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ON THE MOMENTS OF HECKE SERIES AT CENTRAL POINTS

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Abstract: We prove, in standard notation from spectral theory, the following asymptotic formulas:

$$\sum_{\kappa_j \leqslant K} \alpha_j H_j^3(\frac{1}{2}) = K^2 P_3(\log K) + O(K^{5/4}\log^{37/4} K)$$

and

$$\sum_{\kappa_j \leqslant K} \alpha_j H_j^4(\frac{1}{2}) = K^2 P_6(\log K) + O(K^{3/2}\log^{25/2} K),$$

where $P_3(x)$ and $P_6(x)$ are polynomials of degree three and six, whose coefficients may be explicitly evaluated.
Keywords: Hecke series, Maass wave forms, hypergeometric function, exponential sums

1. Introduction and statement of results

The purpose of this paper is to obtain asymptotic formulas for sums of $H_j^3(\frac{1}{2})$ and $H_j^4(\frac{1}{2})$, where $H_j(s)$ is the Hecke series, to be defined below. Sums with $H_j(\frac{1}{2})$ are important for several reasons, one of which is that they appear in the spectral decomposition of weighted integrals involving $|\zeta(\frac{1}{2}+it)|^4$, which is of fundamental importance in the theory of the Riemann zeta-function $\zeta(s)$.

We shall first present the relevant notation involving the spectral theory of the non-Euclidean Laplacian will be given below. For a competent and extensive account of spectral theory the reader is referred to Y. Motohashi's monograph [15].

Let $\{\lambda_j = \kappa_j^2 + \frac{1}{4}\} \cup \{0\}$ be the eigenvalues (discrete spectrum) of the hyperbolic Laplacian

$$\Delta = -y^2\left(\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2\right)$$

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acting over the Hilbert space composed of all Γ-automorphic functions which are square integrable with respect to the hyperbolic measure ($\Gamma = \mathrm{PSL}(2,\mathbb{Z})$). Let $\{\psi_j\}_{j=1}^{\infty}$ be a maximal orthonormal system such that $\Delta\psi_j = \lambda_j\psi_j$ for each $j \geqslant 1$ and $T(n)\psi_j = t_j(n)\psi_j$ for each integer $n \in \mathbb{N}$, where

$$(T(n)f)(z) = \frac{1}{\sqrt{n}} \sum_{ad=n} \sum_{b=1}^{d} f\left(\frac{az+b}{d}\right)$$

is the Hecke operator. We shall further assume that $\psi_j(-\bar{z}) = \varepsilon_j\psi_j(z)$ with $\varepsilon_j = \pm 1$. We then define ($s = \sigma + it$ will denote a complex variable)

$$H_j(s) = \sum_{n=1}^{\infty} t_j(n)n^{-s} \qquad (\sigma > 1),$$

which is the Hecke series associated with the Maass wave form $\psi_j(z)$, and which can be continued to an entire function. It satisfies the functional equation

$$H_j(s) = 2^{2s-1}\pi^{2s-2}\Gamma(1-s+i\kappa_j)\Gamma(1-s-i\kappa_j)(\varepsilon_j\cosh(\pi\kappa_j) - \cos(\pi s))H_j(1-s),$$

which by the Phragmén–Lindelöf principle (convexity) implies the bound

$$H_j(\tfrac{1}{2}) \ll_\varepsilon \kappa_j^{\frac{1}{2}+\varepsilon}, \tag{1.1}$$

where here and later ε denotes arbitrarily small, positive constants, not necessarily the same ones at each occurrence. It is also important to note that, from the work of Katok–Sarnak [9], it is known that $H_j(\frac{1}{2}) \geqslant 0$.

The sharpest asymptotic formula for sums of $\alpha_j H_j^2(\frac{1}{2})$ is due to Y. Motohashi [14]. His result is

$$\sum_{\kappa_j \leqslant T} \alpha_j H_j^2(\tfrac{1}{2}) = 2\pi^{-2}T^2(\log T + \gamma - \tfrac{1}{2} - \log(2\pi)) + O(T\log^6 T), \tag{1.2}$$

where γ is Euler's constant,

$$\alpha_j = |\varrho_j(1)|^2(\cosh\pi\kappa_j)^{-1},$$

and $\varrho_j(1)$ is the first Fourier coefficient of $\psi_j(z)$.

In what concerns known results on sums of $\alpha_j H_j^3(\frac{1}{2})$ and $\alpha_j H_j^4(\frac{1}{2})$ we have (see [15, Chapter 3])

$$\sum_{\kappa_j \leqslant K} \alpha_j H_j^4(\tfrac{1}{2}) \ll K^2\log^{15} K \tag{1.3}$$

and

$$\sum_{j=1}^{\infty} \alpha_j H_j^3(\tfrac{1}{2})h_0(\kappa_j) = \left(\frac{8}{3} + O\left(\frac{1}{\log K}\right)\right)\pi^{-3/2}K^3G\log^3 K \tag{1.4}$$

with

$$K^{\frac{1}{2}}\log^5 K \leqslant G \leqslant K^{1-\varepsilon}, \tag{1.5}$$

$$h_0(r) = (r^2+\tfrac14)\left\{\exp\left(-\left(\frac{r-K}{G}\right)^2\right) + \exp\left(-\left(\frac{r+K}{G}\right)^2\right)\right\}. \tag{1.6}$$

In [5] the author proved that

$$\sum_{K\leqslant\kappa_j\leqslant K+1} \alpha_j H_j^3(\tfrac12) \ll_\varepsilon K^{1+\varepsilon}. \tag{1.7}$$

This result immediately implies, since $H_j(\frac12) \geqslant 0$ and $\alpha_j \gg \kappa_j^{-\varepsilon}$ (see H. Iwaniec [6]), that

$$H_j(\tfrac12) \ll_\varepsilon \kappa_j^{\frac13+\varepsilon}, \tag{1.8}$$

which improves the convexity bound (1.1), and represents hitherto the sharpest known unconditional upper bound for $H_j(\frac12)$. The bound (1.8) also follows from the result of M. Jutila [7], namely

$$\sum_{K\leqslant\kappa_j\leqslant K+K^{1/3}} \alpha_j H_j^4(\tfrac12) \ll_\varepsilon K^{\frac43+\varepsilon}, \tag{1.9}$$

and an extension of the bound (1.9) to sums of $|H_j(\frac12+it)|^4$ has been attained by Jutila–Motohashi [8].

Note that (1.7) and (1.9) do not seem to apply one another, and that for the derivation of (1.8) from (1.9) the non-negativity of $H_j(\frac12)$ is not needed.

Our new results on sums of sums of $\alpha_j H_j^3(\frac12)$ and $\alpha_j H_j^4(\frac12)$ are contained in

Theorem 1. *We have*

$$\sum_{\kappa_j\leqslant K} \alpha_j H_j^3(\tfrac12) = K^2 P_3(\log K) + O(K^{5/4}\log^{37/4} K), \tag{1.10}$$

where $P_3(x)$ is a polynomial of degree three with leading coefficient $4/(3\pi^2)$, whose remaining coefficients may be explicitly evaluated.

Theorem 2. *We have*

$$\sum_{\kappa_j\leqslant K} \alpha_j H_j^4(\tfrac12) = K^2 P_6(\log K) + O(K^{3/2}\log^{25/2} K), \tag{1.11}$$

where $P_6(x)$ is a polynomial of degree six with leading coefficient $16/(15\pi^4)$, whose remaining coefficients may be explicitly evaluated.

The proofs of (1.10) and (1.11), which will be given in subsequent sections, depend on several ingredients. Besides the transformation formulas for sums of

$\alpha_j H_j^k(\frac{1}{2})$ (see Section 3), two salient ones are the short interval bounds (1.7) and (1.9), and the estimates for the sixth and eighth moments of $|\zeta(\frac{1}{2}+it)|$. Indeed, it is a deep and beautiful fact that sums of $\alpha_j H_j^k(\frac{1}{2})$ and moments of $|\zeta(\frac{1}{2}+it)|^{2k}$ $(k \in \mathbb{N})$ are closely related, at least for $k \leqslant 4$. Both quantities tend to increase in complexity as k increases. One of the reasons why Motohashi was able to get the sharp error term $O(T\log^6 T)$ in (1.2) was that the continuous part of his relevant formula, namely the integral on the left-hand side of (1.12) below, contained $|\zeta(\frac{1}{2}+it)|^4$. However, for $\int_0^T |\zeta(\frac{1}{2}+it)|^4\,dt$ we know that the correct order of magnitude is $T\log^4 T$, and actually the asymptotic formula with error term $O(T^{2/3}\log^C T)$ is known (see e.g., [4] and [15]). Unfortunately, to this day such type of result is not known for any power moment of $|\zeta(\frac{1}{2}+it)|$ greater than the fourth.

