l,q}^{\omega}\eta = \mathcal{M}_{l,q}^{\omega,*}\eta + b(\eta)\rho\mathcal{B}_\omega\varphi_{l,q}(1,0), \tag{9.7}$$

and $[\mathcal{M}_{l,q}^{\omega}]\eta$ by

$$[\mathcal{M}_{l,q}^{\omega}]\eta = [\mathcal{M}_{l,q}^{\omega,*}]\eta + b(\eta)[\rho\mathcal{B}_\omega]\varphi_{l,q}(1,0), \tag{9.8}$$

where

$$[\mathcal{M}_{l,q}^{\omega,*}]\eta = \frac{1}{2}\sum_{\gamma\in\Gamma_N\backslash\Gamma} \mathcal{M}_{l,q}^{\omega,*}\eta(\gamma\mathrm{g}). \tag{9.9}$$

By the construction we have that $\mathcal{M}_{l,q}^{\omega,*}\eta(\mathrm{g}) \ll r^{2+\varepsilon}$ with an $\varepsilon > 0$ as $r \downarrow 0$, and $\ll r^{-A}$ for any $A > 0$ as $r \uparrow \infty$. Thus $[\mathcal{M}_{l,q}^{\omega,*}]\eta(\mathrm{g}) \ll r^{-\varepsilon}$ as $r \uparrow \infty$. This and (9.6) give

$$[\mathcal{M}_{l,q}^{\omega}]\eta(\mathrm{g}) \ll 1. \tag{9.10}$$

Now, let η, θ satisfy the three conditions given in Theorem 7.1, and let us consider the inner-product $\langle[\mathcal{M}_{l,q}^{\omega_1}]\eta, [\mathcal{M}_{l,q}^{\omega_2}]\theta\rangle_{\Gamma\backslash G}$ with $\omega_1, \omega_2 \in \mathbb{Z}[i]$, $\omega_1\omega_2 \ne 0$. We are going to apply Theorem 8.1 to it. To this end we note first that the above discussion implies

$$\langle[\mathcal{M}_{l,q}^{\omega}]\eta, f\rangle_{\Gamma\backslash G} = \langle[\mathcal{M}_{l,q}^{\omega,*}]\eta, f\rangle_{\Gamma\backslash G} + b(\eta)\lim_{\nu\to 1+0}\langle[\rho\mathcal{B}_\omega]\varphi_{l,q}(\nu,0), f\rangle_{\Gamma\backslash G} \tag{9.11}$$

for any left Γ-automorphic f which is integrable over $\Gamma\backslash G$. In this we have, by the unfolding argument,

$$\langle [\mathcal{M}_{l,q}^{\omega,*}]\eta, f\rangle_{\Gamma\backslash G} = \frac{1}{2}\int_{N\backslash G} \mathcal{M}_{l,q}^{\omega,*}\eta(\mathrm{g})\overline{F_\omega f(\mathrm{g})}d\dot{\mathrm{g}}, \tag{9.12}$$

$$\langle [\rho\mathcal{B}_\omega]\varphi_{l,q}(\nu,0), f\rangle_{\Gamma\backslash G} = \frac{1}{2}\int_{N\backslash G} \rho(\mathrm{g})\mathcal{B}_\omega\varphi_{l,q}(\nu,0)(\mathrm{g})\overline{F_\omega f(\mathrm{g})}d\dot{\mathrm{g}}.$$

Thus, assuming that

$$\lim_{\nu\to 1+0}\int_{N\backslash G} \rho(\mathrm{g})\mathcal{B}_\omega\varphi_{l,q}(\nu,0)(\mathrm{g})\overline{F_\omega f(\mathrm{g})}d\dot{\mathrm{g}} = \int_{N\backslash G} \rho(\mathrm{g})\mathcal{B}_\omega\varphi_{l,q}(1,0)(\mathrm{g})\overline{F_\omega f(\mathrm{g})}d\dot{\mathrm{g}}, \tag{9.13}$$

we have

$$\langle [\mathcal{M}_{l,q}^{\omega}]\eta, f\rangle_{\Gamma\backslash G} = \frac{1}{2}\int_{N\backslash G} \mathcal{M}_{l,q}^{\omega}\eta(\mathrm{g})\overline{F_\omega f(\mathrm{g})}d\dot{\mathrm{g}}. \tag{9.14}$$

The functions $T_V\varphi_{l,q}(\nu_V,p_V)$ and $e_{l,q}(\nu,p)$ with $\mathrm{Re}\,\nu = 0$ are integrable over $\Gamma\backslash G$ and satisfy (9.13). Hence we have, on noting (7.4), (8.8), and (8.17),

$$\begin{aligned}&\langle [\mathcal{M}_{l,q}^{\omega}]\eta, T_V\varphi_{l,q}(\nu_V,p_V)\rangle_{\Gamma\backslash G} \\ &= \frac{1}{2}(-i)^{p_V}\pi^{2-\nu_V}\overline{c_V(1)}t_V(\omega)\frac{\|\Phi^l_{p_V,q}\|_K}{\Gamma(l+1-\nu_V)}\mathcal{L}^\omega_{l,q}\mathcal{M}^\omega_{l,q}\eta(\nu_V,p_V) \\ &= (-i)^{p_V}\pi^{-\nu_V}\overline{c_V(1)}t_V(\omega)\|\Phi^l_{p_V,q}\|_K\Gamma(l+1+\nu_V)\frac{\nu_V^{\epsilon(p_V)}\sin\pi\nu_V}{p_V^2-\nu_V^2}\eta(\nu_V,p_V)\end{aligned} \tag{9.15}$$

by virtue of Theorem 7.1 or rather (7.17). Similarly we have, from (5.32),

$$\begin{aligned}E_{l,q}[\nu,p;[\mathcal{M}^\omega_{l,q}]\eta] &= (-1)^{p/2}\frac{(\pi|\omega|)^{-\nu}(\omega/|\omega|)^p\sigma_\nu(\omega,-p/2)}{\zeta_{\mathrm{F}}(1-\nu,-p/2)} \\ &\quad\times\|\Phi^l_{p,q}\|_K\Gamma(l+1+\nu)\frac{\nu^{\epsilon(p)}\sin\pi\nu}{p^2-\nu^2}\eta(\nu,p)\end{aligned} \tag{9.16}$$

for $\mathrm{Re}\,\nu = 0$, $p\in 2\mathbb{Z}$. Further, we have obviously $\langle [\mathcal{M}^\omega_{l,q}]\eta, 1\rangle_{\Gamma\backslash G} = 0$.

Collecting these we get, by Theorem 8.1,

$$\begin{aligned}&\langle [\mathcal{M}^{\omega_1}_{l,q}]\eta, [\mathcal{M}^{\omega_2}_{l,q}]\theta\rangle_{\Gamma\backslash G} \\ &= \sum_V |c_V(1)|^2 t_V(\omega_1)t_V(\omega_2)\lambda_l(\nu_V,p_V)\eta(\nu_V,p_V)\overline{\theta(\nu_V,p_V)} \\ &\quad+\sum_{\substack{|p|\le l\\ p\in 2\mathbb{Z}}}\frac{(\omega_1\omega_2/|\omega_1\omega_2|)^p}{2\pi i}\int_{(0)}\frac{\sigma_\nu(\omega_1,-p/2)\sigma_\nu(\omega_2,-p/2)}{|\omega_1\omega_2|^\nu|\zeta_{\mathrm{F}}(1+\nu,p/2)|^2}\lambda_l(\nu,p)\eta(\nu,p)\overline{\theta(\nu,p)}\,d\nu,\end{aligned} \tag{9.17}$$

where $V \cap L^2(\Gamma\backslash G)_{l,q} \neq \{0\}$, and

$$\lambda_l(\nu, p) = \Gamma(l+1+\nu)\Gamma(l+1-\nu)\left(\frac{\nu^{\epsilon(p)}\sin \pi\nu}{\nu^2 - p^2}\right)^2. \tag{9.18}$$

Next, we move to the geometric computation of the inner-product. To this end we make a trivial observation that

$$F_{\omega_1}[\mathcal{M}_{l,q}^{\omega_2}]\theta = F_{\omega_1}[\mathcal{M}_{l,q}^{\omega_2,*}]\theta + b(\theta) \lim_{\nu\to 1+0} F_{\omega_1}[\rho\mathcal{B}_{\omega_2}]\varphi_{l,q}(\nu,0). \tag{9.19}$$

Here we have, by (5.6),

$$\begin{aligned} F_{\omega_1}[\mathcal{M}_{l,q}^{\omega_2,*}]\theta = {} & \frac{1}{2}\left(\delta_{\omega_1,\omega_2}\mathcal{M}_{l,q}^{\omega_2,*}\theta + \delta_{\omega_1,-\omega_2}\ell_i\mathcal{M}_{l,q}^{\omega_2,*}\theta\right) \\ & + \frac{1}{4}\sum_{c\neq 0} S_{\mathrm{F}}(\omega_1,\omega_2;c)\mathcal{A}_{\omega_1}\ell_{1/c}\mathcal{M}_{l,q}^{\omega_2,*}\theta \end{aligned} \tag{9.20}$$

as well as

$$\begin{aligned} F_{\omega_1}[\rho\mathcal{B}_{\omega_2}]\varphi_{l,q}(\nu,0) = {} & \frac{1}{2}\left(\delta_{\omega_1,\omega_2}\rho\mathcal{B}_{\omega_2}\varphi_{l,q}(\nu,0) + \delta_{\omega_1,-\omega_2}\ell_i\rho\mathcal{B}_{\omega_2}\varphi_{l,q}(\nu,0)\right) \\ & + \frac{1}{4}\sum_{c\neq 0} S_{\mathrm{F}}(\omega_1,\omega_2;c)\mathcal{A}_{\omega_1}\ell_{1/c}\rho\mathcal{B}_{\omega_2}\varphi_{l,q}(\nu,0). \end{aligned} \tag{9.21}$$

We have, by definition,

$$\begin{aligned} & \mathcal{A}_{\omega_1}\ell_{1/c}\rho\mathcal{B}_{\omega_2}\varphi_{l,q}(\nu,0)(\mathrm{g}) \\ & = (c/|c|)^{2p}|c|^{-2(1+\nu)}\int_N \psi_{\omega_1}(\mathrm{n})^{-1}\rho(\mathrm{h}[1/c]\mathrm{wng})\mathcal{B}_{\omega_2/c^2}\varphi_{l,q}(\nu,0)(\mathrm{wng})\,dn. \end{aligned} \tag{9.22}$$

In view of (5.9) this integral is bounded by a constant multiple of

$$\int_{N^*} |\mathcal{B}_{\omega_2/c^2}\varphi_{l,q}(\nu,0)(\mathrm{wn}[z]\mathrm{g})|\,d_+z, \tag{9.23}$$

where $N^* = \{z : r \le 2r_0|c|^2(r^2+|z|^2)\}$ with $\mathrm{g} = \mathrm{a}[r]\mathrm{k}$. By Lemma 6.1 we have, for $z \in N^*$,

$$\mathcal{B}_{\omega_2/c^2}\varphi_{l,q}(\nu,0)(\mathrm{wn}[z]\mathrm{g}) \ll \left(\frac{r}{|c|^2(r^2+|z|^2)}\right)^{1+\mathrm{Re}\,\nu}, \tag{9.24}$$

where the implicit constant depends on ν, ω_2 but neither on c nor on r. In this way we get the estimate

$$\mathcal{A}_{\omega_1}\ell_{1/c}\rho\mathcal{B}_{\omega_2}\varphi_{l,q}(\nu,0)(\mathrm{g}) \ll r|c|^{-4-2\mathrm{Re}\,\nu} \tag{9.25}$$

uniformly for $r \geq 0$, $\mathrm{Re}\,\nu > 0$, and $\mathbb{Z}[i] \ni c \neq 0$. This means that we may take the limit of (9.19) inside the sum of (9.21). Then, by virtue of Lemma 7.2, we have

$$\begin{aligned} F_{\omega_1}[\mathcal{M}_{l,q}^{\omega_2}]\theta &= \frac{1}{2}\left(\delta_{\omega_1,\omega_2}\mathcal{M}_{l,q}^{\omega_2}\theta + \delta_{\omega_1,-\omega_2}\ell_i\mathcal{M}_{l,q}^{\omega_2}\theta\right) \\ &\quad + \frac{\pi^2}{4}\sum_{c\neq 0}\frac{1}{|c|^2}S_{\mathrm{F}}(\omega_1,\omega_2;c)\mathcal{M}_{l,q}^{\omega_1}\kappa(\omega_1,\omega_2,1/c)\theta. \end{aligned} \tag{9.26}$$

The formulas (7.22) and (7.24) with appropriate changes of notation imply that

$$\begin{aligned} &\sum_{c\neq 0}\frac{1}{|c|^2}|S_{\mathrm{F}}(\omega_1,\omega_2;c)||\mathcal{M}_{l,q}^{\omega_1}\kappa(\omega_1,\omega_2,1/c)\theta(\nu,p)(\mathrm{a}[r]\mathrm{k})| \\ &\ll \begin{cases} r^{-A} & \text{as } r\uparrow\infty, \\ r^{\frac{1}{2}} & \text{as } r\downarrow 0, \end{cases} \end{aligned} \tag{9.27}$$

where $A > 0$ is arbitrary. Thus $F_{\omega_1}[\mathcal{M}_{l,q}^{\omega_2}]\theta$ satisfies the condition (9.13) with $\omega = \omega_1$, and (9.14) gives

$$\langle[\mathcal{M}_{l,q}^{\omega_1}]\eta, [\mathcal{M}_{l,q}^{\omega_2}]\theta\rangle_{\Gamma\backslash G} = \frac{1}{2}\int_{N\backslash G}\mathcal{M}_{l,q}^{\omega_1}\eta(\mathrm{g})\overline{F_{\omega_1}[\mathcal{M}_{l,q}^{\omega_2}]\theta(\mathrm{g})}\,d\dot{\mathrm{g}}. \tag{9.28}$$

Moreover, we may insert (9.26) into this and perform the integration inside the infinite sum, getting

$$\begin{aligned} &\langle[\mathcal{M}_{l,q}^{\omega_1}]\eta, [\mathcal{M}_{l,q}^{\omega_2}]\theta\rangle_{\Gamma\backslash G} \\ &= \frac{1}{4}(\delta_{\omega_1,\omega_2} + \delta_{\omega_1,-\omega_2})\int_{N\backslash G}\mathcal{M}_{l,q}^{\omega_1}\eta(\mathrm{g})\overline{\mathcal{M}_{l,q}^{\omega_1}\theta(\mathrm{g})}\,d\dot{\mathrm{g}} \\ &\quad + \frac{\pi^2}{8}\sum_{c\neq 0}\frac{1}{|c|^2}S_{\mathrm{F}}(\omega_1,\omega_2;c)\int_{N\backslash G}\mathcal{M}_{l,q}^{\omega_1}\eta(\mathrm{g})\overline{\mathcal{M}_{l,q}^{\omega_1}\kappa(\omega_1,\omega_2,1/c)\theta(\mathrm{g})}\,d\dot{\mathrm{g}}. \end{aligned} \tag{9.29}$$

Invoking Lemma 7.1, we have, from (9.12) and (9.29),

Lemma 9.1. *Let η, θ satisfy the three conditions given in Theorem 7.1. Then we have, for any non-zero $\omega_1, \omega_2 \in \mathbb{Z}[i]$,*

$$\sum_V |c_V(1)|^2 t_V(\omega_1) t_V(\omega_2) \lambda_l(\nu_V, p_V)\eta(\nu_V, p_V)\overline{\theta(\nu_V, p_V)} \tag{9.30}$$

$$+ \sum_{\substack{|p|\le l\\ p\in 2\mathbb{Z}}} \frac{1}{2\pi i}\left(\frac{\omega_1\omega_2}{|\omega_1\omega_2|}\right)^p \int_{(0)} \frac{\sigma_\nu(\omega_1,-p/2)\sigma_\nu(\omega_2,-p/2)}{|\omega_1\omega_2|^\nu |\zeta_{\mathrm{F}}(1+\nu,p/2)|^2} \lambda_l(\nu,p)\eta(\nu,p)\overline{\theta(\nu,p)}\, d\nu$$

$$= \frac{\delta_{\omega_1,\omega_2}+\delta_{\omega_1,-\omega_2}}{4\pi^3 i} \sum_{|p|\le l} \int_{(0)} \lambda_l(\nu,p)\eta(\nu,p)\overline{\theta(\nu,p)}(p^2-\nu^2)d\nu$$

$$+ \sum_{c\neq 0} \frac{S_{\mathrm{F}}(\omega_1,\omega_2;c)}{8\pi i|c|^2} \sum_{|p|\le l} \int_{(0)} \mathcal{K}_{\nu,p}\left(\frac{2\pi}{c}\sqrt{\omega_1\omega_2}\right) \lambda_l(\nu,p)\eta(\nu,p)\overline{\theta(\nu,p)}(p^2-\nu^2)d\nu,$$

where $V \cap L^2(\Gamma\backslash G)_{l,q} \neq \{0\}$.

Remark. For the idea of Hecke see Section 2.2 of [25]. The inner product of two Poincaré series is the basis of almost all proofs of the sum formula. In Kuznetsov's original proof in [19], the seed function is explicit; and the same is in [26], where the K-trivial case is treated. A more general class of seed functions is used in [1] for $\mathrm{PSL}_2(\mathbb{R})$, and in [23] for the K-trivial case on Lie groups of real rank one. Our discussion in this section is different from that of Miatello and Wallach [23] in that we positively exploit the arithmetical situation. Any non-trivial estimate of Kloosterman sums suffices for the continuation of $[\mathcal{B}_\omega]\varphi_{l,q}(\nu,p)$ to a neighbourhood of $\nu = 1$, as has been indicated already. In the general situation considered by Miatello and Wallach a spectral decomposition is needed for analytic continuation. It gives in fact a meromorphic continuation to $\mathbb{C}$. In this respect our argument is specific.

10. Sum formula. I

Based on the above discussion, we shall establish the first version of our sum formula, in which a given bilinear sum of Hecke eigenvalues, or equivalently Fourier coefficients, of cuspidal irreducible subspaces of $L^2(\Gamma\backslash G)$ is expressed in terms of the arithmetic sums S_{F} :

Theorem 10.1 (Spectral–Kloosterman sum formula). *Let $h(\nu,p)$ be a function defined on a set $\{\nu \in \mathbb{C} \,:\, |\mathrm{Re}\,\nu| \le \frac{1}{2}+a\}\times\mathbb{Z}$ for some small $a>0$, satisfying the following conditions:*

1. *$h(\nu,p) = h(-\nu,-p)$,*
2. *$h(\nu,p)$ is regular,*
3. *$h(\nu,p) \ll (1+|\nu|+|p|)^{-4-b}$ with a small $b>0$.*

Then we have, for any non-zero $\omega_1, \omega_2 \in \mathbb{Z}[i]$,

$$\begin{aligned} &\sum_V |c_V(1)|^2 t_V(\omega_1) t_V(\omega_2) h(\nu_V, p_V) \\ &\quad + \sum_{p\in 2\mathbb{Z}} \frac{1}{2\pi i}\left(\frac{\omega_1\omega_2}{|\omega_1\omega_2|}\right)^p \int_{(0)} \frac{\sigma_\nu(\omega_1,-p/2)\sigma_\nu(\omega_2,-p/2)}{|\omega_1\omega_2|^\nu |\zeta_{\mathrm{F}}(1+\nu,p/2)|^2} h(\nu,p)\,d\nu \\ &= \frac{\delta_{\omega_1,\omega_2}+\delta_{\omega_1,-\omega_2}}{4\pi^3 i} \sum_{p\in\mathbb{Z}} \int_{(0)} h(\nu,p)(p^2-\nu^2)d\nu \\ &\quad + \sum_{c\neq 0} \frac{1}{|c|^2} S_{\mathrm{F}}(\omega_1,\omega_2;c) \mathrm{B}h\left(\frac{2\pi}{c}\sqrt{\omega_1\omega_2}\right). \end{aligned} \tag{10.1}$$

Here V *runs over all Hecke invariant right-irreducible cuspidal subspaces of* $L^2(\Gamma\backslash G)$ *together with the specifications in Section 8; and*

$$\mathrm{B}h(u) = \sum_{p\in\mathbb{Z}} \frac{1}{8\pi i} \int_{(0)} \mathcal{K}_{\nu,p}(u) h(\nu,p)(p^2-\nu^2)d\nu \tag{10.2}$$

with $\mathcal{K}_{\nu,p}$ *as in* (7.21). *Convergence of these expressions is absolute throughout.*

Proof. We denote the left and right sides of (10.1) by $L_{\omega_1,\omega_2}h$ and $R_{\omega_1,\omega_2}h$, respectively. We may regard L_{ω_1,ω_2}, R_{ω_1,ω_2} as linear functionals on the space of functions defined on $i\mathbb{R}\times\mathbb{Z}$. The eigenvalues of Hecke operators $t_V(\omega)$ are real, and so are the quantities $(\omega/|\omega|)^p \sigma_\nu(\omega,-p/2)|\omega|^{-\nu}$. Thus $L_{\omega,\omega}$ is positive definite for any non-zero $\omega\in\mathbb{Z}[i]$. We put

$$\eta_0(\nu,p) = (1-\nu^2)^2(4-\nu^2)^{-2}(4-\nu^2+p^2)^{-2-b/2}. \tag{10.3}$$

In (9.30) we may set $\eta(\nu,p) = \lambda_l(\nu,p)^{-1}\eta_0(\nu,p)e^{\delta\nu^2}$, and $\theta(\nu,p) = e^{\delta\nu^2}$ with $\delta > 0$. Thus we have

$$L_{\omega_1,\omega_2}\eta_\delta^l = R_{\omega_1,\omega_2}\eta_\delta^l, \tag{10.4}$$

where $\eta_\delta^l(\nu,p) = \eta_0(\nu,p)e^{2\delta\nu^2}$, if $|p|\le l$, and $=0$ otherwise. Using this we are going to show that

$$\lim_{l\to\infty}\lim_{\delta\to 0+} L_{\omega_1,\omega_2}\eta_\delta^l = L_{\omega_1,\omega_2}\eta_0. \tag{10.5}$$

Since $|L_{\omega_1,\omega_2}\eta_\delta^l| \le \left(L_{\omega_1,\omega_1}\eta_\delta^l \cdot L_{\omega_2,\omega_2}\eta_\delta^l\right)^{1/2}$, and η_δ^l is increasing on $i\mathbb{R}\times\mathbb{Z}$ as $\delta\downarrow 0$, $l\uparrow\infty$, it is obviously sufficient to show that $L_{\omega,\omega}\eta_\delta^l$ is uniformly bounded for $l\ge 0$, $\delta\ge 0$. To this end we shall prove that uniformly for $l\ge 0$, $\delta\ge 0$

$$\mathrm{B}\eta_\delta^l(u) \ll |u|^{1+\varepsilon} \tag{10.6}$$

as $|u|\downarrow 0$. Here $\varepsilon>0$ is small and may depend on b. Then, by the bound (8.14), one may confirm our claim by showing that $\lim_{l\to\infty}\lim_{\delta\to 0+} R_{\omega,\omega}\eta_\delta^l = R_{\omega,\omega}\eta_0$.

By definition we have

$$\begin{aligned}
&\mathrm{B}\eta_\delta^l(u) &(10.7)\\
&= \sum_{|p|\le l} \frac{1}{8\pi i}\int_{(0)} \mathcal{K}_{\nu,p}(u)\eta_0(\nu,p)\mathrm{e}^{2\delta\nu^2}(p^2-\nu^2)d\nu\\
&= -\sum_{|p|\le l}\frac{1}{4\pi i}\int_{(\alpha)}\frac{\mathcal{J}_{\nu,p}(u)}{\sin\pi\nu}\eta_0(\nu,p)\mathrm{e}^{2\delta\nu^2}(p^2-\nu^2)d\nu + \frac{1}{4\pi}\sum_{|p|\le l} p^2\mathcal{J}_{0,p}(u)\eta_0(0,p),
\end{aligned}$$

where $\frac{1}{2} < \alpha < 1$. Since we have

$$\mathcal{J}_{0,p}(u) = (-1)^p|J_{|p|}(u)|^2 \ll (|u|/2)^{2|p|}/(|p|!)^2, \tag{10.8}$$

the last sum of (10.7) is negligible compared with (10.6). Also, we have, as $|u| \downarrow 0$, $\mathrm{Re}\,\nu = \alpha$,

$$\begin{aligned}
\frac{\mathcal{J}_{\nu,p}(u)}{\sin\pi\nu} &\ll |u|^{2\alpha}\frac{|\Gamma(|p|-\nu)|}{|\Gamma(|p|+1+\nu)|} &(10.9)\\
&= |u|^{2\alpha}\frac{|\Gamma(-\nu)|}{|(|p|+\nu)\Gamma(\nu)|}\prod_{j=0}^{|p|-1}\frac{|j-\nu|}{|j+\nu|} \ll \frac{|u|^{2\alpha}}{||p|+\nu||\nu|^{2\alpha}}.
\end{aligned}$$

Inserting this into (10.7) we indeed get (10.6).

Next, we put

$$h_\delta(\nu,p) = -ie^{\delta\nu^2}\sqrt{\frac{\delta}{\pi}}\int_{(0)}\frac{h(\xi,p)}{\eta_0(\xi,p)}\mathrm{e}^{\delta(\xi-\nu)^2}d\xi. \tag{10.10}$$

We repeat the above discussion with $\theta = h_\delta$ and the same η. We have first $L_{\omega_1,\omega_2}f_\delta^l = R_{\omega_1,\omega_2}f_\delta^l$, where $f_\delta^l(\nu,p) = \eta_0(\nu,p)h_\delta(\nu,p)\mathrm{e}^{\delta\nu^2}$ for $|p| \le l$, and $= 0$ otherwise. Then we note that $f_\delta^l(\nu,p) \ll \eta_0(\nu,p)$ uniformly for $\delta \ge 0$, $l \ge 0$, and that $f_\delta^l \to h$ on $i\mathbb{R}\times\mathbb{Z}$ as $\delta \downarrow 0$, $l \uparrow \infty$. Thus, by (10.5), we get

$$\lim_{l\to\infty}\lim_{\delta\to 0+} L_{\omega_1,\omega_2}f_\delta^l = L_{\omega_1,\omega_2}h. \tag{10.11}$$

Corresponding to (10.6) we have to estimate $\mathrm{B}f_\delta^l(u)$. In (10.7) we replace η_δ^l by f_δ^l and set $\alpha = \frac{1}{2}+a$ with $a > 0$ given in the condition 2. above. Accordingly, we shift the contour in (10.10) to (α), and see that $f_\delta^l(\nu,p) \ll |\eta_0(\nu,p)|$ uniformly for $l \ge 0$, $\delta \ge 0$ with $\mathrm{Re}\,\nu = \alpha$ and arbitrary integer p. Hence we have the counterpart of (10.6) for f_δ^l. This gives

$$\lim_{l\to\infty}\lim_{\delta\to 0+} R_{\omega_1,\omega_2}f_\delta^l = R_{\omega_1,\omega_2}h, \tag{10.12}$$

which ends the proof. ∎

As the first application of the sum formula (10.1) we shall prove

Corollary 10.1. *There exist infinitely many V's, and we have, uniformly for N, $P \geq 1$ and non-zero $\omega \in \mathbb{Z}[i]$,*

$$\sum_{|\nu_V| \leq N, |p_V| \leq P} |c_V(1)|^2 t_V(\omega)^2 \ll (NP + |\omega|^{1+\varepsilon})(N^2 + P^2) \tag{10.13}$$

with any fixed $\varepsilon > 0$. In particular, we have the bound (8.15).

Proof. The deduction of the last assertion from (10.13) is analogous to the case of $\mathrm{PSL}_2(\mathbb{Z})$; see the proof of Lemma 3.3 of [25]. To prove the first and the second assertions we put in (10.1) $\omega_1 = \omega_2 = \omega$ and

$$h(\nu, p) = h(\nu, p; N, P) = \exp((\nu/N)^2 - (p/P)^2). \tag{10.14}$$

We note that $\zeta_{\mathrm{F}}(1 + \nu, p) \gg \log^{-1}(|\nu| + |p| + 2)$ for $\mathrm{Re}\,\nu = 0$, with the implied constant being absolute. This can be proved as in Sections 3.10–3.11 of [30]. The necessary uniform upper bound for $\zeta_{\mathrm{F}}(s, p)$ in the critical strip follows from the functional equation (5.34) and the convexity argument of Phragmén and Lindelöf. Thus we have

$$\begin{aligned}&\sum_V |c_V(1)|^2 t_V(\omega)^2 h(\nu_V, p_V; N, P) + O(NP\sigma_0(\omega)^2 \log^2(NP+2)) \\ &= \frac{1}{8\pi^{5/2}} \sum_{p\in\mathbb{Z}} (2Np^2 + N^3) e^{-(p/P)^2} + \sum_{c\neq 0} \frac{1}{|c|^2} S_{\mathrm{F}}(\omega, \omega; c) \mathrm{B}h(2\pi\omega/c).\end{aligned} \tag{10.15}$$

We are going to show

$$\mathrm{B}h(2\pi\omega/c) \ll \min(1, |\omega/c|^2)(N^2 + P^2). \tag{10.16}$$

This and the bound (8.14) give (10.13), as well as an asymptotic formula for the first sum in (10.15), which gives the first assertion.