As to the true order of sums of $\alpha_j H_j^k(\frac{1}{2})$, perhaps it is true that, for $k \in \mathbb{N}$ fixed,

$$\sum_{\kappa_j \leqslant T} \alpha_j H_j^k(\tfrac{1}{2}) + \frac{2}{\pi}\int_0^T \frac{|\zeta(\frac{1}{2}+it)|^{2k}}{|\zeta(1+2it)|^2}\,dt = T^2 P_{\frac{1}{2}(k^2-k)}(\log T) + O(T^{1+c_k+\varepsilon}), \quad (1.12)$$

where $P_{\frac{1}{2}(k^2-k)}(z)$ is a suitable polynomial of degree $\frac{1}{2}(k^2-k)$ in z whose coefficients depend on k, and $0 \leqslant c_k < 1$; perhaps even $c_k = 0$ is true. We actually have $c_2 = 0$ in view of (1.2), and from the proofs of Theorem 1 and Theorem 2 it follows that we may take $c_3 = 1/7, c_4 = 1/3$. For example, (5.5) and (5.6) (for $k = 4$) clearly show why the left-hand side of (1.12) appears, and in view of $H_j(\frac{1}{2}) \geqslant 0$ it is positive. It would be interesting to evaluate (or estimate) the sum in (1.12) when $k = 1$ and $k \geqslant 5$. The case $k = 1$ will be briefly discussed at the end of the paper, while $k \geqslant 5$ lies outside the scope of this work. However the latter case is of potential importance since it could yield upper bounds for the $2k$-th moment of $|\zeta(\frac{1}{2}+it)|$. Namely if for some $k \geqslant 6$ the right-hand side of (1.12) is bounded by $T^{2+\varepsilon}$, this would essentially give a bound at least as strong as the (known) twelfth moment of $|\zeta(\frac{1}{2}+it)|$ (see (4.2)). If this bound holds for every k, then this implies both $H_j(\frac{1}{2}) \ll_\varepsilon \kappa_j^\varepsilon$ and the Lindelöf hypothesis that $\zeta(\frac{1}{2}+it) \ll_\varepsilon |t|^\varepsilon$. It is yet unknown what is the connection between these two conjectures, namely whether one of them implies the other one.

Conjectures for moments of various L-functions have been recently proposed by considerations from Random matrix theory (see J.B. Conrey [1] and the comprehensive work by J.B. Conrey, D.W. Farmer, J.P. Keating, M.O. Rubinstein and N.C. Snaith [2]). In all cases which can be predicted by this theory and where the asymptotic formula in question was rigorously proved, the main terms coincide. In our context this theory says that one should have

$$\sum_{\kappa_j \leqslant K} \alpha_j H_j^k(\tfrac{1}{2}) = K^2 P_{\frac{1}{2}(k^2-k)}(\log K) + o(K^2) \quad (1.13)$$

for $k \in \mathbb{N}$ fixed. The leading coefficient of $P_{\frac{1}{2}(k^2-k)}(x)$ equals

$$c_k = \frac{a_k g_k}{\pi^2 \left(\frac{k(k-1)}{2}\right)!}. \tag{1.14}$$

In the notation of this theory g_k is the so-called *geometric* part. In our case it is

$$g_k = \left(\tfrac{1}{2}k(k-1)\right)!\, 2^{k(k+1)/2-1} \prod_{j=1}^{k-1} \frac{j!}{(2j)!}, \tag{1.15}$$

so that $g_1 = 1, g_2 = 2, g_3 = 8, g_4 = 128$. The constant a_k is the *arithmetic* part. It equals

$$a_k = \prod_p \left(1 - \frac{1}{p}\right)^{k(k-1)/2} \sum_{j=0}^{\infty} \binom{k+j-1}{j} \binom{k+j-2}{j} \frac{1}{(j+1)p^j}. \tag{1.16}$$

We have $a_1 = a_2 = a_3 = 1, a_4 = 1/\zeta(2) = \frac{6}{\pi^2}$. In general, a_k can be expressed in terms of hypergeometric functions. Note that

$$\sum_{j=0}^{\infty} \binom{k+j-1}{j} \binom{k+j-2}{j} \frac{x^j}{j+1} \qquad (|x| < 1)$$

is a rational function of x whose denominator is $(1-x)^{2k-3}$ and numerator is 1 for $k = 2, 3$, and is equal to $1 + x\,(k = 4)$, $1 + 3x + x^2\,(k = 5)$ etc. This shows that, for $k \geqslant 5$, a_k will not be expressible in a simple closed form, but as an Euler product over the primes. We have the values $c_1 = 1/\pi^2, c_2 = 2/\pi^2, c_3 = 4/(3\pi^2), c_4 = 16/(15\pi^4)$, which coincide for $k = 2, 3, 4$ with the ones that follow from (1.2), Theorem 1 and Theorem 2. Note that Random matrix theory also predicts the asymptotic formula for the sum in (1.13) without the normalizing factor α_j. The shape of the conjectured formula will be similar to the above one, only the constants will be different, and somewhat more complicated. Unfortunately, the methods at hand permit one to deal only with the sum in (1.13).

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2. Kuznetsov's work on sums of $H_j^4(\frac{1}{2})$

N.V. Kuznetsov's preprint [12] states as the main result (Theorem 1 on p. 5) the asymptotic formula

$$\sum_{\kappa_j \leqslant T} \alpha_j H_j^4(\tfrac{1}{2}) = T^2 P_6(\log T) + O(T^{4/3+\varepsilon}), \tag{2.1}$$

where $P_6(x)$ is a polynomial in x of degree six whose leading coefficient is equal to $2^9/(15\pi^3)$. This is actually stronger than our (1.11) of Theorem 2. Unfortunately, Kuznetsov did not prove (2.1), and even the leading coefficient of $P_6(x)$ is not correctly stated (it equals $16/(15\pi^4)$, see Section 9 for details). We shall analyze his preprint and substantiate our claim, using certain valid parts of his work, namely the derivation of the main term to shorten the proof of our Theorem 2. A complete list of misprints, errors etc. of [12] is not given, but just some of the important ones will be stated here. Further discussion concerning [12] will be given in subsequent sections.

Page 8, line after (21) it is not shown why $\hat{\psi}(2w)$ is regular for $\Re e\, w > -5/2$, which is claimed in the text. Namely

$$\hat{\psi}(2w) = \int_{-\infty}^{\infty} \pi^{-1} 2^{2w-1} \Gamma(w-iu)\Gamma(w+iu)h(u)u \sinh(\pi u)\, du \quad (\Re e\, w > 0), \tag{2.2}$$

where $(Q \asymp T^{1/3})$

$$\begin{aligned} h(r) &= q(r)\left\{\exp\left(-\left(\frac{r-T}{Q}\right)^2\right) + \exp\left(-\left(\frac{r+T}{Q}\right)^2\right)\right\}, \\ q(r) &= \frac{(r^2+\frac{1}{4})(r^2+\frac{9}{4})}{(r^2+\frac{1}{4})(r^2+\frac{9}{4})+626}, \end{aligned} \tag{2.3}$$

so that $h(r)$ is even, regular for $|\Im m\, w| \leqslant 3$, $h(\pm\frac{i}{2}) = h(\pm\frac{3i}{2}) = 0$, and $h(r)$ decays like $\exp(-c|r|^2)$. To analyze the function $\hat{\psi}(2w)$, note that from

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \tag{2.4}$$

one obtains the identity

$$\Gamma(w+iu)\Gamma(w-iu) = \frac{\pi i}{2\sinh(\pi u)\cos(\pi w)}\left\{\frac{\Gamma(w+iu)}{\Gamma(1-w+iu)} - \frac{\Gamma(w-iu)}{\Gamma(1-w-iu)}\right\}.$$

Since $h(r)$ is even, this gives

$$\hat{\psi}(2w) = \frac{i2^{2w}}{\cos(\pi w)} h^*(w), \tag{2.5}$$

where (see Y. Motohashi [14, eq. (2.12)])

$$h^*(s) := \int_{-\infty}^{\infty} uh(u)\frac{\Gamma(s+iu)}{\Gamma(1-s+iu)}\,du = -\int_{(0)} wh(iw)\frac{\Gamma(s+w)}{\Gamma(1-s+w)}\,dw \qquad (2.6)$$

is regular for $\Re e\, s > 0$, where $\int_{(\alpha)}$ means integration over the line $\Re e\, w = \alpha$. If $h(r)$ is entire (cf. (1.6)), then in (2.6) the line of integration may be shifted to $\Re e\, w = C > 0$. Thus $h^*(s)$ is seen to be regular for $\Re e\, s > -C$, and since C may be arbitrary, it follows that $h^*(s)$ is entire and of polynomial growth in $|s|$ for σ in a fixed strip. In the case of (2.3) $h^*(s)$ is regular at least for $\Re e\, s > -3$, and we have $h^*(\pm\frac12) = h^*(\pm\frac32) = 0$. For example, by using taking $\Re e\, w = 2$ in (2.6) and using the functional equation $s\Gamma(s) = \Gamma(s+1)$ one obtains

$$\frac{\Gamma(-\frac32 + w)}{\Gamma(\frac52 + w)} = \frac{1}{(w^2 - \frac14)(w^2 - \frac94)}.$$

Thus this cancels with the corresponding factor of $h(iw)$, and $h^*(-\frac32) = 0$ follows since $h(r)$ is even. Likewise it follows that $h^*(n + \frac12) = 0$ $(n \in \mathbb{N})$, hence $\hat{\psi}(2w)$ is indeed regular for $\Re e\, w > -5/2$, the first pole at $w = -5/2$ coming from the zero of $\cos(\pi w)$ in the numerator in (2.5).

Page 9, in the formulation of Theorem 2 the numbering (27) is missing, and the condition (contradicting $\Re e\, \mu = \Re e\, \nu = \frac12$) $\Re e\, \mu, \Re e\, \nu \neq \frac12$ should be $\mu, \nu \neq \frac12$.