By (12.1) below we have

$$\mathrm{B}h(2\pi\omega/c) = \frac{N^3}{2\pi^{3/2}} \int_1^\infty C_{\omega/c}(y; N, P) \exp(-(N\log y)^2) \frac{dy}{y}, \tag{10.17}$$

where

$$\begin{aligned}C_{\omega/c}(y; N, P) = &\sum_{p\in\mathbb{Z}} (-1)^p ((p/N)^2 - (N\log y)^2 + \tfrac{1}{2}) \\ &\times J_{2p}(2\pi|\omega/c||x|) \exp(-(p/P)^2 + 2pi\psi)\end{aligned} \tag{10.18}$$

with $x = |x|e^{i\psi} = ye^{i\vartheta} + (ye^{i\vartheta})^{-1}$, $\vartheta = \arg(\omega/c)$. Using an integral representation for J_{2p} we have also

$$\begin{aligned} C_{\omega/c}(y;N,P) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(2\pi i|\omega/c||x|\cos(\theta+\psi)) \\ &\qquad \times \sum_{p\in\mathbb{Z}}((p/N)^2 - (N\log y)^2 + \tfrac{1}{2})\exp(-(p/P)^2 - 2pi\theta)d\theta \\ &= \frac{P}{2\sqrt{\pi}}\int_{-\pi}^{\pi} \exp(2\pi i|\omega/c||x|\cos(\theta+\psi)) \\ \times \sum_{q\in\mathbb{Z}} &((P/N)^2(\tfrac{1}{2} - (P(\theta+q\pi))^2) - (N\log y)^2 + \tfrac{1}{2})\exp(-(P(\theta+q\pi))^2)d\theta, \end{aligned} \tag{10.19}$$

where the last line is due to Poisson's sum formula. This gives

$$C_{\omega/c}(y;N,P) \ll (P/N)^2 + (N\log y)^2 + 1 \tag{10.20}$$

uniformly for all parameters involved. Now, the case $|\omega/c| \geq 1$ is settled by inserting (10.20) into (10.17). If $|\omega/c| < 1$ then we divide the integral in (10.17) at $y = \sqrt{|c/\omega|}$. For the infinite integral thus obtained we use again (10.20), and see that the contribution is negligible compared with (10.16). To estimate the remaining part we use (10.18) together with $J_{2p}(a) = (1+O(a^2))(a/2)^{2p}/(2p)!$ for small $a > 0$. We have, for $1 \leq y \leq \sqrt{|c/\omega|}$,

$$C_{\omega/c}(y;N,P) = -(N\log y)^2 + \frac{1}{2} + O\left(|\omega/c|^2x^2((N\log y)^2+1)\right). \tag{10.21}$$

The contribution of this error term to (10.17) is $\ll |\omega/c|^2N^2$. As to the main term, we note that

$$\int_1^{\infty} (-(N\log y)^2 + \tfrac{1}{2})\exp(-(N\log y)^2)\frac{dy}{y} = 0. \tag{10.22}$$

Thus the relevant contribution to (10.17) is easily seen to be negligible. This ends the proof. ■

Remark. The class of test functions in Theorem 10.1 is as large as possible. The strip on which the test functions are required to be defined is narrow, due to the Weil bound (8.14) (cf. (3.6.24) of [25]). The use of general seed functions in Lemma 9.1 leads to an extension step with a functional analytic flavour. The proof is similar to those in [1] (for $\mathrm{PSL}_2(\mathbb{R})$), and [3] (for PSL_2 over the product of the archimedean completions of a number field). The proof in [26] for the K-trivial case is an extension of Kuznetsov's original treatment [19] of the rational case.

The corollary is a counterpart of Kuznetsov's estimate for the spectral mean square of the Fourier coefficients of Maass forms over the modular group; see

Lemma 2.4 of [25]. In Lemma 11 of [26] the K-trivial case of the corollary is given. The bound (10.13) is essentially the best possible. We could prove an asymptotic result in which the main term is a constant multiple of $NP(N^2+P^2)$. Note that the proof of the corollary requires the rather deep integral representation of $\mathcal{K}_{\nu,p}$ in Theorem 12.1, whereas the series expansion defining $\mathfrak{J}_{\nu,p}$ suffices for the theorem.

Generalization of the Spectral–Kloosterman sum formula and the bound (10.13) to other imaginary quadratic number fields and congruence subgroups seems possible, as long as we have a counterpart of (8.14). Without such a bound, one has to be content with test functions which are holomorphic for $|\mathrm{Re}\,\nu| \le 1+a$ with an $a>0$, and have prescribed zeros at $\nu=\pm1$ for most values of p (see (9.18) and Lemma 9.1). In contrast to what we have seen in the above there might be, in general, irreducible subspaces V of complementary series type as well, corresponding to exceptional eigenvalues. We note also that the assumption that the spaces V are Hecke invariant is not essential for the spectral sum formula; that is, the sums over V in (10.1) and (13.1) could be formulated in terms of Fourier coefficients in place of Hecke eigenvalues.

11. A Bessel inversion

The aim of this section is to demonstrate a one-sided inversion of the transform B defined by (10.2). Results of the present and the next sections will play basic rôles in the proof of the second version of our sum formula for S_{F}, which is to be developed in Section 13.

Theorem 11.1. *We put*

$$\mathrm{K}f(\nu,p)=\int_{\mathbf{C}^\times}\mathcal{K}_{\nu,p}(u)f(u)d^\times u. \tag{11.1}$$

Then, for any f that is even, smooth and compactly supported on $\mathbb{C}^\times$, we have

$$2\pi\mathrm{BK}f=f. \tag{11.2}$$

Proof. We shall prove, instead, the Parseval identity

$$\int_{\mathbf{C}^\times}f(u)g(u)d^\times u=\sum_{p\in\mathbf{Z}}\frac{1}{4i}\int_{(0)}\mathrm{K}f(\nu,p)\mathrm{K}g(\nu,p)(p^2-\nu^2)d\nu, \tag{11.3}$$

where f, g are to satisfy the condition given in the theorem. This implies (11.2), since a simple manipulation shows that the right side is equal to

$$\int_{\mathbf{C}^\times}\mathrm{BK}f(u)\cdot g(u)d^\times u. \tag{11.4}$$

Here the necessary absolute convergence follows from the estimate

$$\mathrm{K}f(\nu,p) \ll (1+|\nu|+|p|)^{-A}, \tag{11.5}$$

where $A>0$ is arbitrary, and the implied constant depends on $\mathrm{Re}\,\nu$, A, and the support of f. To show this we put

$$\mathrm{J}f(\nu,p) = \int_{\mathbf{C}^\times} f(u)\mathcal{J}_{\nu,p}(u)d^\times u, \tag{11.6}$$

so that

$$\mathrm{K}f(\nu,p) = \frac{1}{\sin\pi\nu}\left\{\mathrm{J}f(-\nu,-p) - \mathrm{J}f(\nu,p)\right\}. \tag{11.7}$$

By definition we have

$$\mathrm{J}f(\nu,p) = 2\pi\sum_{m,\,n\ge 0}\frac{(-1)^{m+n}2^{-2(\nu+m+n)}\mathrm{M}f(\nu+m+n,p-m+n)}{m!n!\Gamma(\nu-p+m+1)\Gamma(\nu+p+n+1)}, \tag{11.8}$$

where

$$\mathrm{M}f(\nu,p) = \int_0^\infty f_p(r)r^{2\nu-1}dr, \quad f_p(r) = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})e^{-2pi\theta}d\theta. \tag{11.9}$$

A multiple application of partial integration gives, for any integers $A,\ B\ge 0$,

$$f_p(r) \ll \frac{r_f^A}{(1+|p|)^A}, \quad \mathrm{M}f(\nu,p) \ll \frac{1}{(1+|p|)^A}\left|\frac{\Gamma(2\nu)}{\Gamma(2\nu+B)}\right| r_f^{2|\mathrm{Re}\,\nu|+A+B}, \tag{11.10}$$

where the constant r_f depends on the compact support of f, and the implied constants only on A, B and f. Collecting these, we get (11.5). Thus we see also that the right side of (11.3) is equal to

$$\begin{aligned}
&\lim_{T\to\infty}\sum_{|p|\le P}\frac{1}{4i}\int_{-iT}^{iT}\mathrm{K}f(\nu,p)\mathrm{K}g(\nu,p)(p^2-\nu^2)d\nu \qquad (11.11)\\
&= \lim_{T\to\infty}\left\{\int_{|u|<|v|}+\int_{|u|>|v|}\right\} f(u)g(v)R_{P,T}(u,v)d^\times u d^\times v,
\end{aligned}$$

where $P=P(T)\in\mathbb{Z}$ is to be chosen later, and

$$R_{P,T}(u,v) = \sum_{|p|\le P}\frac{1}{4i}\int_{-iT}^{iT}\mathcal{K}_{\nu,p}(u)\mathcal{K}_{\nu,p}(v)(p^2-\nu^2)d\nu. \tag{11.12}$$

We shall consider the case $|u|<|v|$. We indent the contour $[-iT,iT]$ with the right half of a small circle centered at $\nu=0$. Denoting the new contour by L_T, we have, by (11.7),

$$R_{P,T}(u,v) = \sum_{|p|\le P}\frac{1}{4i}\left\{\int_{L_T^-} - \int_{L_T}\right\}\mathcal{J}_{\nu,p}(u)\mathcal{K}_{\nu,p}(v)\frac{p^2-\nu^2}{\sin\pi\nu}d\nu, \tag{11.13}$$

where $L_T^- = \{\nu : -\nu \in L_T\}$. This implies that

$$R_{P,T}(u,v) = \sum_{|p|\le P} \frac{i}{2} \int_{C_{P,T}} \mathcal{J}_{\nu,p}(u)\mathcal{K}_{\nu,p}(v)\frac{p^2-\nu^2}{\sin \pi\nu} d\nu \qquad (11.14)$$
$$+ \frac{1}{2} \sum_{|p|\le P} \sum_{0\le q\le P} (-1)^q a_q \mathcal{J}_{q,p}(u)\mathcal{K}_{q,p}(v)(p^2-q^2),$$

where $a_0 = 1$, $a_q = 2$ for $q > 0$, and $C_{P,T}$ is the oriented polygonal line connecting the points $-iT$, $P+\frac{1}{2}-iT$, $P+\frac{1}{2}+iT$, iT in this order. This double sum vanishes, because of the identities, for any $p, q \in \mathbb{Z}$,

$$\mathcal{J}_{q,p} = (-1)^{p+q}\mathcal{J}_{p,q} = (-1)^{p+q}\mathcal{J}_{|p|,q\,\mathrm{sign}(p)}, \quad \mathcal{K}_{q,p} = \mathcal{K}_{|p|,q\,\mathrm{sign}(p)}. \qquad (11.15)$$

In fact, the first identity is trivial, and the second follows from the expression

$$\mathcal{K}_{q,p}(u) = \frac{(-1)^{q+1}}{\pi}\left\{\mathbf{Y}_{q-p}(u)J_{p+q}(\bar{u}) + J_{p-q}(u)\mathbf{Y}_{p+q}(\bar{u})\right\}. \qquad (11.16)$$

Here $\mathbf{Y}_n$ with $n \in \mathbb{Z}$ is the Hankel function, which satisfies $\mathbf{Y}_n = (-1)^n\mathbf{Y}_{-n}$ (see p. 59 of [35]). Hence we have

$$R_{P,T}(u,v) = \sum_{|p|\le P} \frac{i}{2} \int_{C_{P,T}} \mathcal{J}_{\nu,p}(u)\mathcal{K}_{\nu,p}(v)\frac{p^2-\nu^2}{\sin \pi\nu} d\nu \qquad (11.17)$$
$$= \sum_{|p|\le P} \frac{i}{2} \int_{C_{P,T}} \left[\mathcal{J}_{\nu,p}(u)\mathcal{J}_{-\nu,-p}(v) - \mathcal{J}_{\nu,p}(u)\mathcal{J}_{\nu,p}(v)\right]\frac{p^2-\nu^2}{(\sin \pi\nu)^2} d\nu$$
$$= R^-_{P,T}(u,v) - R^+_{P,T}(u,v),$$

say, in an obvious mode of division.

Now we have trivially

$$\int_{|u|<|v|} f(u)g(v)R^{\pm}_{P,T}(u,v)d^{\times}u\, d^{\times}v = \int_{r_1<r_2} (r_1r_2)^{-1}Q^{\pm}_{P,T}(r_1,r_2;f,g)dr_1\, dr_2, \qquad (11.18)$$

where

$$Q^{\pm}_{P,T}(r_1,r_2;f,g) = \int_0^{2\pi}\int_0^{2\pi} f(r_1e^{i\theta_1})g(r_2e^{i\theta_2})R^{\pm}_{P,T}(r_1e^{i\theta_1},r_2e^{i\theta_2})d\theta_1\, d\theta_2. \qquad (11.19)$$

The series expansion of $\mathcal{J}_{\nu,p}$ implies that

$$\mathcal{J}_{\nu,p}(u)\mathcal{J}_{-\nu,-p}(v)\frac{p^2-\nu^2}{(\sin \pi\nu)^2} = -\pi^{-2}\left|\frac{u}{v}\right|^{2\nu}\left(\frac{u}{|u|}\right)^{-2p}\left(\frac{v}{|v|}\right)^{2p} \qquad (11.20)$$
$$\times \sum_{k,l,m,n\ge 0} \frac{(-1)^{k+l+m+n}(u/2)^{2k}(\bar{u}/2)^{2l}(v/2)^{2m}(\bar{v}/2)^{2n}}{k!l!m!n!\lambda(\nu,p;k,l,m,n)},$$

where $\lambda(\nu,p;k,l,m,n)=(\nu-p+1)_k(\nu+p+1)_l(-\nu+p+1)_m(-\nu-p+1)_n$ with $(\alpha)_k$ as in (5.19). Thus we have

$$\begin{aligned} &Q^-_{P,T}(r_1,r_2;f,g)\\ &=\sum_{|p|\le P}\sum_{k,l,m,n\ge 0}\frac{(-1)^{k+l+m+n}(r_1/2)^{2(k+l)}(r_2/2)^{2(m+n)}}{k!l!m!n!}\\ &\quad\times f_{p-k+l}(r_1)g_{-p-m+n}(r_2)S^-_{P,T}(r_1/r_2,p;k,l,m,n), \end{aligned}\tag{11.21}$$

where g_q is analogous to f_q, and

$$S^-_{P,T}(\rho,p;k,l,m,n)=-2i\int_{C_{P,T}}\frac{\rho^{2\nu}}{\lambda(\nu,p;k,l,m,n)}d\nu.\tag{11.22}$$