More importantly, Kuznetsov did not prove Theorem 2 (which yields the spectral decomposition for the sum in (3.2), and is the basis of [12]) in [10] as he claimed. The result was used there in his unsuccessful attempt to prove the eighth moment for the Riemann zeta-function, namely

$$\int_0^T |\zeta(\tfrac12 + it)|^8\,dt \ll T \log^C T.$$

The same formula was also used in [11] in his failure to prove the Lindelöf hypothesis that $\zeta(\frac12 + it) \ll_\varepsilon |t|^\varepsilon$. A corrected version of the formula is due to Y. Motohashi [13] in 1991, and recently this was updated and improved in [16]. Hence due to Motohashi's work [16] this important obstacle in dealing with the asymptotic evaluation of the sum in (2.1) has been removed but, unfortunately, this is not the only shortcoming of [12] as will be clear from the sequel.

Page 10, l. 8. Kuznetsov chooses $s = \nu = \varrho = \mu = \frac12$, which violates the assumptions of Theorem 2, without mentioning that first one has to take $\mu = \frac12 + it, \nu = \frac12 + i\tau$ and then to take $t, \tau \to 0$. In (30), in the first line, $1 - 2s$ should be $1 - 2\varrho$.

Page 14, in l. 10 (32) should be (38), in (44) $4^5 = 1012$ is false.

Page 24, l. -5,6 it should be sh $\eta/2 = \frac{r}{T}$.

Page 25. l. 2,4 of (91), ξ is repeatedly written in place of ζ. Formula (92) is incorrect, detailed discussion will be given below in Section 4. In (93), on the right-hand side, Q is missing twice. In (94), in the exponent in the O-term, ve should be replaced by ε. In (95), dt should be dT, $r^{3/2}$ should be $\kappa_j^{3/2}$. Line below (91), ξ should be (6).

Page 26, in (97) Q is missing once on both sides, ve should be ε.

3. Formulas for products of three and four Hecke series

The essence to the approach of dealing with sums of $H_j^3(\frac{1}{2})$ and $H_j^4(\frac{1}{2})$ are the transformation formulas for the sums

$$C(K,G) := \sum_{j=1}^{\infty} \alpha_j H_j^3(\tfrac{1}{2}) h_0(\kappa_j) \tag{3.1}$$

with $h_0(r)$ given by (1.6), and

$$\sum_{j=1}^{\infty} \alpha_j H_j^4(\tfrac{1}{2}) h(\kappa_j), \tag{3.2}$$

with $h(r)$ given by (2.3). The notation in (3.2) corresponds to Motohashi [15], while that of (2.3) is from Kuznetsov [12]. We shall adhere to this for practical reasons, but of course it would have been possible to use $h_0(\kappa_j)$ instead of $h(\kappa_j)$ etc. To obtain transformation formulas for the weighted sums (this facilitates the resolution of the problems involving analytic continuation) one starts from general expressions, namely $H_j(u)H_j(v)H_j(\frac{1}{2})$ in (3.1) and $H_j(u)H_j(v)H_j(w)H_j(z)$ in (3.2) in the region of absolute convergence. In the former one replaces $H_j(\frac{1}{2})$ by an approximate functional equation (e.g., [15, Lemma 3.9]) which reduces it to suitable sums of $t_j(f)f^{-1/2}$. The product of two Hecke series is transformed by the use of the identity (in the region of absolute convergence; see [15, (3.2.7)])

$$H_j(s)H_j(s-\alpha) = \zeta(2s-\alpha)\sum_{n=1}^{\infty} \sigma_\alpha(n) t_j(n) n^{-s} \quad \Big(\sigma_\alpha(n) = \sum_{d|n} d^\alpha\Big),$$

which is the analytic equivalent of the multiplicativity of the arithmetic function $t_j(n)$, namely (see e.g., [15, eq. (3.1.14)])

$$t_j(m)t_j(n) = \sum_{d|(m,n)} t_j\left(\frac{mn}{d^2}\right). \tag{3.3}$$

After this there is summation of $t_j(m)t_j(f)$ in both cases, which is effected by applying the Kuznetsov trace formula (see [15, Theorem 2.4]). It is here that delicate questions of analytic continuation arise. In [7] M. Jutila used a variation of this approach in proving (1.9). Namely he used ([14, pp. 266-267] or [15, Lemma 3.8]) Motohashi's formula for

$$\sum_{j=1}^{\infty} \alpha_j H_j^2(\tfrac{1}{2}) t_j(f) h(\kappa_j) \qquad (f \in \mathbb{N}),$$

combined with his explicit expression for $H_j^2(\frac{1}{2})$ (see (9.2)).

We shall present now Motohashi's explicit formula for sums of $H_j^3(\frac{1}{2})$, needed for the proof of Theorem 1. We have (see (3.1)) with $\lambda = C \log K$ $(C > 0)$ (this is [15, (3.5.18)], with the extraneous factor $(1-(\kappa_j/K)^2)^\nu$ omitted)

$$\begin{aligned} C(K,G) = & \sum_{f \leqslant 3K} f^{-\frac{1}{2}} \exp\Big(-\Big(\frac{f}{K}\Big)^\lambda\Big) \mathcal{H}(f;h_0) \\ & - \sum_{\nu=0}^{N_1} \sum_{f \leqslant 3K} f^{-\frac{1}{2}} U_\nu(fK) \mathcal{H}(f;h_\nu) + O(1), \end{aligned} \tag{3.4}$$

with ($h_0(r)$ is given by (1.6))

$$h_\nu(r) = h_0(r)\left(1 - \left(\frac{r}{K}\right)^2\right)^\nu \qquad (\nu = 0,1,2\ldots), \tag{3.5}$$

$$\begin{aligned} \mathcal{H}(f;h) &= \sum_{\nu=1}^{7} \mathcal{H}_\nu(f;h), \\ \mathcal{H}_1(f;h) &= -2\pi^{-3} i \Big\{(\gamma - \log(2\pi\sqrt{f}))(\hat{h})'(\tfrac{1}{2}) + \tfrac{1}{4}(\hat{h})''(\tfrac{1}{2})\Big\} d(f) f^{-\frac{1}{2}}, \\ \mathcal{H}_2(f;h) &= \pi^{-3} \sum_{m=1}^{\infty} m^{-\frac{1}{2}} d(m) d(m+f) \Psi^+(\frac{m}{f};h) \quad \Big(d(n) = \sum_{\delta | n} 1\Big), \\ \mathcal{H}_3(f;h) &= \pi^{-3} \sum_{m=1}^{\infty} (m+f)^{-\frac{1}{2}} d(m) d(m+f) \Psi^-(1 + \frac{m}{f};h), \\ \mathcal{H}_4(f;h) &= \pi^{-3} \sum_{m=1}^{f-1} m^{-\frac{1}{2}} d(m) d(f-m) \Psi^-(\frac{m}{f};h), \\ \mathcal{H}_5(f;h) &= -(2\pi^3)^{-1} f^{-\frac{1}{2}} d(f) \Psi^-(1;h), \\ \mathcal{H}_6(f;h) &= -12\pi^{-2} i \sigma_{-1}(f) f^{\frac{1}{2}} h'(-\tfrac{1}{2}i), \\ \mathcal{H}_7(f;h) &= -\pi^{-1} \int_{-\infty}^{\infty} \frac{|\zeta(\frac{1}{2}+ir)|^4}{|\zeta(1+2ir)|^2} \sigma_{2ir}(f) f^{-ir} h(r)\, dr, \end{aligned} \tag{3.6}$$

where (see (2.6))

$$\Psi^+(x;h) = \int_{(\beta)} \Gamma^2(\tfrac{1}{2} - s) \tan(\pi s) h^*(s) x^s \, ds,$$

and

$$\Psi^-(x;h) = \int_{(\beta)} \Gamma^2(\tfrac{1}{2} - s)\frac{h^*(s)}{\cos(\pi s)} x^s \, ds$$

with $-\frac{3}{2} < \beta < \frac{1}{2}$. In (3.4) N_1 is a sufficiently large integer and

$$U_\nu(x) = \frac{1}{2\pi i\lambda}\int_{(-\lambda^{-1})} (4\pi^2 K^{-2}x)^w u_\nu(w)\Gamma(\frac{w}{\lambda})\, dw \ll \left(\frac{x}{K^2}\right)^{-\frac{C}{\log K}} \log^2 K, \quad (3.7)$$

where $u_\nu(w)$ is a polynomial in w of degree $\leqslant 2N_1$, whose coefficients are bounded. A prominent feature of Motohashi's explicit expression for $C(K,G)$ is that it contains series and integrals with the classical divisor function $d(n)$ only, with no quantities from spectral theory. Therefore the problem of evaluating $C(K,G)$ is a problem of classical analytic number theory.

As for (3.2), we adopt the notation of [12], primarily since we intend to correct Kuznetsov's proof. As already stated, a correct and rigorous proof of the spectral decomposition for (3.2) is given by Y. Motohashi [13] and [16]. The formulation is technically complicated, and for the sake of brevity will not be reproduced here.

4. The asymptotic formula for sums of $H_j^4(\frac{1}{2})$

We shall provide in fact two completely different proofs of Theorem 2. The first is obtained by correcting and simplifying the proof given by N.V. Kuznetsov in [12]. The second approach consists of elaborating the method of M. Jutila [7], used in the proof of the bound (1.9), which is one of the crucial ingredients in the proof of Theorem 2. It will be outlined in Section 9.

We shall begin now with the proof of (1.11) of Theorem 2, correcting and simplifying [12]. We remark first that one obtains (1.11) from

$$\sum_{\kappa_j \leqslant T} \alpha_j H_j^4(\tfrac{1}{2}) + O\left(\log^2 T \int_0^T |\zeta(\tfrac{1}{2}+it)|^8\, dt\right) = T^2 P_6(\log T) + O(T^{4/3+\varepsilon}). \quad (4.1)$$

Namely one has (e.g., see [3]) the bounds

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^4\, dt \ll T\log^4 T, \qquad \int_0^T |\zeta(\tfrac{1}{2}+it)|^{12}\, dt \ll T^2\log^{17} T. \quad (4.2)$$

Hence by the Cauchy-Schwarz inequality for integrals it follows from (4.2) that

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^8\, dt \ll T^{3/2}\log^{21/2} T, \quad (4.3)$$

which is still the sharpest known upper bound estimate for the integral in (4.3). In [12] N.V. Kuznetsov assumed that the bound

$$\int_0^T |\zeta(\tfrac12 + it)|^8\,dt \ll T\log^C T \tag{4.4}$$

holds for some $C > 0$. This is what he claimed to have proved in [10]. Although he never officially withdrew the claim (the proof was faulty), this fact was mentioned in the review in the Zentralblatt (Zbl.745.11040). The asymptotic formula (4.1) shows clearly that one cannot attain the exponent $4/3 + \varepsilon$ in (4.1) unless it is attained in (4.3). This, however, would be a big achievement in zeta-function theory.

The plan of the proof is as follows: from the fundamental formula for sums of products of four of Hecke series ([12, Theorem 2] or [16, Theorem]) one obtains first the formula

$$\begin{aligned}
&\sum_{j\geqslant 1}\alpha_j H_j^4(\tfrac12)h(\kappa_j) + \frac{2}{\pi}\int_0^\infty \frac{|\zeta(\frac12+ir)|^8}{|\zeta(1+2ir)|^2}h(r)\,dr\\
&\quad + \sum_{k\geqslant 12,\,k\equiv 0(\mathrm{mod})2} g(k)\sum_{1\leqslant j\leqslant \nu_k}\alpha_{j,k}H_{j,k}^4(\tfrac12)\\
&= \sum_{j\geqslant 1}\alpha_j H_j^4(\tfrac12)(h_0(\kappa_j) + \varepsilon_j h_1(\kappa_j))\\
&\quad + \frac{1}{\pi}\int_{-\infty}^\infty \frac{|\zeta(\frac12+ir)|^8}{|\zeta(1+2ir)|^2}(h_0(r)+h_1(r))\,dr + R + O(Q\log^6 T).
\end{aligned} \tag{4.5}$$

Here $h(r)$ is given by (2.3), the quantities in

$$\sum_{k\geqslant 12,\,k\equiv 0(\mathrm{mod})2} g(k)\sum_{1\leqslant j\leqslant \nu_k}\alpha_{j,k}H_{j,k}^4(\tfrac12), \tag{4.6}$$

which are associated with holomorphic cusp forms are precisely defined in [12] or [15],

$$Q = T^{1/3}, \tag{4.7}$$

$$\begin{aligned}
g(k) &= \frac{1}{2\pi^3 i}\int_{(\delta)} \frac{\Gamma(k-\frac12+w)}{\Gamma(k+\frac12+w)}\Gamma^4(\tfrac12 - w)\sin(\pi w)\hat\psi(2w)\,dw,\\
h_0(r) &= \frac{1}{\pi^3 i}\int_{(\delta)} \Gamma(w+ir)\Gamma(w-ir)\Gamma^4(\tfrac12 - w)\sin(2\pi w)\hat\psi(2w)\,dw,\\
h_1(r) &= \frac{1}{2\pi^3 i}\int_{(\delta)} \Gamma(w+ir)\Gamma(w-ir)\Gamma^4(\tfrac12 - w)\cosh(\pi r)\frac{\sin^2(\pi w)+1}{\cos(\pi w)}\hat\psi(2w)\,dw,
\end{aligned} \tag{4.8}$$

where $\hat{\psi}$ is given by (2.2) and $\delta > 0$ is a small constant. The choice of Q in (4.7) seems optimal, and any improvements (namely $Q = T^\alpha$ with $\alpha < 1/3$) will require the use of new methods. Actually, instead of (4.7) the correct choice of Q is $Q = CT^{1/3}$ with some $C > 0$, since we shall integrate (4.5) over the interval $[T_0, 2T_0]$, so Q should ultimately depend on T_0 and not on T; e.g., one can take $Q = T_0^{1/3}$ (this fact is not mentioned in [12]). The symbol R in (4.5) stands for the residual (main) terms. This has been calculated by Kuznetsov in [12] to be equal to

$$\sum_{k=0}^{6} a_k \hat{\psi}^{(6-k)}(1) + O(Q\log^6 T) \qquad \Big(a_0 = \frac{1024}{15\pi^3}\Big). \tag{4.9}$$

It can be shown that the contribution of (4.6) is $O(Q\log^6 T)$ (note that the sum with $\alpha_{j,k}H_{j,k}^4(\frac12)$ is easily majorized; see [15]) and so is the contribution of $g(k)$ and h_0 in (4.5) (see (6.2) and (6.3)). What remains then is the basic formula

$$\begin{aligned}\sum_{j\geqslant1}&\alpha_j H_j^4(\tfrac12)\exp\Big(-\Big(\frac{\kappa_j-T}{Q}\Big)^2\Big)\\ &+\frac{2}{\pi}\int_0^\infty \frac{|\zeta(\frac12+ir)|^8}{|\zeta(1+2ir)|^2}\exp\Big(-\Big(\frac{r-T}{Q}\Big)^2\Big)\,dr\\ &=\sum_{k=0}^{6} a_k\hat{\psi}^{(6-k)}(1)+\sum_{j\geqslant1}\alpha_j H_j^4(\tfrac12)\widetilde{h}(\kappa_j)\\ &+\frac{1}{\pi}\int_0^\infty \frac{|\zeta(\frac12+ir)|^8}{|\zeta(1+2ir)|^2}\widetilde{h}(r)\,dr + O(Q\log^6 T),\end{aligned} \tag{4.10}$$

where $\widetilde{h}(r)$ is the oscillatory integral transform obtained by replacing $\sin^2(\pi w)+1$ in the definition of $h_1(r)$ (see (4.8)) by $\sin^2(\pi w)$. The terms containing this function will be small, while $\hat{\psi}$ will give rise to the main term $T^3P_6(\log T)$ in (4.1).

In the relevant range one has (this follows from Kuznetsov's Lemma 4.7)

$$\widetilde{h}(r) \ll Qr^{-1/2}\exp(-CQ^2r^2T^{-2}) \qquad (C>0). \tag{4.11}$$

Hence by the non-negativity of the integral on the left-hand side of (4.10), (1.3) and (4.11) it follows that

$$\begin{aligned}\sum_{j\geqslant1}&\alpha_j H_j^4(\tfrac12)\exp\Big(-\Big(\frac{\kappa_j-T}{Q}\Big)^2\Big)\\ &\ll QT\log^6 T + Q\sum_{\kappa_j\leqslant TQ^{-1}\log T}\alpha_j H_j^4(\tfrac12)\kappa_j^{-1/2}\\ &\ll QT\log^6 T + T^{3/2}Q^{-1/2}\log^{16} T \ll T^{4/3}\log^{16} T.\end{aligned} \tag{4.12}$$

Observe that (4.12) is a sharpened variant, in view of (4.7), of (1.9), as it gives (1.9) with the right-hand side replaced by $K^{4/3}\log^{16} K$. By using (4.3) and (4.11) it follows that the integral on the right-hand side of (4.5) is $\ll T^{1+\epsilon}$. Also note that we have ($Q = T^{1/3}$)

$$\int_0^\infty \frac{|\zeta(\frac{1}{2}+ir)|^8}{|\zeta(1+2ir)|^2} \exp\left(-\left(\frac{r-T}{Q}\right)^2\right) dr \ll T^{4/3}\log^{16} T, \tag{4.13}$$

which can be easily obtained from the mean square bounds for $\zeta(\frac{1}{2}+it)$ over short intervals (see [3, Chapter 15]) and the classical bound $\zeta(\frac{1}{2}+it) \ll |t|^{1/6}$. One also has to use e.g. the standard bound

$$\frac{1}{|\zeta(1+it)|} \ll \log|t|. \tag{4.14}$$

After these considerations it remains to integrate the basic formula (4.10) over T from T_0 to $2T_0$ and then to replace T_0 by $T_0 2^{-j}$, and sum the resulting expressions for $j \geqslant 1$. This will lead to (4.1). The technical details are given in the next section, as well as the calculation of the main term.

We have restrained ourselves from analyzing the difficult lemmas of [12, Section 4], especially of the Lemma 4.7 which claims an asymptotic formula for the crucial function $\widetilde{h}(r)$ appearing in (4.10). The function $h_1(r)$ in [12, Lemma 3.2] is first transformed into a complicated expression involving the hypergeometric function. This is said to follow from the use of Parseval's formula for Mellin transforms. The author was unable to follow the proof of Lemma 4.7, which claims an asymptotic expansion of $\widetilde{h}(r)$. However, this asymptotic expansion will be proved, in Section 6, by a method which is different and simpler than Kuznetsov's.