Similarly we have

$$\begin{aligned} &Q^+_{P,T}(r_1,r_2;f,g)\\ &=\sum_{|p|\le P}\sum_{k,l,m,n\ge 0}\frac{(-1)^{k+l+m+n}(r_1/2)^{2(k+l)}(r_2/2)^{2(m+n)}}{k!l!m!n!}\\ &\quad\times f_{p-k+l}(r_1)g_{p-m+n}(r_2)S^+_{P,T}(r_1r_2,p;k,l,m,n), \end{aligned}\tag{11.23}$$

where

$$\begin{aligned} &S^+_{P,T}(\rho,p;k,l,m,n)\\ &=2\pi^2 i\int_{C_{P,T}}\frac{(p^2-\nu^2)(\rho/4)^{2\nu}}{(\sin\pi\nu)^2\Gamma(\nu-p+k+1)\Gamma(\nu+p+l+1)}\\ &\quad\times\frac{1}{\Gamma(\nu-p+m+1)\Gamma(\nu+p+n+1)}d\nu. \end{aligned}\tag{11.24}$$

Assuming that $2P\le T$, we shall estimate $Q^\pm_{P,T}(r_1,r_2;f,g)$; implicit constants may depend only on the parameter A and the supports of f and g. To this end we stress that the first estimate in (11.10) implies readily that for any $A>0$

$$\sum_{k,l,m,n\ge 0}\frac{(r_1/2)^{2(k+l)}(r_2/2)^{2(m+n)}}{k!l!m!n!}|f_{p-k+l}(r_1)g_{\pm p-m+n}(r_2)|\ll\frac{1}{(1+|p|)^A}.\tag{11.25}$$

Arguing as in (10.9) we see that the integrand in (11.24) is $O((\rho/4)^{2\mathrm{Re}\,\nu}|\Gamma(-\nu)/\Gamma(\nu)|^2)$. Hence we get immediately

$$Q^+_{P,T}(r_1,r_2;f,g)\ll\frac{1}{\log T}+\frac{1}{P!},\tag{11.26}$$

where the terms on the right come from the horizontal and the vertical parts of $C_{P,T}$, respectively. As to $Q^{-}_{P,T}(r_1, r_2; f, g)$ we note first that

$$S^{-}_{P,T}(\rho, p; 0, 0, 0, 0) = -2i \int_{-iT}^{iT} \rho^{2\nu} d\nu. \tag{11.27}$$

If $(k, l, m, n) \neq (0, 0, 0, 0)$, then on the horizontal part of $C_{P,T}$ we have $|\lambda(\nu, p; k, l, m, n)| \gg T$, and if $\rho \leq 1$, then the corresponding contribution in (11.22) is $\ll \min(1, (T|\log\rho|)^{-1})$. We restrict ourselves to the vertical part of $C_{P,T}$. We may assume naturally $p \geq 0$. Then, if either $l > 0$ or $n > 0$, we have $|\lambda(\nu, p; k, l, m, n)| \gg P + |\nu|$. By partial integration we see that the corresponding contribution to (11.22) is

$$\ll (k + l + m + n) \log T \min\left(1, \frac{1}{P|\log\rho|}\right), \tag{11.28}$$

provided $\rho \leq 1$. If $l = n = 0$, and $0 \leq p \leq P/2$, then obviously we get the same conclusion. Otherwise the contribution in question is $\ll \log T$, provided $\rho \leq 1$. Collecting these, we have, for $r_1 < r_2$, $2P \leq T$,

$$Q^{-}_{P,T}(r_1, r_2; f, g) = -2i \sum_{|p|\leq P} f_p(r_1) g_{-p}(r_2) \int_{-iT}^{iT} (r_1/r_2)^{2\nu} d\nu \tag{11.29}$$
$$+ O\left(\log T \min\left(1, \frac{1}{P|\log(r_1/r_2)|}\right)\right) + O\left(\frac{\log T}{P^A}\right),$$

which ends the discussion of the case $|u| < |v|$.

The case $|u| > |v|$, i.e., $r_1 > r_2$, can be treated in just the same way. We return to (11.12), and this time we shift the relevant contours to the left, getting the same assertions as (11.26) and (11.29). In this way we now have

$$\int_{\mathbf{C}^\times \times \mathbf{C}^\times} f(u) g(v) R_{P,T}(u, v) d_* u \, d_* v \tag{11.30}$$
$$= -2i \sum_{|p|\leq P} \int_{-iT}^{iT} \mathrm{M}f(\nu, p) \mathrm{M}g(-\nu, -p) d\nu + O\left(\frac{1}{\log T} + \frac{\log^2 T}{P}\right).$$

Hence we set $P = [\log^3 T]$. We find that the right side of (11.3) is equal to

$$-2i \sum_{p\in\mathbb{Z}} \int_{(0)} \mathrm{M}f(\nu, p) \mathrm{M}g(-\nu, -p) d\nu. \tag{11.31}$$

With this and the Parseval formulas for Mellin transform and Fourier series expansion, we finish the proof. ∎

We shall also need the following property of the transform K:

Lemma 11.1. *Let f be an even smooth function on $\mathbb{C}^\times$ with a compact support. Then we have*

$$\sum_{p\in\mathbb{Z}} \int_{(0)} \mathrm{K}f(\nu,p)(p^2-\nu^2)d\nu = 0. \tag{11.32}$$

Proof. Let $\nu \in \mathbb{C}$ be such that $\mathrm{Re}\,\nu \geq 0$ and $|\nu - k| \geq \frac{1}{2}$ for all $k \in \mathbb{Z}$. Then (11.8) and (11.10) give

$$\mathrm{J}f(\nu,p) \ll \sum_{m,n\geq 0} \frac{(r_f/2)^{2\mathrm{Re}\,\nu+2(m+n)}}{m!n!|\Gamma(\nu-p+m+1)||\Gamma(\nu+p+n+1)|} \times \frac{r_f^{A+B}}{(1+|p-m+n|)^A|\nu+m+n|^B}. \tag{11.33}$$

If $|p-m+n| \leq \frac{1}{2}|p|$, then we have $n+m \geq \frac{1}{2}|p|$, and consequently $|\nu+m+n| \gg |\nu|+|p|$; otherwise we have $|p-m+n||\nu+m+n| \gg |p\nu|$. Thus we have, for any fixed large $C>0$,

$$\mathrm{J}f(\nu,p) \ll \frac{(r_f/2)^{2\mathrm{Re}\,\nu}}{|\Gamma(\nu-|p|+1)\Gamma(\nu+|p|+1)|}(|\nu|+|p|)^{-C}. \tag{11.34}$$

This estimate allows us to carry out the same procedure as in (11.13)–(11.14): We have, for any positive integer P,

$$\sum_{p=-P}^{P} \int_{(0)} \mathrm{K}f(\nu,p)(p^2-\nu^2)d\nu = 2\sum_{p=-P}^{P} \int_{(P+1/2)} \mathrm{J}f(\nu,p)\frac{p^2-\nu^2}{\sin\pi\nu}d\nu. \tag{11.35}$$

As before, the sum of residues arising from this shift of contour vanishes because of the first relation in (11.15). The last integral is, in view of (10.9) and (11.34),

$$\ll \left(\frac{r_f}{2P}\right)^{2P} \int_{-\infty}^{\infty} (P+|t|)^{2-C}dt \tag{11.36}$$

uniformly for $|p| \leq P$. This obviously ends the proof. ■

Remark. The inversion formula (11.2) could be formulated as a discontinuous integral of new type in the theory of Bessel functions. The idea of the proof is to view the transformation K in (11.1) as a perturbation of the Mellin–Fourier transformation on $\mathbb{C}^\times$. Insert the power series expansion of $\mathcal{J}_{\pm\nu,\pm p}$ into the integrals hidden in $\mathrm{K}f(\nu,p)\mathrm{K}g(\nu,p)$ on the right of (11.3). Two of the four lowest order terms describe the Mellin–Fourier transformation on $\mathbb{C}^\times$ in polar coordinates, as is well indicated by (11.20). The proof of the inversion consists of showing that all other terms do not contribute. The key to achieve this is the vanishing of the double sum in (11.14). That is, a certain rearrangement of products of J-Bessel functions of various orders is taking place behind our argument, which further points to a relation with the Neumann expansion (see Chapter XVI of [35]). The basic idea is present in Section 2.5 of Kuznetsov's preprint [18], which deals with the Bessel inversion for the modular case, and is indeed the first instance of such investigations (see also Section 2.4 of [25]).

Our proof is, however, admittedly technical, and one may wish to find a more structural proof that takes into account the way through which the functions $\mathcal{J}_{\nu,p}$ and $\mathcal{K}_{\nu,p}$ come into our discussion. They correspond to functions on the big cell in the Bruhat decomposition of G, transforming on the left and the right according to non-trivial characters of the subgroup N, and turn out to be a basis of the solutions of $\Omega_\pm f = \frac{1}{8}((\nu \mp p)^2 - 1)f$. These and the adjoint formulation (11.3) suggest that the inversion should be a part of the spectral theory on the big cell. A proof along such a line might work for other Lie groups of rank one as well. A further discussion is given in the final section.

As to Lemma 11.1, we remark that there are test functions h such as the one introduced in (10.14), for which

$$\sum_{p\in\mathbb{Z}} \int_{(0)} h(\nu,p)(p^2-\nu^2)d\nu \neq 0. \tag{11.37}$$

Thus Lemma 11.1 shows that (11.2) gives only a one-sided inversion of the transformation B. Also see Remark at the end of Section 13 for an alternative argument.

12. The Bessel kernel $\mathcal{K}_{\nu,p}$

The main feature of Theorem 10.1 rests precisely in the integral transform B defined by (10.2), and thus in the kernel $\mathcal{K}_{\nu,p}$. In this section we shall prove an integral formula for $\mathcal{K}_{\nu,p}$, which has a practical value for our purpose, and an interest of its own.

Theorem 12.1. *Let $|\mathrm{Re}\,\nu| < \frac{1}{4}$. Then we have, for any $p \in \mathbb{Z}$ and non-zero $u \in \mathbb{C}$,*

$$\begin{aligned}&\mathcal{K}_{\nu,p}(u)\\&=(-1)^p\frac{2}{\pi}\int_0^\infty y^{2\nu-1}\left(\frac{ye^{i\vartheta}+(ye^{i\vartheta})^{-1}}{|ye^{i\vartheta}+(ye^{i\vartheta})^{-1}|}\right)^{2p} J_{2p}\left(|u||ye^{i\vartheta}+(ye^{i\vartheta})^{-1}|\right)dy,\end{aligned} \tag{12.1}$$

where $u = |u|e^{i\vartheta}$.

Proof. Since $\overline{\mathcal{K}_{\bar\nu,p}(u)} = \mathcal{K}_{\nu,-p}(u)$, we may assume that p is non-negative. We shall first show that we have, for $\mathrm{Re}\,\nu > -p$,

$$\begin{aligned}\mathcal{J}_{\nu,p}(2\pi u) &= (-1)^p|u/2|^{2p}\sum_{|m|\le p}\binom{2p}{p+m}(iu/|u|)^{-2m}\\&\quad\times\frac{1}{2\pi i}\int_{(1)}\exp(2\pi\xi-\pi\mathrm{Re}\,(u^2)/\xi)I_{\nu-m}(\pi|u|^2/\xi)\xi^{-2p-1}d\xi.\end{aligned} \tag{12.2}$$

The basis for this formula is (6.20), where the function $\mathcal{J}_{\nu,p}$ enters into our investigations. We set there $\omega_1 = 1$, $\omega_2 = u^2$, $l = p$, and $q = p$. On noting (6.18), we have, for $\mathrm{Re}\,\nu > 0$, $r > 0$, and $\mathrm{k} \in K$,

$$\begin{aligned}\int_{\mathbf{C}} e^{-2\pi i \mathrm{Re}\, z} \mathcal{B}_1 \varphi_{p,p}(\nu, p)(\mathrm{h}[u]\mathrm{wn}[z]\mathrm{a}[r]\mathrm{k}) d_+ z \\ = \pi^{-2\nu} |u|^2 \mathcal{J}_{\nu,p}(2\pi u) \mathcal{A}_1 \varphi_{p,p}(\nu, p)(\mathrm{a}[r]\mathrm{k}),\end{aligned} \tag{12.3}$$

where, by (5.9),

$$\begin{aligned}&\mathrm{h}[u]\mathrm{wn}[z]\mathrm{a}[r] \\ &= \mathrm{n}\left[\frac{-u^2\bar{z}}{r^2+|z|^2}\right] \mathrm{a}\left[\frac{|u|^2 r}{r^2+|z|^2}\right] \mathrm{h}\left[\frac{u}{|u|}\right] \mathrm{k}\left[\frac{\bar{z}}{\sqrt{r^2+|z|^2}}, \frac{-r}{\sqrt{r^2+|z|^2}}\right].\end{aligned} \tag{12.4}$$

Using (3.24), (5.26)–(5.27), and (6.13)–(6.14), we equate the coefficients of $\Phi^p_{0,p}(\mathrm{k})$ on both sides of (12.3), getting

$$K_\nu(2\pi r)\mathcal{J}_{\nu,p}(2\pi u) = \frac{|u|^{2p} r^{-2p}}{2\pi} \sum_{|m|\le p} i^m (u/|u|)^{-2m} U_{\nu,m}(u,r). \tag{12.5}$$