5. Integration of the basic formula and the main term

We shall deal first with the main term in (1.11). One way to obtain this expression is to go through Kuznetsov's paper [12]. Therein he claimed (eq. (92) on p. 25) that

$$\hat{\psi}^{(m)}(1) = \frac{2}{\sqrt{\pi}} QT\log^m T \cdot \left(1 + O\left(\frac{1}{T}\right)\right) \qquad (m = 0, 1, 2, \cdots), \tag{5.1}$$

where $\hat{\psi}$ is defined by (2.2). On the right-hand side of (4.10) there appears

$$\sum_{k=0}^{6} a_k \hat{\psi}^{(6-k)}(1),$$

which will give rise to the main term $K^2P_6(\log K)$ in (2.1). Hence we have to evaluate explicitly $\hat\psi^{(m)}(1)$ for $m = 0, \cdots, 6$.

The case $m = 0$. From (2.2) we have, on using (2.4) and recalling that $h(r)$ is given by (2.3),

$$\begin{aligned}\hat\psi(1) &= \frac{2}{\pi}\int_0^\infty \Gamma(\tfrac12 + iu)\Gamma(\tfrac12 - iu)uh(u)\sinh(\pi u)\,du \\ &= 2\int_0^\infty \frac{uh(u)}{\sin\pi(\frac12 + iu)}\sinh(\pi u)\,du = 2\int_0^\infty uh(u)\tanh(\pi u)\,du \\ &= 2\int_{T-Q\log T}^{T+Q\log T} \exp(-(u-T)^2Q^{-2})u\bar h(u)\tanh(\pi u)\,du + O(\mathrm{e}^{-\frac12\log^2 T}),\end{aligned}$$

where

$$\bar h(r) = 1 + O(r^{-4}).$$

Change of variable $u = T + Qx$ gives then

$$\begin{aligned}\hat\psi(1) &= 2Q\int_{-\log T}^{\log T} \mathrm{e}^{-x^2}(T + Qx)\tanh\pi(T+Qx)\,dx + O(1) \\ &= 2Q(\sqrt{\pi}T(1 + O(1/T)) + O(Q)) = 2\sqrt{\pi}QT\left(1 + O\left(\frac{Q}{T}\right)\right).\end{aligned} \tag{5.2}$$

The case $m \geqslant 1$. We need the formula (see e.g., [4, p. 272])

$$\frac{\Gamma^{(k)}(s)}{\Gamma(s)} = \sum_{j=0}^{k} b_{j,k}(s)\log^j s + c_{-1,k}s^{-1} + \cdots + c_{-1,r}s^{-r} + O_r(|s|^{-r-1}) \tag{5.3}$$

for fixed integers $k \geqslant 1$, $r \geqslant 0$, where each of the functions $b_{j,k}(s)$ ($\sim b_{j,k}$ for a suitable constant $b_{j,k}$ as $s \to \infty$) has an asymptotic expansion in non-positive powers of s. As in the case $m = 0$ the main contribution to $\hat\psi^{(m)}(2w)$ will come from an interval of length $\ll Q\log T$, when w lies in a neighbourhood of $\frac12$. Namely we have

$$\begin{aligned}2^m\hat\psi^{(m)}(2w) &= \frac{1}{\pi}\int_{T-Q\log T}^{T+Q\log T} \frac{d^m}{dw^m}\left(2^{2w}\Gamma(w+iu)\Gamma(w-iu)\right)uh(u)\sinh(\pi u)\,du \\ &\quad + O(\mathrm{e}^{-\frac12\log^2 T}).\end{aligned}$$

To calculate the derivatives in the above integral we apply Leibniz's rule. We have to evaluate ($r = 0, 1, \cdots, m$)

$$\frac{d^r}{dw^r}\Gamma(w+iu)\Gamma(w-iu)\Big|_{w=\frac12}, \qquad u = T + O(Q\log T).$$

By using (5.2), (5.3) and (2.4) it is seen that this expression equals

$$\sum_{j=0}^{r}\binom{r}{j}\Gamma^{(j)}(\tfrac12+iu)\Gamma^{(r-j)}(\tfrac12-iu)$$
$$=\frac{\pi}{\cosh(\pi u)}\left(\sum_{\ell=0}^{r}d_{\ell,r}\log^r u+O(T^{-1}\log^r T)\right)$$

with suitable constants $d_{\ell,r}$. Proceeding as in the case $m=0$, we obtain

$$\hat\psi^{(m)}(1)=QT\left(\sum_{j=0}^{m}c_{j,m}\log^m T+O_m(QT^{-1}\log^m T)\right)\quad(m\in\mathbb{N})\tag{5.4}$$

with suitable constants $c_{j,m}$, which may be explicitly evaluated ($c_{m,m}=2^{2-m}\sqrt{\pi}$). From (5.2) and (5.4) we see that Kuznetsov's claim (5.1) is incorrect.

Now we integrate (4.10) over T from T_0 to $2T_0$, taking $Q=T_0^{1/3}$ (cf. (3.7)), which clearly may be done. We have first

$$\sum_{j\geqslant1}\alpha_jH_j^4(\tfrac12)\int_{T_0}^{2T_0}\exp\left(-\left(\frac{\kappa_j-T}{Q}\right)^2\right)dT$$
$$=\sum_{T_0-Q\log T_0\leqslant\kappa_j\leqslant2T_0+Q\log T_0}\alpha_jH_j^4(\tfrac12)\int_{T_0}^{2T_0}\exp\left(-\left(\frac{\kappa_j-T}{Q}\right)^2\right)dT+o(1).$$

By change of variable and (4.12) (or (1.9)) the sum on the right-hand side equals

$$Q\sum_{T_0-Q\log T_0\leqslant\kappa_j\leqslant2T_0+Q\log T_0}\alpha_jH_j^4(\tfrac12)\int_{(T_0-\kappa_j)/Q}^{(2T_0-\kappa_j)/Q}e^{-x^2}\,dx$$
$$=O(Q^2T_0^{1+\varepsilon})+Q\sum_{T_0+Q\log T_0\leqslant\kappa_j\leqslant2T_0-Q\log T_0}\alpha_jH_j^4(\tfrac12)\int_{(T_0-\kappa_j)/Q}^{(2T_0-\kappa_j)/Q}e^{-x^2}\,dx$$
$$=O(Q^2T_0^{1+\varepsilon})+\sqrt{\pi}Q\sum_{T_0+Q\log T_0\leqslant\kappa_j\leqslant2T_0-Q\log T_0}\alpha_jH_j^4(\tfrac12)+O(e^{-\frac12\log^2T})\tag{5.5}$$
$$=\sqrt{\pi}Q\sum_{T_0\leqslant\kappa_j\leqslant2T_0}\alpha_jH_j^4(\tfrac12)+O(Q^2T_0^{1+\varepsilon}).$$

In a similar fashion, by using (4.3) and (4.13), it follows that

$$\int_{T_0}^{2T_0}\int_0^\infty\frac{|\zeta(\frac12+ir)|^8}{|\zeta(1+2ir)|^2}\exp\left(-\left(\frac{r-T}{Q}\right)^2\right)dr\,dT$$
$$=\sqrt{\pi}Q\int_{T_0}^{2T_0}\frac{|\zeta(\frac12+ir)|^8}{|\zeta(1+2ir)|^2}\,dr+O(Q^2T^{1+\varepsilon})\tag{5.6}$$
$$\ll Q\log^2T_0\int_{T_0}^{2T_0}|\zeta(\tfrac12+ir)|^8\,dr+Q^2T^{1+\varepsilon}\ll QT_0^{3/2}\log^{25/2}T_0.$$

To bound the second sum on the right-hand side of (4.10) we use Lemma 4.7 of [12], or the discussion on $h_1(r)$ in Section 6. We need especially the terms $(T-r)\log(T-r)-(T+r)\log(T+r)$ in (6.10), in conjunction with the first derivative test (Lemma 2.1 of [3]) and (1.3). The derivative in question is $\gg r/T$, and we shall obtain ($\widetilde{h}(\kappa_j)=\widetilde{h}(\kappa_j,T)$)

$$\begin{aligned}
&\sum_{j\geqslant 1}\alpha_j H_j^4(\tfrac12)\int_{T_0}^{2T_0}\widetilde{h}(\kappa_j,T)\,dT\\
&\ll \sum_{j\geqslant 1}\alpha_j H_j^4(\tfrac12) QT_0\kappa_j^{-3/2}\exp\left(-\frac{Q^2\kappa_j^2}{4T_0^2}\right)+1\\
&\ll QT_0\sum_{\kappa_j\leqslant T_0Q^{-1}\log T}\alpha_j H_j^4(\tfrac12)\kappa_j^{-3/2}+1\\
&\ll QT_0^{\frac43+\varepsilon}.
\end{aligned}\tag{5.7}$$

Finally from (5.2) and (5.4) we have

$$\begin{aligned}
&\int_{T_0}^{2T_0}\sum_{k=0}^{6}a_k\hat{\psi}^{(6-k)}(1)\,dT\\
&=Q\int_{T_0}^{2T_0}T\sum_{k=0}^{6}a_k\sum_{j=0}^{6-k}e_{j,m}\log^m T\,dT+O(QT_0^{\frac43+\varepsilon})\\
&=QT^2\sum_{k=0}^{6}f_k\log^k T\Big|_{T_0}^{2T_0}+O(QT_0^{\frac43+\varepsilon}).
\end{aligned}\tag{5.8}$$

with effectively computable constants $e_{j,m}$ and f_k. Therefore (4.1) will follow from (5.5)–(5.8) when we divide by Q, replace T_0 by $T_0 2^{-j}$ and sum over j.

6. The estimates for the oscillatory terms

In this section we shall complete the proof of Theorem 2 by estimating the oscillatory functions defined by (4.8). We shall use the function $h^*(s)$, defined by (2.5)–(2.6) to simplify the functions in (4.8). We obtain

$$\begin{aligned}
g(k)&=\frac{1}{\pi^3}\int_{(\delta)}2^{2w-1}\frac{\Gamma(k-\frac12+w)}{\Gamma(k+\frac12+w)}\Gamma^4(\tfrac12-w)\tan(\pi w)h^*(w)\,dw\quad(k\geqslant 12),\\
h_0(r)&=\frac{1}{\pi^3}\int_{(\delta)}2^{2w+1}\Gamma(w+ir)\Gamma(w-ir)\Gamma^4(\tfrac12-w)\sin(\pi w)h^*(w)\,dw,\\
h_1(r)&=\frac{1}{2\pi^3}\int_{(\delta)}2^{2w}\Gamma(w+ir)\Gamma(w-ir)\Gamma^4(\tfrac12-w)\cosh(\pi r)\frac{\sin^2(\pi w)+1}{\cos(\pi w)}h^*(w)\,dw,
\end{aligned}\tag{6.1}$$

where $\delta > 0$ is a small constant, and we may assume $r > 0$, since both h_0 and h_1 are even. From $s\Gamma(s) = \Gamma(s+1)$ and Stirling's formula it follows that

$$\frac{\Gamma(k - \frac{5}{2} + iv)}{\Gamma(k + \frac{5}{2} + iv)} \ll k^{-5} \qquad (12 \leqslant k \leqslant k_0).$$

In the integral for $g(k)$ we shift the line of integration to $\Re e\, w = -2$, taking $\Re e\, w = 2 + \varepsilon$ as the line of integration in (2.6). Using Stirling's formula and the above bound we obtain

$$g(k) \ll QT^{-7/2}k^{-5}, \tag{6.2}$$

and this bound can be further sharpened. Moreover directly from (6.1) we have

$$h_0(r) \ll Q\mathrm{e}^{-\pi r}. \tag{6.3}$$

From (6.2) and (6.3) it is easily seen that the expressions in (4.5) containing the functions $g(k)$ and $h_0(r)$ contribute $O(Q\log^6 T)$. It remains to deal with the contribution of $h_1(r)$. Since $h^*(w)$ is entire (to be rigorous, one has either to work with h defined by (1.6), or replace the constant 626 in (2.3) by a larger constant), it transpires from (6.1) that in the expression for $h_1(r)$ the poles of the integrand are at $w = \frac{1}{2} - n$ $(n = 3, 4, 5, \ldots)$ and at $w = m \pm ir$ $(m = 0, -1, -2, \ldots)$. The former ones are harmless and could be avoided by inserting factors $r^2 + n^2 + \frac{1}{4}$ in the numerator and denominator of $q(z)$ in (2.3). We shift the line of integration in the expression for $h_1(r)$ to $\Re e\, w = -N$, letting eventually $N \to \infty$. The main contribution will then come from the poles at $w = \pm ir$ (these contributions are evaluated analogously), since the residues at other poles are evaluated similarly, only they will be of a lower order of magnitude. The residue at $w = -ir$ will be

$$\ll |\Gamma(2ir)|e^{2\pi r}|\Gamma(\tfrac{1}{2} + ir)|^4 |h^*(-ir)| \ll e^{-\pi r} r^{-1/2} |h^*(-ir)| \tag{6.4}$$

with

$$h^*(-ir) = \int_{\Im m\, z = -\varepsilon} z h(z) \frac{\Gamma(-ir + iz)}{\Gamma(1 + ir + iz)}\, dz,$$

where h^* is given by (2.6). Since $q(z) = 1 + O(|z|^{-4})$, it is seen that $h^*(-ir)$ is majorized by two similar expressions, one of which is ($z = T + Qy - i\varepsilon$)

$$Q \left| \int_{-\infty}^{\infty} \frac{T + Qy}{T + r + Qy}\, \mathrm{e}^{-y^2 + 2i\varepsilon Q^{-1} y}\, \frac{\Gamma(iT - ir + iQy + \varepsilon)}{\Gamma(iT + ir + iQy + \varepsilon)}\, dy \right|, \tag{6.5}$$

where we used $s\Gamma(s) = \Gamma(s+1)$. For $|y| \geqslant \log(rT)$ the portion of the above integral is negligible, as is also the portion for $r \geqslant T + T^{\varepsilon} Qy$, by Stirling's formula. Also note that $|T + r + Qy| - |T - r + Qy| \leqslant 2r$, so that the exponential function coming from $\mathrm{e}^{-\pi r}$ in (6.4) and the gamma factors will have a non-positive exponent. If

$$T - T^{\varepsilon} Q \leqslant r \leqslant T + T^{\varepsilon} Q \tag{6.6}$$

holds, then from (2.2) and (4.8) we have

$$h_1(r) \ll r^{-1/2}e^{-\pi r}\left|\int_{\mathcal{L}}\Gamma(-ir-iz)\Gamma(-ir+iz)zh(z)\sinh(\pi z)\,dz\right|, \tag{6.7}$$

where $\mathcal{L}$ is the real line with small indentations above and below the points $z=-r$ and $z=r$, respectively. It follows (by Stirling's formula) that the right-hand side of (6.7) is of exponential decay if (6.6) holds. Hence we are left with the most interesting range, namely

$$1 \ll r \leqslant T - T^{\varepsilon}Q. \tag{6.8}$$

Recall that the gamma-function admits an asymptotic expansion, for $t \geqslant t_0 > 0$, whose first two terms are

$$\Gamma(\sigma+it) = \sqrt{2\pi}t^{-\sigma-\frac12}\exp\{-\tfrac12\pi t + i(t\log t - t + \tfrac12\pi(\sigma-\tfrac12))\}\cdot \\ \cdot\left(1+\tfrac12 it^{-1}(\sigma-\sigma^2-\tfrac16)+O_\sigma(t^{-2})\right).$$

The quotient of gamma factors in (6.5) thus equals

$$\left(1+O(\frac{1}{T})\right)\left(\frac{T-r+Qy}{T+r+Qy}\right)^{\varepsilon-\frac12}e^{\pi r}\exp(i\varphi(T,r,Q,y)), \tag{6.9}$$

where the term $O(1/T)$ admits an asymptotic expansion, and by Taylor's formula we obtain

$$\begin{aligned}
&\varphi(T,r,Q,y)\\
&= 2r + (T-r)\log(T-r) - (T+r)\log(T+r)\\
&\quad - 2Qy\left(\frac{r}{T}+\frac13\left(\frac{r}{T}\right)^3+\frac15\left(\frac{r}{T}\right)^5+\cdots\right)\\
&\quad + \frac{(Qy)^2}{T-r} + (T-r+Qy)\left(-\frac12\frac{(Qy)^2}{(T-r)^2}+\frac13\frac{(Qy)^3}{(T-r)^3}+\cdots\right)\\
&\quad - \frac{(Qy)^2}{T+r} + (T+r+Qy)\left(-\frac12\frac{(Qy)^2}{(T+r)^2}+\frac13\frac{(Qy)^3}{(T+r)^3}+\cdots\right).
\end{aligned} \tag{6.10}$$

By (6.8) we have $Q|y|/(T\pm r) \leqslant T^{-\frac12\varepsilon}$ for $|y| \leqslant \log T$, so that we may truncate the contribution of the last two series above in such a way that the tails will make a negligible contribution. The remaining terms are inserted in

$$\int_{-\log T}^{\log T} e^{-y^2+2i\varepsilon Q^{-1}y}\frac{\Gamma(iT-r+iQy+\varepsilon)}{\Gamma(iT+r+iQy+\varepsilon)}\,dy = \int_{-\infty}^{\infty} + O\left(e^{-\frac12\log^2 T}\right),$$

where the term in (6.9) with the exponent $\varepsilon-\frac12$ is again simplified by Taylor's formula. The integrals with the remaining terms are evaluated by using the formula

$$\int_{-\infty}^{\infty} y^j e^{Ay-y^2}\,dy = P_j(A)e^{\frac14 A^2} \qquad (j=0,1,2,\ldots,\ P_0(A)=\sqrt{\pi}),$$

where $P_j(z)$ is a polynomial in z of degree j, which may be explicitly evaluated by successive differentiation of the classic formula

$$\int_{-\infty}^{\infty} e^{Ay-y^2}\,dy = \sqrt{\pi}e^{\frac14 A^2},$$

considered as a function of A. The major contribution will come from the term

$$-2Qy\left(\frac{r}{T}+\frac{1}{3}\left(\frac{r}{T}\right)^3+\frac{1}{5}\left(\frac{r}{T}\right)^5+\cdots\right)$$

in $\varphi(T,r,Q,y)$, hence the total contribution will be, in view of (6.9),

$$\ll e^{-\pi r}r^{-1/2}|h^*(-ir)| \ll Qr^{-1/2}\exp\left(-\frac{Cr^2Q^2}{T^2}\right) \qquad (C>0).$$

The analogous bound follows for the residue at $w = ir$. In fact, it follows that by the above procedure we obtain not only an upper bound, but an asymptotic expansion of the $h_1(r)$ in the range (6.8). This proves then the key bound (4.11), establishes (5.7), and completes the proof of Theorem 2.