Here

$$\begin{aligned}U_{\nu,m}(u,r) = \int_{\mathbf{C}} \exp\Big(-2\pi i r \mathrm{Re}\, z - 2\pi i \frac{\mathrm{Re}\, u^2 \bar{z}}{r(1+|z|^2)}\Big) \\ \times I_{\nu-m}\Big(\frac{2\pi|u|^2}{r(1+|z|^2)}\Big) \Phi^{p,*}_{m,0}\left(\begin{bmatrix} \bar{z} & -1 \\ 1 & z\end{bmatrix}\right) \frac{d_+ z}{(1+|z|^2)^{2p+1}},\end{aligned} \tag{12.6}$$

where the asterisk denotes that we have extended the definition (3.18) in an obvious way. We are going to compute $U_{\nu,p}(u,r)$ asymptotically when r tends to infinity, so that the result yields the cancellation of the factor $K_\nu(2\pi r) \sim \frac{1}{2} r^{-1/2} e^{-2\pi r}$ on the left side of (12.5). We put $z = x + iy$ with $x, y \in \mathbb{R}$ in (12.6), and then regard the integral as a double complex integral. Studying partial derivatives of the argument of the exponentiated factor, we see that a saddle point exists at $x = -i + c_1/r$, $y = c_2/\sqrt{r}$, where c_1, c_2 are asymptotically constant as $r \uparrow \infty$. Because of this we make the change of variables $(x, y) \mapsto (-i(1-1/r)+x/r,\ y/\sqrt{r})$. Here the factor $1 - 1/r$ is to avoid the singularity at $(-i, 0)$. We have

$$\begin{aligned}&U_{\nu,m}(u,r) = r^{2p-\frac{1}{2}} e^{-2\pi(r-1)} \\ &\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\Big(-2\pi i x + 2\pi i \frac{ai - a(x+i)/r - by/\sqrt{r}}{2+y^2-2ix+(x+i)^2/r}\Big) \\ &\times \Phi^{p,*}_{m,0}\left(\begin{bmatrix} -i+(x+i)/r - iy/\sqrt{r} & -1 \\ 1 & -i+(x+i)/r+iy/\sqrt{r}\end{bmatrix}\right) \\ &\times I_{\nu-m}\Big(\frac{2\pi|u|^2}{2+y^2-2ix+(x+i)^2/r}\Big) \frac{dx\,dy}{(2+y^2-2ix+(x+i)^2/r)^{2p+1}},\end{aligned} \tag{12.7}$$

where $a = \operatorname{Re} u^2$, $b = \operatorname{Im} u^2$. It is easy to check that this double integral converges absolutely and uniformly as $r \uparrow \infty$, provided $\operatorname{Re}\nu > 0$. Thus we have

$$\lim_{r\to\infty} \frac{U_{\nu,p}(u,r)}{r^{2p}K_\nu(2\pi r)} = 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(2\pi(1-ix) - \frac{2\pi a}{2+y^2-2ix}\right) \tag{12.8}$$
$$\times\, \Phi_{m,0}^{p,*}\left(\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\right) I_{\nu-m}\left(\frac{2\pi|u|^2}{2+y^2-2ix}\right)\frac{dx\,dy}{(2+y^2-2ix)^{2p+1}}.$$

We shift the contour of the x-integral to $\operatorname{Im} x = -y^2/2$. We then find that

$$\lim_{r\to\infty} \frac{U_{\nu,p}(u,r)}{r^{2p}K_\nu(2\pi r)} \tag{12.9}$$
$$= -i2^{-2p}\int_{(1)} \exp\left(2\pi\xi - \frac{\pi a}{\xi}\right)\Phi_{m,0}^{p,*}\left(\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\right) I_{\nu-m}\left(\frac{\pi|u|^2}{\xi}\right)\frac{d\xi}{\xi^{2p+1}}.$$

On noting that $\Phi_{m,0}^{p,*}\left(\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\right) = (-1)^p i^m \binom{2p}{p-m}$, we get, from (12.5) and (12.9), the representation (12.2) at least for $\operatorname{Re}\nu > 0$, and then we use analytic continuation with respect to ν.

We move to the proof of (12.1). We shall treat first the case $p > 0$. Since $\mathfrak{J}_{-\nu,-p}(u) = \overline{\mathfrak{J}_{-\bar\nu,p}(u)}$, we see that (12.2) gives, for $|\operatorname{Re}\nu| < p$,

$$\mathcal{K}_{\nu,p}(2\pi u) = (-1)^p\frac{2}{\pi}\left|\frac{u}{2}\right|^{2p}\sum_{|m|\le p}\binom{2p}{p+m}\left(\frac{u}{|u|}\right)^{-2m} \tag{12.10}$$
$$\times\, \frac{1}{2\pi i}\int_{(1)} \exp\left(2\pi\xi - \frac{\pi \operatorname{Re}(u^2)}{\xi}\right) K_{\nu-m}\left(\frac{\pi|u|^2}{\xi}\right)\xi^{-2p-1}d\xi.$$

Observing that $\operatorname{Re}\xi^{-1} > 0$, we have

$$\sum_{|m|\le p}\binom{2p}{p+m}\left(\frac{u}{|u|}\right)^{-2m} K_{\nu-m}\left(\frac{\pi|u|^2}{\xi}\right) \tag{12.11}$$
$$= \frac{1}{2}\int_0^\infty \left(\sqrt{y}e^{i\vartheta} + \frac{1}{\sqrt{y}e^{i\vartheta}}\right)^{2p}\exp\left(-\frac{\pi|u|^2}{2\xi}\left(y+\frac{1}{y}\right)\right)y^{\nu-1}dy$$
$$= I_{\nu,p}(u,\xi) + I_{-\nu,p}(\bar u,\xi),$$

say, where ϑ is as in (12.1), and $I_{\nu,p}(u,\xi)$ is the part corresponding to $y \ge 1$. We put

$$\mathcal{K}_{\nu,p}^*(2\pi u)$$
$$= (-1)^p\frac{1}{\pi^2 i}\left|\frac{u}{2}\right|^{2p}\int_{(1)} \exp\left(2\pi\xi - \frac{\pi\operatorname{Re}(u^2)}{\xi}\right) I_{\nu,p}(u,\xi)\xi^{-2p-1}d\xi. \tag{12.12}$$

We insert into this the integral representation of $I_{\nu,p}(u,\xi)$. The resulting double integral is absolutely convergent for $\operatorname{Re}\nu<0$, and $\mathcal{K}^*_{\nu,p}(2\pi u)$ is regular there. To get analytic continuation to $\operatorname{Re}\nu\ge 0$, we turn the line of integration of $I_{\nu,p}(u,\xi)$ around the point $y=1$ through a small angle which has the same sign as $\operatorname{Im}\xi$. We get immediately the bound $I_{\nu,p}(u,\xi)\ll|\xi|^{\operatorname{Re}\nu+p}$, which means that $\mathcal{K}^*_{\nu,p}(2\pi u)$ is regular for $\operatorname{Re}\nu<p$. Thus we have the decomposition

$$\mathcal{K}_{\nu,p}(2\pi u)=\mathcal{K}^*_{\nu,p}(2\pi u)+\mathcal{K}^*_{-\nu,p}(2\pi\bar{u}), \tag{12.13}$$

provided $|\operatorname{Re}\nu|<p$. We now assume that $-p<\operatorname{Re}\nu<0$. Then, because of the absolute convergence mentioned above, we may exchange the order of integration in (12.12). We get

$$\begin{aligned}\mathcal{K}^*_{\nu,p}(2\pi u)&=(-1)^p\frac{1}{2\pi^2 i}\left|\frac{u}{2}\right|^{2p}\int_1^\infty y^{\nu-1}\left(\sqrt{y}e^{i\vartheta}+\frac{1}{\sqrt{y}e^{i\vartheta}}\right)^{2p}\\&\quad\times\int_{(1)}\exp\left(2\pi\xi-\frac{\pi|u|^2}{2\xi}\left|\sqrt{y}e^{i\vartheta}+\frac{1}{\sqrt{y}e^{i\vartheta}}\right|^2\right)\xi^{-2p-1}d\xi\,dy\\&=\frac{(-1)^p}{\pi}\int_1^\infty y^{\nu-1}\left(\frac{\sqrt{y}e^{i\vartheta}+(\sqrt{y}e^{i\vartheta})^{-1}}{|\sqrt{y}e^{i\vartheta}+(\sqrt{y}e^{i\vartheta})^{-1}|}\right)^{2p}\\&\quad\times J_{2p}\left(2\pi|u|\left|\sqrt{y}e^{i\vartheta}+\frac{1}{\sqrt{y}e^{i\vartheta}}\right|\right)dy.\end{aligned} \tag{12.14}$$

By the asymptotic property of the J-Bessel function, the last integral converges absolutely for $\operatorname{Re}\nu<\frac14$. This fact and the identity (12.13) gives rise to (12.1), if $p\neq 0$.

We shall next consider the case $p=0$. We are unable to use the formula (12.2). Nonetheless, the right side of (12.1) converges absolutely for $p=0$ and $|\operatorname{Re}\nu|<\frac14$. By Neumann's addition theorem for J_0 we see that it is equal to

$$\frac{2}{\pi}\sum_{m\in\mathbb{Z}}(-1)^m e^{2mi\vartheta}\int_0^\infty y^{2\nu-1}J_m(|u|y)J_m(|u|/y)dy, \tag{12.15}$$

where the necessary absolute convergence is easy to check. By Lemma 6 of [27] this is transformed into

$$\begin{aligned}&\frac{8}{\pi^2}\cos\pi\nu\sum_{m\in\mathbb{Z}}(-1)^m e^{2mi\vartheta}\int_0^{\pi/2}J_{2m}(|u|\sin\tau)K_{2\nu}(2|u|\cos\tau)d\tau\\&=\frac{8}{\pi^2}\cos\pi\nu\int_0^{\pi/2}\cos(2|u|\cos\vartheta\sin\tau)K_{2\nu}(2|u|\cos\tau)d\tau,\end{aligned} \tag{12.16}$$

where the second line depends on the definition of the J-Bessel function of an integral order. According to Lemma 8 of [26], the last integral is equal to $\frac18\pi^2\mathcal{K}_{\nu,0}(u)/\cos\pi\nu$. This ends the proof. ∎

Next, we turn to the mean of $\mathcal{K}_{\nu,p}$ over $\mathbb{C}^\times$. We shall later require that it is not too large.

Lemma 12.1. *Let*

$$2|\mathrm{Re}\,\nu| < \rho < \frac{1}{2}. \tag{12.17}$$

Then we have

$$\int_{\mathbb{C}^\times} |\mathcal{K}_{\nu,p}(u)|^2 |u|^{2\rho} d^\times u \ll (1+|p|)^{2\rho-1}, \tag{12.18}$$

where the implied constant depends only on ρ *and* $\mathrm{Re}\,\nu$.

Proof. The formula (12.1) and the Cauchy–Schwartz inequality give

$$\begin{aligned}|\mathcal{K}_{\nu,p}(re^{i\vartheta})|^2 \le &\frac{4}{\pi^2}\int_1^\infty (y^{2\mathrm{Re}\,\nu} + y^{-2\mathrm{Re}\,\nu})^2 y^{-\eta-1} dy \\ &\times \int_1^\infty y^{\eta-1} J_{2|p|}(r|ye^{i\vartheta} + (ye^{i\vartheta})^{-1}|)^2 dy,\end{aligned} \tag{12.19}$$

where we assume $4|\mathrm{Re}\,\nu| < \eta < 2\rho$. We multiply both sides by $r^{2\rho-1}$ and integrate with respect to r and ϑ, getting

$$\begin{aligned}&\int_{\mathbb{C}^\times} |\mathcal{K}_{\nu,p}(u)|^2 |u|^{2\rho} d^\times u \\ &\ll \int_1^\infty \int_0^{2\pi} y^{\eta-1}|ye^{i\vartheta} + (ye^{i\vartheta})^{-1}|^{-2\rho} d\vartheta\, dy \int_0^\infty J_{2|p|}(r)^2 r^{2\rho-1} dr.\end{aligned} \tag{12.20}$$

The right side converges. Then we invoke

$$\int_0^\infty J_{2|p|}(r)^2 r^{2\rho-1} dr = 2^{2\rho-1}\frac{\Gamma(1-2\rho)\Gamma(2|p|+\rho)}{\Gamma(1-\rho)^2\Gamma(2|p|+1-\rho)} \tag{12.21}$$

(see p. 403 of [35]). This ends the proof.

Remark. The integral formula (12.1) appears to be new; despite its classical outlook we have not been able to find its tabulation. An alternative proof of (12.1) is indicated in Section 15.

It seems worth remarking that $\mathcal{K}_{\nu,p}$ appears in the context of Section 2 as well: We consider, more generally than (1.1), the mean value

$$\int_{-\infty}^\infty |\zeta_{\mathrm{F}}(\tfrac{1}{2}+it)\zeta_{\mathrm{F}}(\tfrac{1}{2}+it, \tfrac{1}{2}p)|^2 g(t) dt, \tag{12.22}$$

where $p \in 2\mathbb{Z}$. Then we need to treat

$$\sum_{n(n+m)\neq 0} \sigma_\alpha(n, \tfrac{1}{2}p)\sigma_\beta(n+m, -\tfrac{1}{2}p) g^*(n/m; \gamma, \delta) \tag{12.23}$$

with $m \neq 0$. Corresponding to (2.31) we have the sum of Kloosterman sums

$$\sum_{c\neq 0} \frac{1}{|c|^2} S_{\mathrm{F}}(m,n;c)\,[g]_p\Big(\frac{2\pi}{c}\sqrt{mn};\alpha,\beta,\gamma,\delta\Big), \tag{12.24}$$

where

$$\begin{aligned}[g]_p(u;\alpha,\beta,\gamma,\delta) &= (|u|/2)^{-2(1+\alpha+\beta)} \sum_{q\in\mathbb{Z}} (-1)^{\max(|p|,|q|)} (iu/|u|)^{2(p+q)} \\ &\times \int_{(\eta)} \frac{\Gamma(1-s+\frac12|p+q|)\Gamma(1+\alpha-s+\frac12|p-q|)}{\Gamma(s+\frac12|p+q|)\Gamma(s-\alpha+\frac12|p-q|)} \tilde{g}_{p+q}(s;\gamma,\delta)(|u|/2)^{4s} ds.\end{aligned} \tag{12.25}$$

with $\eta < 1+\min(0,\mathrm{Re}\,\alpha)$. We compare this with (14.13), and are led to the expression

$$\begin{aligned}&[g]_p(u;\alpha,\beta,\gamma,\delta) \\ &= \pi i(u/|u|)^{2p}(|u|/2)^{2(1-\beta)} \int_{\mathbb{C}^\times} g^*(v;\gamma,\delta)\mathcal{K}_{-\alpha,p}(iu\sqrt{v})(v/|v|)^p|v|^{2+\alpha}d^\times v.\end{aligned} \tag{12.26}$$

This belongs to the family of Voronoï transforms in the theory of lattice points.