7. The asymptotic formula for sums of $H_j^3(\frac12)$

We shall present now the proof of the asymptotic formula (1.10) of Theorem 1. We start from (3.4)–(3.6), restricting ourselves as to the range

$$K^{\varepsilon} \leqslant G \leqslant K^{\frac12-\varepsilon}, \tag{7.1}$$

and follow the approach developed in [5]. It is seen that it is the term $\nu = 0$ in (3.4) whose contributions should be considered, because the bound for the ν-th term will be essentially the same as the bound for the term $\nu = 0$, only it will be multiplied by $(G/K)^{\nu}$. We note that the factors $\exp(-(f/K)^{\lambda})$ and $U_\nu(fK)$ in (2.1) can be conveniently removed by partial summation. Next we follow the analysis carried out in [15, pp. 120 and 128-129] to show that the contribution of $\nu = 3,5,6$ in (3.4) to (2.1) will be small. Indeed, we have

$$\mathcal{H}_3(f;h_0) \ll e^{-C\log^2 K} \quad (C>0)$$

and

$$\mathcal{H}_5(f;h_0) \ll d(f)f^{-1/2}, \quad \mathcal{H}_6(f;h_0) \ll \sigma_{-1}(f)f^{1/2}K.$$

The contribution of $\mathcal{H}_4(f;h_\nu)$ was shown in [5] to be $\ll GK^{1+\varepsilon}$. To estimate the contribution of $\mathcal{H}_7(f;h_0)$ we note (see [3, Chapter 1]) that

$$\sum_{n=1}^{\infty}\sigma_{2ir}(n)n^{-ir-s} = \zeta(s-ir)\zeta(s+ir) \qquad (r\in\mathbb{R},\ \Re e\, s>1).$$

Consequently by the Perron inversion formula (see e.g., [3, p. 486])

$$\sum_{f\leqslant 3K} \sigma_{2ir}(f)f^{-\frac12-ir} \ll_\varepsilon K^{2\mu(\frac12)+\varepsilon} \qquad (K \ll |r| \ll K), \tag{7.2}$$

where as usual the Lindelöf function $\mu(\sigma)$ is given by

$$\mu(\sigma) = \limsup_{t\to\infty} \frac{\log|\zeta(\sigma+it)|}{\log t}.$$

Instead of using directly (7.2) it is more expedient to use the main contribution to the left-hand side of (7.2), which is

$$\frac{1}{2\pi i}\int_{\varepsilon-iU}^{\varepsilon+iU} \zeta(s+\tfrac12-ir)\zeta(s+\tfrac12+ir)K^s\frac{ds}{s} \quad (K^\varepsilon \ll U \ll K^{1-\varepsilon}),$$

and obtain a contribution which is, by the residue theorem,

$$\begin{aligned}&\int_{\varepsilon-iU}^{\varepsilon+iU}\int_{-\infty}^{\infty} \frac{|\zeta(\frac12+ir)|^4}{|\zeta(1+2ir)|^2} h(r)\zeta(s+\tfrac12-ir)\zeta(s+\tfrac12+ir)K^s\frac{ds}{s}\,dr\\ &= \int_{-\infty}^{\infty}\frac{|\zeta(\frac12+ir)|^6}{|\zeta(1+2ir)|^2}h(r)\,dr + \int_{-\varepsilon-iU}^{-\varepsilon+iU}\int_{-\infty}^{\infty}\cdots\,ds\,dr\\ &= J_1+J_2,\end{aligned} \tag{7.3}$$

say. After evaluating (3.1), we shall integrate it over K from K_0 to $2K_0$, similarly as was done in Section 5. The integral J_1 in (7.3) is the analogue of the integral on the left-hand side of (4.5). Its total contribution will be $O(GK_0^{13/4}\log^{37/4}K_0)$, since (4.14) holds and we use the best known estimate

$$\int_0^T |\zeta(\tfrac12+it)|^6\,dt \ll T^{5/4}\log^{37/4}T, \tag{7.4}$$

which follows by Hölder's inequality from (4.2). The contribution coming from J_2 will be analogous. Namely note that the relevant range of r in $\mathcal{H}_7(f;h_0)$ is $|r\pm K| \leqslant G\log K$, hence it follows from (7.3) and the argument given below that the total contribution of $\mathcal{H}_7(f;h_0)$ to the integrated version of (3.1) is

$$\ll K_0^{3/2+\varepsilon}GU^{-1}(G+K_0^{2/3}) + GK_0^{13/4}\log^{37/4}K_0$$

plus a quantity which is

$$\begin{aligned}\ll \int_{-U}^{U}\Big\{\int_{K_0}^{2K_0}\int_{-\infty}^{\infty} K^{2-\varepsilon}\exp\Big(-(r-K)^2G^{-2}\Big)\log^2 K_0\times\\ |\zeta(\tfrac12+ir)|^4|\zeta(\tfrac12-\varepsilon+iu-ir)\zeta(\tfrac12-\varepsilon+iu+ir)|\,dr\,dK\Big\}\frac{du}{1+|u|}.\end{aligned} \tag{7.5}$$

We shall take the maximum over u in the integral in (7.5) and then integrate; this will account for a loss of a log-factor in the final bound. The integral in curly brackets resembles the one in (5.6), only it has six and not eight zeta values, since now we are dealing with $H_j^3(\frac{1}{2})$ and not with $H_j^4(\frac{1}{2})$. It equals $O(\exp(-c\log^2 K_0))$ plus

$$
\begin{aligned}
&\int_{K_0-G\log K_0}^{2K_0+G\log K_0} |\zeta(\tfrac{1}{2}+ir)|^4|\zeta(\tfrac{1}{2}-\varepsilon+iu-ir)\zeta(\tfrac{1}{2}-\varepsilon+iu+ir)|\times \\
&\int_{K_0}^{2K_0} \exp\left(-(r-K)^2G^{-2}\right)\, dK\, dr \\
&= \left\{\int_{K_0+G\log K_0}^{2K_0-G\log K_0} + \int_{K_0-G\log K_0}^{K_0+G\log K_0} + \int_{2K_0-G\log K_0}^{2K_0+G\log K_0}\right\}\cdots dr \\
&= I_1 + I_2 + I_3,
\end{aligned}
$$

say. The integrals I_2 and I_3 are estimated similarly. By Hölder's inequality for integrals we have

$$
I_2 \ll G\Big(\int_a^b |\zeta(\tfrac{1}{2}+ir)|^6\, dr\Big)^{\frac{2}{3}}\Big(\int_a^b |\zeta(\tfrac{1}{2}-\varepsilon+iu+ir)|^6\, dr\Big)^{\frac{1}{6}}\Big(\int_a^b |\zeta(\tfrac{1}{2}-\varepsilon+iu-ir)|^6\, dr\Big)^{\frac{1}{6}}
$$

with $a = K_0 - G\log K_0$, $b = K_0 + G\log K_0$. Therefore we have to estimate the integral of $|\zeta(\frac{1}{2}+it)|^6$ over a short interval. By using the trivial estimate for $|\zeta(\frac{1}{2}+it)|^2$ and the asymptotic formula for the integral of $|\zeta(\frac{1}{2}+it)|^4$ ([4, Chapter 5]) it follows that

$$
I_2 + I_3 \ll GK_0^{2\mu(\frac{1}{2})+\varepsilon}(G + K_0^{2/3}). \tag{7.6}
$$

There remains (on this occasion we fix ε)

$$
\begin{aligned}
I_1 &= G\int_{K_0+G\log K_0}^{2K_0-G\log K_0} |\zeta(\tfrac{1}{2}+ir)|^4|\zeta(\tfrac{1}{2}-\varepsilon+iu-ir)\zeta(\tfrac{1}{2}-\varepsilon+iu+ir)|\times \\
&\quad\times\int_{(K_0-r)/G}^{(2K_0-r)/G} e^{-x^2}\, dx\cdot dr \\
&= \sqrt{\pi}G\int_{K_0+G\log K_0}^{2K_0-G\log K_0} |\zeta|^4|\zeta||\zeta|\, dr + O(\exp(-c\log^2 K_0)) \\
&\ll G\left(\int_{K_0}^{2K_0} |\zeta(\tfrac{1}{2}+ir)|^6\, dr\right)^{2/3}\left(\int_{K_0}^{2K_0} |\zeta(\tfrac{1}{2}-\varepsilon+iu+ir)|^6\, dr\right)^{1/6}\times \\
&\quad\times\left(\int_{K_0}^{2K_0} |\zeta(\tfrac{1}{2}-\varepsilon+iu-ir)|^6\, dr\right)^{1/6} \\
&\ll GK_0^{\frac{5}{4}+\frac{\varepsilon}{3}}\log^{37/4} K_0,
\end{aligned}
$$