13. Sum formula. II

We are now ready to invert the sum formula (10.1). The result is one of the main assertions of the present article, and is embodied in

Theorem 13.1 (Kloosterman–Spectral sum formula). *Let f be an even function on $\mathbb{C}^\times$. Let us suppose that there exist constants ρ and σ such that $0<\rho<\frac12<\sigma$, and*

1. $f(u) = O(|u|^{2\sigma})$ *as* $|u| \downarrow 0$,
2. f *is six times continuously differentiable, and for* $a+b \le 6$

$$\int_{\mathbb{C}^\times} |(u\partial_u)^a f(u)|^2 |u|^{b-2\rho} d^\times u < \infty.$$

Then we have, for any non-zero $\omega_1,\omega_2 \in \mathbb{Z}[i]$,

$$\begin{aligned}&\sum_{c\neq 0} \frac{1}{|c|^2} S_{\mathrm{F}}(\omega_1,\omega_2;c) f\left(\frac{2\pi}{c}\sqrt{\omega_1\omega_2}\right) \\ &= 2\pi \sum_V |c_V(1)|^2 t_V(\omega_1)t_V(\omega_2)\mathrm{K}f(\nu_V,p_V) \\ &\quad - i \sum_{p\in 2\mathbb{Z}} \left(\frac{\omega_1\omega_2}{|\omega_1\omega_2|}\right)^p \int_{(0)} \frac{\sigma_\nu(\omega_1,-p/2)\sigma_\nu(\omega_2,-p/2)}{|\omega_1\omega_2|^\nu |\zeta_{\mathrm{F}}(1+\nu,p/2)|^2} \mathrm{K}f(\nu,p)\,d\nu,\end{aligned} \tag{13.1}$$

where the transformation K *is defined in* (11.1), *and* V *runs over all Hecke invariant right-irreducible cuspidal subspaces of* $L^2(\Gamma\backslash G)$ *together with the specifications in Section 8. The contour is the imaginary axis, and the convergence is absolute throughout.*

Proof. We denote the left and the right sides of (13.1) by $L^*_{\omega_1,\omega_2}(f)$ and $R^*_{\omega_1,\omega_2}(f)$, respectively. Let $X>0$ be large, and let $\chi(r)$ be a smooth function which is equal to 1 for $1/X \le r \le X$, and 0 for $r \le 1/(2X)$ and $r \ge 2X$, and monotonic otherwise. Also let ϕ be a smooth function on $\mathbb{C}$ such that

$$\int_{\mathbb{C}^\times} \phi\, d^\times u = 1, \quad \phi(u) \ge 0, \text{ and } \phi(u) = 0 \text{ if } |u-1| \ge X^{-1}. \tag{13.2}$$

With these we put

$$f_X(u) = \int_{\mathbb{C}^\times} \phi(u/v)\chi(|v|)f(v)\, d^\times v. \tag{13.3}$$

This function is smooth and compactly supported on $\mathbb{C}^\times$, and converges pointwise to f as $X \uparrow \infty$. By virtue of Theorems 10.1 and 11.1 coupled with Lemma 11.1, we have

$$L^*_{\omega_1,\omega_2}(f_X) = R^*_{\omega_1,\omega_2}(f_X). \tag{13.4}$$

By the condition 1. and by the bound (8.14) we have readily

$$\lim_{X\to\infty} L^*_{\omega_1,\omega_2}(f_X) = L^*_{\omega_1,\omega_2}(f). \tag{13.5}$$

To deal with the right side of (13.4), we observe that

$$u\partial_u f_X(u) = \int_{\mathbb{C}^\times} \phi(u/v) v\partial_v[\chi(|v|)f(v)]\, d^\times v, \tag{13.6}$$

and that

$$\mathrm{K}[\mathrm{b}_u f_X](\nu,p) = (\nu-p)^2 \mathrm{K} f_X(\nu,p) \tag{13.7}$$

with $\mathrm{b}_u = (u\partial_u)^2 + u^2$. The latter is due to the fact that $\mathcal{J}_{\nu,p}$ and thus $\mathcal{K}_{\nu,p}$ are eigenfunctions of Bessel's differential operator b_u with eigenvalue $(\nu-p)^2$. Invoking Lemma 12.1, we have

$$\mathrm{K} f_X(\nu,p) \ll \min\left(\|f_X\|_\rho,\ \|\mathrm{b}_u^3 f_X\|_\rho |\nu-p|^{-6}\right) \tag{13.8}$$

for any $\nu \in i\mathbb{R}$, $p \in \mathbb{Z}$, where $\|\cdot\|_\rho$ is the norm of the Hilbert space $L^2(\mathbb{C}^\times, |u|^{-2\rho} d^\times u)$. By the definition we have $\|f_X\|_\rho \ll \|f\|_\rho$. A multiple use of (13.6) gives that

$$\|u^a(u\partial_u)^b f_X\|_\rho \ll \|u^a(u\partial_u)^b \chi f\|_\rho \ll \sum_{j=0}^{b} \|u^a(u\partial_u)^j f\|_\rho, \tag{13.9}$$

since $(u\partial_u)^j\chi$ is bounded. This and the condition 2. imply that $\|\mathrm{b}_u^3 f_X\|_\rho$ is uniformly bounded. That is, we have, from (13.8),

$$\mathrm{K}f_X(\nu,p) \ll (1+|\nu|+|p|)^{-6} \tag{13.10}$$

uniformly in all involved parameters. Following the argument leading to (10.5), we find that

$$\lim_{X\to\infty} R^*_{\omega_1,\omega_2}(f_X) = R^*_{\omega_1,\omega_2}(f). \tag{13.11}$$

This ends the proof of the theorem. ■

Remark. Note the similarity between Theorem 13.1 and Theorem 2.3 of [25]. Comparing (11.1) with (2.4.8) of [25], the impression will be enhanced. One may improve Theorem 13.1 by relaxing the second condition. It appears, however, that our assertion is sufficiently precise for practical purposes. For a possible alternative approach to Theorem 13.1; see the final section.

The identity (13.4) does not contain the delta-term corresponding to that in (10.1). This is of course due to Lemma 11.1. Then, it might be worth remarking that the assertion (11.32) is a consequence of the spectral sum formula as well. Namely, one may prove it alternatively in the following way: We write (13.4) with the f in Lemma 11.1, in place of f_X, but without using the lemma, so that the delta-term remains in the identity. Specialize it by setting $\omega_1 = 1$ and $\omega_2 = n$, multiply both sides by $\zeta_{\mathrm{F}}(1-\beta)\sigma_{-\alpha}(n)|n|^{\alpha+\beta-1}$ with $|\mathrm{Re}\,\alpha| + \mathrm{Re}\,\beta < -2$, and sum over all non-zero $n \in \mathbb{Z}[i]$. Applying some rearrangement partly depending on the arguments in Sections 2 and 14, one is lead to the conclusion that $\zeta_{\mathrm{F}}(1-\beta)$ times the left side of (11.32) is regular at $\beta = 0$, which is naturally equivalent to (11.32). This is definitely far more complicated than the above proof, but seems to have certain interest of its own.

It seems possible to extend Theorem 13.1 to other discrete subgroups of $\mathrm{PSL}_2(\mathbb{C})$, provided we have a non-trivial bound for the sums corresponding to our S_{F}. Otherwise, the test functions in Theorem 10.1 have to live on a wider strip $|\mathrm{Re}\,\nu| \leqslant 1+\varepsilon$ with an $\varepsilon > 0$, and have prescribed zeros at $\nu = \pm 1$. Then the problem is that the transforms $\mathrm{K}f$, for f even, smooth, compactly supported, need not posses those zeros.

14. An explicit formula

In this final section we shall apply Theorem 13.1 to the sum (2.31) and establish a spectral decomposition of $\mathcal{Z}_2(g,\mathrm{F})$ defined in (1.1). The underlying principle is the same as in the rational case but the procedure is naturally more involved (cf. Sections 4.4–4.7 of [25]).

We have first to examine if the two conditions in Theorem 13.1 are satisfied by the function $[g](u;\alpha,\beta,\gamma,\delta)$ while (2.27) is assumed. The first condition is easy to check. As to the second we observe that $[g]$ is smooth and, together with its derivatives, of rapid decay as $|u| \uparrow \infty$; indeed, by virtue of Lemma 2.1, it is enough to move the contour in (2.32) far to the left. Thus we may restrict ourselves to the vicinity of the point $u = 0$. Then we note that $u\partial_u = \frac{1}{2}(r\partial_r - i\partial_\theta)$ with $u = re^{i\theta}$. This implies that when applied to $[g]$ the operator $u\partial_u$ does not change essentially the asymptotic behaviour of the function as $|u| \downarrow 0$. Note that again Lemma 2.1 plays a rôle. Hence we have to check only the case $a = b = 0$ in the condition 2. of Theorem 13.1. The confirmation is then immediate.

Now, let us put

$$\Phi_p(\nu) = \Phi_p(\nu; \alpha, \beta, \gamma, \delta; g) = \int_{\mathbf{C}^\times} \mathcal{K}_{\nu,p}(u)\, [g](u; \alpha, \beta, \gamma, \delta) d^\times u. \tag{14.1}$$

Theorem 13.1 gives, on (2.27),

$$S_{m,n}(\alpha, \beta, \gamma, \delta; g) = \{S^{(1)}_{m,n} + S^{(2)}_{m,n}\}(\alpha, \beta, \gamma, \delta; g), \tag{14.2}$$

where

$$S^{(1)}_{m,n}(\alpha, \beta, \gamma, \delta; g) = 2\pi \sum_V |c_V(1)|^2 t_V(m) t_V(n) \Phi_{p_V}(\nu_V), \tag{14.3}$$

$$S^{(2)}_{m,n}(\alpha, \beta, \gamma, \delta; g) = -i \sum_{p \in 2\mathbf{Z}} \left(\frac{mn}{|mn|}\right)^p \int_{(0)} \frac{\sigma_\nu(m, -p/2)\sigma_\nu(n, -p/2)}{|mn|^\nu |\zeta_{\mathrm{F}}(1+\nu, p/2)|^2} \Phi_p(\nu)\, d\nu. \tag{14.4}$$

To consider the function Φ_p, let $K_{\nu,p}(r, q)$ and $G(r, q)$ be the $2q$-th Fourier coefficients, in $\arg u$, of the functions $\mathcal{K}_{\nu,p}(u)$ and $[g](u; \alpha, \beta, \gamma, \delta)$, respectively. We have, from (2.32),

$$\begin{aligned} G(r, q) &= (r/2)^{-2(1+\alpha+\beta)} \\ &\times \int_{(\eta)} \frac{\Gamma(1 - s + \frac{1}{2}|q|)\Gamma(1 + \alpha - s + \frac{1}{2}|q|)}{\Gamma(s + \frac{1}{2}|q|)\Gamma(s - \alpha + \frac{1}{2}|q|)} \tilde{g}_q(s; \gamma, \delta)(r/2)^{4s} ds \end{aligned} \tag{14.5}$$

with $\eta < 1 + \min(0, \mathrm{Re}\,\alpha)$. Note that on (12.17) the assertion (12.18) induces

$$\sum_{q \in \mathbf{Z}} \int_0^\infty |K_{\nu,p}(r, q)|^2 r^{2\rho - 1} dr < \infty, \tag{14.6}$$

and that by the above discussion we have, given (2.27) and $0 < \rho < \frac{1}{2}$,

$$\sum_{q \in \mathbf{Z}} \int_0^\infty |G(r, q)|^2 r^{-2\rho - 1} dr < \infty. \tag{14.7}$$

These imply that

$$\Phi_p(\nu) = 2\pi \sum_{q \in \mathbf{Z}} \int_0^\infty K_{\nu,p}(r, q) G(r, q) \frac{dr}{r}. \tag{14.8}$$

Further, let $\tilde{K}_{\nu,p}(s,q)$ and $\tilde{G}(s,q)$ be the Mellin transforms of $K_{\nu,p}(r,q)$ and $G(r,q)$, respectively, as functions of r. Then the last expression is transformed into

$$\Phi_p(\nu) = -i\sum_{q\in\mathbb{Z}}\int_{(\rho)} \tilde{K}_{\nu,p}(s,q)\tilde{G}(-s,q)ds. \tag{14.9}$$

This is the result of an application of the Parseval formula for Mellin transforms to each integral in (14.8) (see Theorem 72 of [31]). The formula (14.5) is obviously equivalent to

$$\begin{aligned}\tilde{G}(s,q) = &\pi i\tilde{g}_q(\tfrac{1}{2}(1+\alpha+\beta)-\tfrac{1}{4}s;\gamma,\delta)\\ &\times 2^{s-1}\frac{\Gamma(\frac{1}{4}s+\frac{1}{2}(1-\alpha-\beta+|q|))\Gamma(\frac{1}{4}s+\frac{1}{2}(1+\alpha-\beta+|q|))}{\Gamma(\frac{1}{2}(1+\alpha+\beta+|q|)-\frac{1}{4}s)\Gamma(\frac{1}{2}(1-\alpha+\beta+|q|)-\frac{1}{4}s)},\end{aligned} \tag{14.10}$$

provided (2.27) and $\mathrm{Re}\, s > 2(\mathrm{Re}\,\beta + |\mathrm{Re}\,\alpha| - 1)$. To find $K_{\nu,p}(r,q)$ we combine Theorem 12.1 with Graf's addition theorem (formula (1) on p. 359 of [35]), getting

$$K_{\nu,p}(r,q) = (-1)^{\max(|p|,|q|)}\frac{2}{\pi}\int_0^\infty y^{2\nu-1}J_{|p+q|}(ry)J_{|p-q|}(r/y)dy \tag{14.11}$$

for $|\mathrm{Re}\,\nu| < \frac{1}{4}$ (cf. (12.15)). Via the formula

$$\int_0^\infty J_\xi(y)y^{s-1}dy = 2^{s-1}\frac{\Gamma(\frac{1}{2}(\xi+s))}{\Gamma(\frac{1}{2}(\xi-s)+1)} \tag{14.12}$$

with $-\mathrm{Re}\,\xi < \mathrm{Re}\, s < \frac{1}{2}$, and the Parseval formula for Mellin transforms, one may express $K_{\nu,p}(r,q)$ by an inverse Mellin transform. Then we get

$$\begin{aligned}\tilde{K}_{\nu,p}(s,q) = &(-1)^{\max(|p|,|q|)}\frac{2^{s-2}}{\pi}\\ &\times\frac{\Gamma(\frac{1}{4}s+\frac{1}{2}(\nu+|p+q|))\Gamma(\frac{1}{4}s-\frac{1}{2}(\nu-|p-q|))}{\Gamma(1-\frac{1}{4}s-\frac{1}{2}(\nu-|p+q|))\Gamma(1-\frac{1}{4}s+\frac{1}{2}(\nu+|p-q|))},\end{aligned} \tag{14.13}$$

provided $2|\mathrm{Re}\,\nu| < \mathrm{Re}\, s < 1-2|\mathrm{Re}\,\nu|$.

The combination of (14.9), (14.10), and (14.13) yields

Lemma 14.1. *The function Φ_p continues meromorphically to $\mathbb{C}^5$. We have the representation*

$$\begin{aligned}\Phi_p(\nu;\alpha,\beta,\gamma,\delta;g) = &\frac{1}{2}\sum_{q\in\mathbb{Z}}(-1)^{\max(|p|,|q|)}\\ &\times\int_{-i\infty}^{i\infty}\tilde{g}_q(s;\gamma,\delta)\Gamma_q(s;\alpha)\Gamma_{p,q}(s,\nu;\alpha,\beta)ds,\end{aligned} \tag{14.14}$$

where

$$\Gamma_q(s;\alpha) = \frac{\Gamma(1-s+\frac{1}{2}|q|)\Gamma(1-s+\alpha+\frac{1}{2}|q|)}{\Gamma(s+\frac{1}{2}|q|)\Gamma(s-\alpha+\frac{1}{2}|q|)}, \tag{14.15}$$

$$\begin{aligned}\Gamma_{p,q}(s,\nu;\alpha,\beta) &= \frac{\Gamma(s-\frac{1}{2}(\alpha+\beta-\nu+1)+\frac{1}{2}|p+q|)}{\Gamma(1-s+\frac{1}{2}(\alpha+\beta-\nu+1)+\frac{1}{2}|p+q|)} \\ &\times \frac{\Gamma(s-\frac{1}{2}(\alpha+\beta+\nu+1)+\frac{1}{2}|p-q|)}{\Gamma(1-s+\frac{1}{2}(\alpha+\beta+\nu+1)+\frac{1}{2}|p-q|)}.\end{aligned} \tag{14.16}$$

In (14.14) *it is supposed that the poles of* $\tilde{g}_q(s;\gamma,\delta)\Gamma_q(s;\alpha)$ *and those of* $\Gamma_{p,q}(s,\nu;\alpha,\beta)$ *are separated by the contour to the right and the left, respectively; and the parameters are such that the contour can be drawn. It follows in particular that if* $\mathrm{Re}\,\nu$, α, β, γ, *and* δ *are bounded, then we have, for any fixed* $A>0$,

$$\Phi_p(\nu;\alpha,\beta,\gamma,\delta;g) \ll (1+|\nu|+|p|)^{-A} \tag{14.17}$$

as $|\nu|+|p|$ *tends to infinity.*

Proof. We assume first (2.27) and (12.17). Then we may insert (14.10) and (14.13) into (14.9). After the change of variable $s \mapsto 4s-2(1+\alpha+\beta)$, we get the expression (14.14) with the contour (0). The expression for the general situation follows by analytic continuation. The meromorphy of Φ_p is an immediate consequence of (14.14). As to the bound (14.17), we need only to shift the contour in (14.14) far to the left. The resulting integral and the residues are estimated by Stirling's formula and (2.12). This ends the proof. ■

We assume (2.27) again, and collect (2.30)–(2.31) and (14.2)–(14.4). We obtain

$$B_m^{(1)}(\alpha,\beta;g^*(\cdot\,;\gamma,\delta)) = \left\{B_m^{(1,1)}+B_m^{(1,2)}\right\}(\alpha,\beta;g^*(\cdot\,;\gamma,\delta)). \tag{14.18}$$

Here we have

$$\begin{aligned}B_m^{(1,1)}(\alpha,\beta;g^*(\cdot\,;\gamma,\delta)) &= -2i\pi^{2\beta}|m|^{\alpha+\beta+1}\sum_V |c_V(1)|^2 t_V(m) \\ &\times H_V(\tfrac{1}{2}(1-\alpha-\beta))H_V(\tfrac{1}{2}(1+\alpha-\beta))\Phi_{p_V}(\nu_V),\end{aligned} \tag{14.19}$$

and

$$\begin{aligned}B_m^{(1,2)}(\alpha,\beta;g^*(\cdot\,;\gamma,\delta)) &= -\pi^{2\beta-1}|m|^{\alpha+\beta+1} \\ &\times \sum_{p\in\mathbb{Z}}\left(\frac{m}{|m|}\right)^{4p}\int_{(0)}\frac{\sigma_\nu(m,-2p)}{|m|^\nu|\zeta_{\mathrm{F}}(1+\nu,2p)|^2} \\ &\times \zeta_{\mathrm{F}}(\tfrac{1}{2}(1-\alpha-\beta+\nu),p)\zeta_{\mathrm{F}}(\tfrac{1}{2}(1-\alpha-\beta-\nu),-p) \\ &\times \zeta_{\mathrm{F}}(\tfrac{1}{2}(1+\alpha-\beta+\nu),p)\zeta_{\mathrm{F}}(\tfrac{1}{2}(1+\alpha-\beta-\nu),-p)\Phi_{4p}(\nu)d\nu.\end{aligned} \tag{14.20}$$

The expression (14.19) depends on (8.15), (8.21) and (10.13). On the other hand, (14.20) depends on the following formula: For any $a, b, c \in \mathbb{Z}$, $\tau, \xi \in \mathbb{C}$ we have, in the region of absolute convergence,

$$\frac{1}{4}\sum_{n\neq 0}(n/|n|)^{4a}\sigma_\tau(n,b)\sigma_\xi(n,c)|n|^{-2s} \tag{14.21}$$
$$= \frac{\zeta_{\rm F}(s,a)\zeta_{\rm F}(s-\tau,a+b)\zeta_{\rm F}(s-\xi,a+c)\zeta_{\rm F}(s-\tau-\xi,a+b+c)}{\zeta_{\rm F}(2s-\tau-\xi,2a+b+c)}.$$

This is an extension of Ramanujan's well-known identity for the product of four values of the Riemann zeta-function, and the proof is similar (see (1.3.3) of [30]).

Returning to (2.6), we specialize (2.29) and (14.18) with $\alpha = z_1 - z_2$, $\beta = z_3 - z_4$, $\gamma = z_1$, and $\delta = z_3$. We shall assume temporarily that

$$1 < \operatorname{Re} z_1 < \operatorname{Re} z_2 < \operatorname{Re} z_1 + 1, \quad 1 < \operatorname{Re} z_3 < \operatorname{Re} z_4 - 3. \tag{14.22}$$

The condition (2.27) is satisfied. Hence, via (2.28)–(2.29) and (14.18)–(14.20), the formula (2.6) is transformed into

$$\mathfrak{I}(z_1,z_2,z_3,z_4;g) = \left\{\mathfrak{I}^{(r)} + \mathfrak{I}^{(c)} + \mathfrak{I}^{(e)}\right\}(z_1,z_2,z_3,z_4;g) \tag{14.23}$$

in the domain (14.22). Here we have

$$\mathfrak{I}^{(r)}(z_1,z_2,z_3,z_4;g) = \frac{\zeta_{\rm F}(z_1+z_3)\zeta_{\rm F}(z_1+z_4)\zeta_{\rm F}(z_2+z_3)\zeta_{\rm F}(z_2+z_4)}{4\zeta_{\rm F}(z_1+z_2+z_3+z_4)}\hat{g}(0) \tag{14.24}$$
$$+\pi\frac{\zeta_{\rm F}(z_1+z_3-1)\zeta_{\rm F}(z_2+z_4)\zeta_{\rm F}(1+z_2-z_1)\zeta_{\rm F}(1+z_4-z_3)}{2\zeta_{\rm F}(2+z_2+z_4-z_1-z_3)}\tilde{g}_0(1;z_1,z_3)$$
$$+\pi\frac{\zeta_{\rm F}(z_2+z_3-1)\zeta_{\rm F}(z_1+z_4)\zeta_{\rm F}(1+z_1-z_2)\zeta_{\rm F}(1+z_4-z_3)}{2\zeta_{\rm F}(2+z_1+z_4-z_2-z_3)}$$
$$\times\tilde{g}_0(1+z_1-z_2;z_1,z_3),$$

$$\mathfrak{I}^{(c)}(z_1,z_2,z_3,z_4;g) = \frac{\pi^{2(z_3-z_4)}}{2i}\sum_V |c_V(1)|^2 H_V(\tfrac12(z_1+z_2+z_3+z_4-1)) \tag{14.25}$$
$$\times H_V(\tfrac12(z_2+z_4-z_1-z_3+1))H_V(\tfrac12(z_1+z_4-z_2-z_3+1))$$
$$\times\Phi_{p_V}(\nu_V;z_1-z_2,z_3-z_4,z_1,z_3;g),$$

and

$$\mathfrak{I}^{(e)}(z_1,z_2,z_3,z_4;g) = \sum_{p\in\mathbf{Z}}\mathfrak{I}_p^{(e)}(z_1,z_2,z_3,z_4;g), \tag{14.26}$$

where

$$\mathfrak{I}_p^{(e)}(z_1,z_2,z_3,z_4;g) = -\frac{\pi^{2(z_3-z_4)-1}}{4}\int_{(0)}\zeta_{\rm F}(\tfrac12(z_1+z_2+z_3+z_4-1+\nu),p) \tag{14.27}$$
$$\times\zeta_{\rm F}(\tfrac12(z_1+z_2+z_3+z_4-1-\nu),-p)\zeta_{\rm F}(\tfrac12(z_2+z_4-z_1-z_3+1+\nu),p)$$
$$\times\zeta_{\rm F}(\tfrac12(z_2+z_4-z_1-z_3+1-\nu),-p)\zeta_{\rm F}(\tfrac12(z_1+z_4-z_2-z_3+1+\nu),p)$$
$$\times\zeta_{\rm F}(\tfrac12(z_1+z_4-z_2-z_3+1-\nu),-p)\frac{\Phi_{4p}(\nu;z_1-z_2,z_3-z_4,z_1,z_3;g)}{\zeta_{\rm F}(1+\nu,2p)\zeta_{\rm F}(1-\nu,-2p)}d\nu.$$

We then observe that by Lemma 8.1 the function $H_V(s)$ is entire, and of polynomial order in s, p_V and ν_V if Re s is bounded. Thus $\mathfrak{J}^{(c)}$ is meromorphic over $\mathbb{C}^4$ by virtue of Corollary 10.1 and Lemma 14.1. To see the situation at the point $\mathrm{p}_{\frac{1}{2}}$, we note that if (z_1, z_2, z_3, z_4) is sufficiently close to $\mathrm{p}_{\frac{1}{2}}$, and $\nu \in i\mathbb{R}$ (see (8.4)), then we may take $(\frac{3}{4})$ as a contour in (14.14). This implies readily that $\mathfrak{J}^{(c)}$ is regular at $\mathrm{p}_{\frac{1}{2}}$, and we have

$$\mathfrak{J}^{(c)}(\mathrm{p}_{\frac{1}{2}};g) = \frac{1}{2i}\sum_V |c_V(1)|^2 H_V(\tfrac{1}{2})^3 \Phi_{p_V}(\nu_V;0,0,\tfrac{1}{2},\tfrac{1}{2};g). \tag{14.28}$$

As to the sum of $\mathfrak{J}_p^{(e)}$ over $p \neq 0$, it is analogous to $\mathfrak{J}^{(c)}$, and we have

$$\sum_{\substack{p\in\mathbf{Z}\\ p\neq 0}} \mathfrak{J}_p^{(e)}(\mathrm{p}_{\frac{1}{2}};g) = -\frac{1}{4\pi}\sum_{\substack{p\in\mathbf{Z}\\ p\neq 0}} \int_{(0)} \frac{|\zeta_{\mathrm{F}}(\frac{1}{2}(1+\nu),p)|^6}{|\zeta_{\mathrm{F}}(1+\nu,2p)|^2}\Phi_{4p}(\nu;0,0,\tfrac{1}{2},\tfrac{1}{2};g)d\nu. \tag{14.29}$$

It remains to consider the function $\mathfrak{J}_0^{(e)}$. We note first that it is meromorphic over $\mathbb{C}^4$. This can be proved either shifting the contour appropriately or simply observing that all terms, except for $\mathfrak{J}_0^{(e)}$, in (14.23) are already known to be meromorphic over $\mathbb{C}^4$. Thus, to see the nature of $\mathfrak{J}_0^{(e)}$ near the point $\mathrm{p}_{\frac{1}{2}}$, we may let (z_1, z_2, z_3, z_4) approach to $\mathrm{p}_{\frac{1}{2}}$ in a specific way, as we shall do shortly.

We start from the domain defined by (14.22), where we have the representation (14.27) with $p = 0$. We shall move the contour, closely following the discussion on the corresponding part of the rational case (Section 4.7 of [25]). In the process we shall encounter singularities of the integrand, and the difficulty lies in that they depend on z_1, z_2, z_3, z_4. To facilitate the discussion we put

$$\begin{gathered}\nu_1 = z_1 + z_2 + z_3 + z_4 - 3,\ \nu_2 = z_2 + z_4 - z_1 - z_3 - 1,\\ \nu_3 = z_1 + z_4 - z_2 - z_3 - 1.\end{gathered} \tag{14.30}$$

The zeta-part of the integrand in $\mathfrak{J}_0^{(e)}$ has singularities only at the six points $\pm\nu_1$, $\pm\nu_2$, $\pm\nu_3$ and at the zeros of $\zeta_{\mathrm{F}}(1+\nu)\zeta_{\mathrm{F}}(1-\nu)$. Then we make an observation:

Lemma 14.2. *The singularities of $\Phi_0(\nu; z_1 - z_2, z_3 - z_4, z_1, z_3; g)$ as a function of ν is contained in the set*

$$\Big\{\pm(\nu_1 + 2a), \pm(\nu_2 + 2b), \pm(\nu_3 + 2c) : \mathbb{Z} \ni a, b, c \geq 1\Big\}. \tag{14.31}$$

Proof. The singularities can occur only when we are unable to draw the contour in (14.14), that is,

$$\begin{aligned}&\Big\{\tfrac{1}{2}(\alpha+\beta\pm\nu+1) - l_\pm : \mathbb{Z}\ni l_\pm \geq 0\Big\}\\ &\cap\Big\{1+l_1,\ 1+\alpha+l_2,\ \gamma+\delta+l_3 : \mathbb{Z}\ni l_1,l_2,l_3\geq 0\Big\}\neq\emptyset.\end{aligned} \tag{14.32}$$

Such situations are covered by (14.31) under the current specialization. This ends the proof. ∎

We now set

$$z_1 = \frac{1}{2} + t,\ z_2 = \frac{1}{2} + 2t,\ z_3 = \frac{1}{2} + t,\ z_4 = \frac{1}{2} + 6t; \quad |\mathrm{Im}\, t| < \varepsilon_0 \tag{14.33}$$

with a sufficiently small ε_0. If $\frac{2}{3} < \mathrm{Re}\, t < 1$, then (14.22) is satisfied, and moreover the points $\nu_1 = 10t - 1$, $\nu_2 = 6t - 1$, $\nu_3 = 4t - 1$ are not in the set (14.31). Thus, the last lemma implies that we can move the contour in $\mathfrak{J}_0^{(e)}$ so that the points ν_1, ν_2, ν_3 are on the left of the new contour, but none of the points in (14.31) and zeros $\zeta_{\mathrm{F}}(1-\nu)\zeta_{\mathrm{F}}(1+\nu)$ are encountered in the process. Leaving the residues at ν_1, ν_2 and ν_3 for a later discussion, we consider the resulting integral as a function of t as $t \to 0$, while keeping $\mathrm{Re}\, t > 0$ and moving the contour stepwise. We observe, via the last lemma, that, except for the cases $t = \frac{2}{3}, \frac{1}{2}, \frac{1}{3}$, we can draw the contour. These exceptional points obviously make no trouble; for instance we may assume $\mathrm{Im}\, t > 0$. Thus the integral continues analytically to a small right semicircle centered at the origin. Then, having t in this domain, we shift the contour back to the original, i.e., the imaginary axis. This time we encounter singularities at $-\nu_1$, $-\nu_2$, and $-\nu_3$ but none else. Note that at this stage we may leave the specialization (14.33), and suppose, instead, that (z_1, z_2, z_3, z_4) are in a small neighbourhood of $\mathrm{p}_{\frac{1}{2}}$. The integral $\mathfrak{J}_{0,*}^{(e)}$ thus obtained is regular at $\mathrm{p}_{\frac{1}{2}}$, and we see readily that

$$\mathfrak{J}_{0,*}^{(e)}(\mathrm{p}_{\frac{1}{2}}; g) = -\frac{1}{4\pi} \int_{(0)} \frac{|\zeta_{\mathrm{F}}(\frac{1}{2}(1+\nu))|^6}{|\zeta_{\mathrm{F}}(1+\nu)|^2} \Phi_0(\nu; 0, 0, \tfrac{1}{2}, \tfrac{1}{2}; g)\, d\nu. \tag{14.34}$$

Gathering these, we obtain the assertion

$$\mathfrak{J}(\mathrm{p}_{\frac{1}{2}}; g) = \left\{ M + \mathfrak{J}^{(c)} + \mathfrak{J}_*^{(e)} \right\}(\mathrm{p}_{\frac{1}{2}}; g). \tag{14.35}$$

Here $\mathfrak{J}_*^{(e)}$ is the sum of the right side of (14.27) over all $p \in \mathbb{Z}$, but with a different (z_1, z_2, z_3, z_4). On the other hand M is the sum of $\mathfrak{J}^{(r)}$ and the contribution of the poles at $\nu = \pm\nu_1$, $\pm\nu_2$, and $\pm\nu_3$ that we encountered in the above procedure. We stress that M is regular at $\mathrm{p}_{\frac{1}{2}}$. This is because all other functions involved in (14.35) are regular at $\mathrm{p}_{\frac{1}{2}}$.

Hence it remains for us to compute $M(z_1, z_2, z_3, z_4; g)$. We have, in a small neighbourhood of $\mathrm{p}_{\frac{1}{2}}$,

$$M(z_1, z_2, z_3, z_4; g) = \frac{\zeta_{\mathrm{F}}(z_1+z_3)\zeta_{\mathrm{F}}(z_1+z_4)\zeta_{\mathrm{F}}(z_2+z_3)\zeta_{\mathrm{F}}(z_2+z_4)}{4\zeta_{\mathrm{F}}(z_1+z_2+z_3+z_4)} \hat{g}(0) \tag{14.36}$$

$$+ \pi \frac{\zeta_{\mathrm{F}}(z_1+z_3-1)\zeta_{\mathrm{F}}(z_2+z_4)\zeta_{\mathrm{F}}(1+z_2-z_1)\zeta_{\mathrm{F}}(1+z_4-z_3)}{2\zeta_{\mathrm{F}}(2+z_2+z_4-z_1-z_3)} \tilde{g}_0(1; z_1, z_3)$$

$$+\pi\frac{\zeta_{\rm F}(z_2+z_3-1)\zeta_{\rm F}(z_1+z_4)\zeta_{\rm F}(1+z_1-z_2)\zeta_{\rm F}(1+z_4-z_3)}{2\zeta_{\rm F}(2+z_1+z_4-z_2-z_3)}\tilde{g}_0(1+z_1-z_2;z_1,z_3)$$

$$+i\pi^{2(z_3-z_4)+1}\frac{\zeta_{\rm F}(z_2+z_4-1)\zeta_{\rm F}(2-z_1-z_3)\zeta_{\rm F}(z_1+z_4-1)\zeta_{\rm F}(2-z_2-z_3)}{4\zeta_{\rm F}(4-z_1-z_2-z_3-z_4)}$$

$$\times\Phi_0(z_1+z_2+z_3+z_4-3;z_1-z_2,z_3-z_4,z_1,z_3;g)$$

$$+i\pi^{2(z_3-z_4)+1}\frac{\zeta_{\rm F}(z_2+z_4-1)\zeta_{\rm F}(z_1+z_3)\zeta_{\rm F}(z_4-z_3)\zeta_{\rm F}(z_1-z_2+1)}{4\zeta_{\rm F}(2-z_2-z_4+z_1+z_3)}$$

$$\times\Phi_0(z_2+z_4-z_1-z_3-1;z_1-z_2,z_3-z_4,z_1,z_3;g)$$

$$+i\pi^{2(z_3-z_4)+1}\frac{\zeta_{\rm F}(z_1+z_4-1)\zeta_{\rm F}(z_2+z_3)\zeta_{\rm F}(z_4-z_3)\zeta_{\rm F}(z_2-z_1+1)}{4\zeta_{\rm F}(2-z_1-z_4+z_2+z_3)}$$

$$\times\Phi_0(z_1+z_4-z_2-z_3-1;z_1-z_2,z_3-z_4,z_1,z_3;g).$$

This is obviously a linear integral transform of the weight function g. The six members on the right side have singularities at $\mathrm{p}_{\frac12}$, but these have to cancel out each other as M is regular at the point. Thus, what matters actually are the constant terms m_j $(1\le j\le 6)$, respectively, in the Laurent series expansions of these members at $\mathrm{p}_{\frac12}$. That is, we have

$$M(\mathrm{p}_{\frac12};g)=\sum_{j=1}^{6}m_j. \tag{14.37}$$

The computation of m_j can be carried out in much the same way as in the rational case (see pp. 176–178 of [25]). It is possible to write $M(\mathrm{p}_{\frac12};g)$ down explicitly in terms of g and derivatives of the Γ-function, but we stop here to restrict ourselves to the description of the overall structure of our subject.

To state our final result we put $\Lambda_{\nu,p}(g)=(2i)^{-1}\Phi_p(\nu;0,0,\frac12,\frac12;g)$, and $M_{\rm F}(g)=M(\mathrm{p}_{\frac12},g)+a_0g(\frac12 i)+b_0g(-\frac12 i)+a_1g'(\frac12 i)+b_1g'(-\frac12 i)$ with a_0, a_1, b_0, b_1 as in (2.2). We thus have established

Theorem 14.1. *Let g be as in* (1.1). *Then, with the transformations $M_{\rm F}$ and $\Lambda_{\nu,p}$ defined above, we have the identity*

$$\mathcal{Z}_2(g,\mathrm{F})=M_{\rm F}(g)+\sum_V|c_V(1)|^2H_V(\tfrac12)^3\Lambda_{\nu_V,p_V}(g) \tag{14.38}$$

$$+\frac{1}{2\pi i}\sum_{p\in\mathbf{Z}}\int_{(0)}\frac{|\zeta_{\rm F}(\frac12(1+\nu),p)|^6}{|\zeta_{\rm F}(1+\nu,2p)|^2}\Lambda_{\nu,4p}(g)d\nu,$$

where V runs over all Hecke invariant right-irreducible cuspidal subspaces of $L^2(\Gamma\backslash G)$ together with the specifications in Section 8. The contour is the imaginary axis, and the convergence is absolute throughout.

Remark. Despite the special nature of our dissection in (2.6) the arithmetic ingredients in (14.38), i.e., the functions $H_V(s)$ and $\zeta_F(s,p)$ are in fact defined over integral ideals of $\mathbb{Q}(i)$.

15. Concluding remarks

In this final section we shall develop a discussion of elements involved in our main result (14.38), in the light of recent developments made to understand the explicit formula for $\mathcal{Z}_2(g,\mathbb{Q})$, the fourth moment of the Riemann zeta-function $\zeta = \zeta_{\mathbb{Q}}$. We shall also ponder on an intriguing nature of (14.38).

Theorem 4.2 of [25] is now translated into

$$\mathcal{Z}_2(g,\mathbb{Q}) = M_{\mathbb{Q}}(g) + \sum_{\mathrm{V}} |c_{\mathrm{V}}(1)|^2 \mathrm{H}_{\mathrm{V}}(\tfrac{1}{2})^3 \Lambda_{\nu_{\mathrm{V}}}(g) \qquad (15.1)$$
$$+ \frac{1}{2\pi i}\int_{(0)} \frac{|\zeta(\frac{1}{2}(1+\nu))|^6}{|\zeta(1+\nu)|^2}\Lambda_\nu(g)d\nu.$$

Here $M_{\mathbb{Q}}$ has a construction similar to M_{F}, V runs over all Hecke invariant right-irreducible cuspidal subspaces of $L^2(\mathrm{PSL}_2(\mathbb{Z})\backslash\mathrm{PSL}_2(\mathbb{R}))$, and $c_{\mathrm{V}}(n)$ $(\mathbb{Z} \ni n \neq 0)$ are the Fourier coefficients of V, to which the Hecke series H_{V} is associated. The ν_{V} is the spectral parameter of V; that is, being restricted to V, the Casimir operator over $\mathrm{PSL}_2(\mathbb{R})$ becomes the constant multiplication $(\nu_{\mathrm{V}}^2 - \frac{1}{4})\cdot 1$. The functional Λ_ν is to be made precise shortly.

The similarity between the formulas (14.38) and (15.1) appears to the authors to suggest the existence of a geometric structure yet to be discovered. In particular, these results are expected to extend to a wide family of automorphic L-functions (cf. [16]). To enhance this observation, we quote, from [28] with minor changes of notation, the integral representation

$$\Lambda_\nu(g) = \int_0^\infty \frac{\hat{g}(\log(1+1/r))}{(r(r+1))^{\frac{1}{2}}}\Xi_\nu(r)dr, \qquad (15.2)$$

where $\hat{g}$ is as above, and

$$\Xi_\nu(r) = \frac{1}{2}\int_{\mathbb{R}^\times} j_\nu(u/r) j_0(-u)\frac{d^\times u}{\sqrt{|u|}}, \quad \mathbb{R}^\times = \mathbb{R}\backslash\{0\}, \quad d^\times u = du/|u|. \qquad (15.3)$$

Here

$$j_\nu(u) = \pi\frac{\sqrt{|u|}}{\sin \pi\nu}\left\{J_{-\nu}^{\mathrm{sgn}(u)}(4\pi\sqrt{|u|}) - J_\nu^{\mathrm{sgn}(u)}(4\pi\sqrt{|u|})\right\}, \qquad (15.4)$$

with $J_\nu^+ = J_\nu$, $J_\nu^- = I_\nu$. Correspondingly, we have, for $\mathrm{F} = \mathbb{Q}(i)$,

$$\Lambda_{\nu,p}(g) = \int_{\mathbb{C}} \frac{\hat{g}(2\log|1+1/u|)}{|u(u+1)|}\Xi_{\nu,p}(u)d_+u, \qquad (15.5)$$

where

$$\Xi_{\nu,p}(u) = \frac{1}{16\pi}\int_{\mathbb{C}^\times} j_{\nu,p}\left(\sqrt{v/u}\right) j_{0,0}\left(\sqrt{-v}\right)\frac{d^\times v}{|v|}, \tag{15.6}$$

with

$$j_{\nu,p}(u) = 2\pi^2|u|^2\mathcal{K}_{\nu,p}(2\pi u) = 2\pi^2\frac{|u|^2}{\sin\pi\nu}\left(\mathcal{J}_{-\nu,-p}(2\pi u) - \mathcal{J}_{\nu,p}(2\pi u)\right). \tag{15.7}$$

The last three formulas follow readily from (12.26), with $p=0$, and (14.1). As a matter of fact, the Bessel kernel j_ν that originates in Kuznetsov's works [18], [19] can be identified as the Bessel function of irreducible representations of the group $\mathrm{PSL}_2(\mathbb{R})$, and so is the $j_{\nu,p}$ in its relation with $\mathrm{PSL}_2(\mathbb{C})$. For these facts see [28] and [7], respectively. This interpretation of j_ν is given in Cogdell and Piatetski-Shapiro [8]; and [28] contains an alternative and rigorous approach to it via the concept of local functional equations of Jacquet and Langlands [15]. Its extension (15.5)–(15.7) is proved in [7]. Hence, the resemblance between (14.38) and (15.1) in fact reaches deeper than the sheer outlook suggests. At any events, the last expressions show how tightly the mean values of zeta-functions are related to the structure of function spaces over linear Lie groups.

This is naturally the same with sum formulas of Kloosterman sums. The work [8] in fact indicates a way to directly connect the Kloosterman–Spectral sum formula with j_ν, without the inversion procedure of the Spectral-Kloosterman sum formula for $\mathrm{PSL}_2(\mathbb{R})$ as Kuznetsov did. Since the principal means on which [8] is based have been extended to the present situation, in [7] as remarked above, one may argue that we could prove our Theorem 13.1 without first establishing Theorem 10.1. That appears to be the case, but in the present work we have chosen the way to extend the argument of [23] to include all K-aspects. This is because the combination of the Jacquet and the Goodman–Wallach operators provides us with a flexibility, perhaps greater than the extension of [8] could. Moreover, the present version of the sum formula for $\mathrm{PSL}_2(\mathbb{C})$ is more suitable than existing ones for applications in the study of Kloosterman sums and in the investigation of the distribution of automorphic spectral data. It should, however, be remarked that Theorem 12.1, the above proof of which depends on the Goodman–Wallach operator, could be derived also from the interpretation of $j_{\nu,p}$ as Bessel functions of representation of $\mathrm{PSL}_2(\mathbb{C})$. For this see [7] again.

Finally, we make a naive comparison between (14.38) and (15.1), in their asymptotic aspects. Here exists a remarkable difference between these formulas. That concerns the nature of the term $M_{\mathrm{F}}(g)$ in (14.38). In (15.1) the $M_{\mathrm{Q}}(g)$ is indeed the main term in the sense that with a specialization of g it gives rise to the main term in the asymptotic formula

$$\int_{-T}^{T} |\zeta(\tfrac{1}{2}+it)|^4 dt = TP_4(\log T) + O(T^{\frac{2}{3}}\log^8 T) \tag{15.8}$$

as $T\uparrow\infty$, where P_4 is a polynomial of order 4 (Theorem 5.2 of [25]). The function that is the counterpart of the above M for ζ has an expression similar to (14.36), and at the point $\mathrm{p}_{\frac{1}{2}}$ the terms involved in it have singularities of order 4 at most; see Section 4.7 of [25]. Analogously the functions on the right of (14.36) have singularities of order 4 at $\mathrm{p}_{\frac{1}{2}}$, and the construction of $M_{\mathrm{F}}(g)$ is similar to that of $M_{\mathrm{Q}}(g)$. We have, however,

$$\int_{-T}^{T} |\zeta_{\mathrm{F}}(\tfrac{1}{2}+it)|^4 dt = \Omega(T\log^8 T), \tag{15.9}$$

which can be proved by the argument in Section 7.19 of [30]. Thus, it is hard to regard $M_{\mathrm{F}}(g)$ as the main term in (14.38). This appears to raise a basic question about the fourth moment of ζ_{F}, and remotely the same about the eighth moment of the Riemann zeta-function.

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