on using the functional equation for $\zeta(s)$ for the factors with " $-\varepsilon$" and the bound (7.4). The gain of $\frac{\varepsilon}{3}$ and one log-factor is more than compensated by $K_0^{2-\varepsilon}\log^2 K_0$ in (7.5). We choose now $U = K_0^{1/2-\varepsilon}$ and note that $\mu(\frac12) < 1/6$ and $G \leqslant K_0^{1/2-\varepsilon}$. It follows from (7.6) and the last bound that the total contribution of $\mathcal{H}_7(f;h_0)$ to the integrated version of (3.1) is

$$\ll GK_0^{13/4}\log^{37/4} K_0. \tag{7.7}$$

It remains to deal yet with the contribution of $\mathcal{H}_2(f;h_0)$ and $\mathcal{H}_1(f;h_0)$, which will produce the main term. We have that the latter contributes

$$4\pi^{-3/2}K^3G\,\{C_1^*(K,G) + C_2^*(K,G)\} + O(K^{1+\varepsilon}G^3), \tag{7.8}$$

where

$$C_1^*(K,G) = \sum_{f\geqslant 1} f^{-1}d(f)(\log K + \gamma - \log(2\pi\sqrt{f})\exp(-(f/K)^\lambda),$$
$$C_2^*(K,G) = -\sum_{f\geqslant 1} f^{-1}d(f)(\log K + \gamma - \log(2\pi\sqrt{f})U_0(fK),$$

and the function U_0 is given by (3.7). As in [15] we note that $C_1^*(K,G)$ equals

$$\frac{1}{2\pi i\lambda}\int_{(1)}\left((\log K + \gamma - \log(2\pi))\zeta^2(w+1) + \zeta'(w+1)\zeta(w+1)\right)K^w\Gamma(w/\lambda)\,dw,$$

and likewise $C_2^*(K,G)$ can be represented by a similar type of integral. The line of integration is shifted to $\Re e\, w = -1$, where the integrand is regular. There is a pole of order three at $w=0$, hence by the residue theorem and Stirling's formula for $\Gamma(s)$ we obtain

$$\begin{aligned} C_1^*(K,G) &= \sum_{j=0}^{3} A_j \log^j K + O(K^{\varepsilon-1}),\\ C_2^*(K,G) &= \sum_{j=0}^{3} B_j \log^j K + O(K^{\varepsilon-1}), \end{aligned} \tag{7.9}$$

with $A_3 = B_3 = 1/3$. The O-term in (7.8) comes from the fact (see the definition of $\mathcal{H}_1(f;h)$ in (3.6)) that we have

$$(\hat{h}_0)'(\tfrac12) = 2i\pi^{3/2}K^3G + O(KG^3),$$
$$(\hat{h}_0)''(\tfrac12) = 8i\pi^{3/2}K^3G\log K + O(KG^3\log K).$$

From (7.3)–(7.9) we obtain ($G = G(K_0)\,(\leqslant K_0^{1/2-\varepsilon})$ will be suitably chosen a little later; see (8.9))

$$\begin{aligned} \int_{K_0}^{2K_0} C(K,G)\,dK &= GK^4\bar{P}_3(\log K)\Big|_{K_0}^{2K_0} \\ &\quad + O(GK_0^{13/4}\log^{37/4} K_0) + O(G^3K_0^{2+\varepsilon}), \end{aligned} \tag{7.10}$$

where $\bar{P}_3$ is another cubic polynomial, this time with leading coefficient $2/(3\pi^{3/2})$. Here we have assumed that the total contribution of $\mathcal{H}_2(f;h)$ can be absorbed in the error terms in (7.10), which will be shown in Section 8 with suitable G.

On the other hand, applying (1.7) in the form

$$\sum_{K\leqslant\kappa_j\leqslant K+H} \alpha_j H_j^3(\tfrac{1}{2}) \ll_\varepsilon K^{1+\varepsilon}H \qquad (1\ll H\leqslant K)$$

and using the method of proof of Section 5, it is seen that

$$\begin{aligned}&\int_{K_0}^{2K_0} C(K,G)\,dK\\ &= \sum_{j\geqslant 1}\alpha_j H_j^3(\tfrac{1}{2})\int_{K_0}^{2K_0}(\kappa_j^2+\tfrac{1}{4})\exp(-(\kappa_j-K)^2G^{-2})\,dK + o(1)\\ &= \sqrt{\pi}G\sum_{K_0\leqslant\kappa_j\leqslant 2K_0}\alpha_j H_j^3(\tfrac{1}{2})\kappa_j^2 + O(K_0^{3+\varepsilon}G^2).\end{aligned} \tag{7.11}$$

Therefore we obtain from (7.10) and (7.11)

$$\begin{aligned}&\sum_{K_0\leqslant\kappa_j\leqslant 2K_0}\alpha_j H_j^3(\tfrac{1}{2})\kappa_j^2\\ &= K^4\left(\frac{2}{3\pi^2}\log^3 K + a_2\log^2 K + a_1\log K + a_0\right)\Bigg|_{K_0}^{2K_0}\\ &\quad + O(K_0^{13/4}\log^{37/4}K_0) + O(GK_0^{3+\varepsilon})\end{aligned} \tag{7.12}$$

plus the contribution of $\mathcal{H}_2(f;h)$. We apply partial summation (to get rid of κ_j^2), replace K_0 by $K_0 2^{-j}$, and sum over j. The O-terms will be absorbed in the O-term of Theorem 1 if $G = K_0^\alpha$ with any $0<\alpha<1/4$.

8. The contribution of $\mathcal{H}_2(f;h)$

To complete the proof of Theorem 1 it remains to show that the total contribution of $\mathcal{H}_2(f;h)$ is absorbed in the O-terms in (7.12) with suitable G. We follow, as before, the proof given in [5]. We use the observation made in [7] which states that the relevant sum to be estimated is, after integration over $[K_0, 2K_0]$,

$$\begin{aligned}&GK_0^{5/2}\sum_{f\leqslant 3K_0} f^{-1/2}\sum_{m\leqslant fG^{-2}\log^2 K}(m/f)^{1/4}d(m)d(m+f)\times\\ &\times\left(\sqrt{\frac{m}{f}}+\sqrt{1+\frac{m}{f}}\right)^{-2iK_0} \mathrm{e}^{-CG^2mf^{-1}}\log\left(\sqrt{\frac{m}{f}}+\sqrt{1+\frac{m}{f}}\right)^{-1}.\end{aligned} \tag{8.1}$$

Note that (8.1) corresponds to (3.1) of [5] with the additional factor $(m/f)^{1/4}$, namely to (16) of [7]. As in (3.2) of [5] we replace $m+f$ by n and consider subsums of the sum in (8.1) where $m \sim M$ (meaning $M < m \leqslant 2M$), $n \sim N$. If we get rid of the last two factors in (8.1) by partial summation and Taylor's formula, respectively, we are left with the sum

$$\begin{aligned} &GK_0^{5/2} \sum_{n\sim N} d(n)n^{-1/4} \sum_{m\sim M} d(m)m^{-3/4} \exp(iF(m,n)), \\ &F(m,n) \ := \ -2K_0 \log\left(\sqrt{\frac{m}{n-m}} + \sqrt{\frac{n}{n-m}}\right), \end{aligned} \tag{8.2}$$

and we have, with effectively computable constants b_j,

$$\log\left(\sqrt{\frac{m}{n-m}} + \sqrt{\frac{n}{n-m}}\right) = \sum_{j=1}^{\infty} b_j \left(\frac{m}{n}\right)^{j/2}. \tag{8.3}$$

As in [5, eq. (3.4)], we have the conditions

$$K_0^\varepsilon \leqslant G \leqslant K_0^{1/2-\varepsilon}, \ MG^2 \log^2 K_0 \ll N \ll K_0. \tag{8.4}$$

By applying the Cauchy-Schwarz inequality we see that the sum in (8.2) is

$$\begin{aligned} &\leqslant \left(\sum_{n\sim N} d^2(n)n^{-1/2}\right)^{1/2} \left(\sum_{n\sim N}\left|\sum_{m\sim M} d(m)m^{-3/4}\exp(iF(m,n))\right|^2\right)^{1/2} \\ &\ll N^{1/4}\log^2 N {\sum}^{1/2}, \end{aligned}$$

where we have set

$$\begin{aligned} \sum \ :&= \sum_{n\sim N}\left|\sum_{m\sim M} d(m)m^{-3/4}\exp(iF(m,n))\right|^2 \\ &= \sum_{m\sim M} d^2(m)m^{-3/2}O(N) \\ &\quad + \sum_{m_1\neq m_2} d(m_1)d(m_2)(m_1m_2)^{-3/4} \sum_{n\sim N} \exp(iF(m_1,n) - iF(m_2,n)) \\ &\ll NM^{\varepsilon-1/2} + M^{\varepsilon-3/2} \sum_{m_1\neq m_2}\left|\sum_{n\sim N} \exp(iF(m_1,n) - iF(m_2,n))\right|. \end{aligned}$$

The effect of this procedure is that the exponential sum over n does not contain the divisor function, and consequently can be estimated by the technique of exponent pairs (see e.g., [3, Chapter 2]). Note that by (8.3) we have (in the relevant range for m, n)

$$\frac{\partial}{\partial n}(F(m_1, n) - F(m_2, n)) \asymp |m_1 - m_2| K_0 M^{-1/2} N^{-3/2}.$$

Thus if (κ, λ) is an exponent pair, then we have

$$\begin{aligned} \sum &\ll N M^{\varepsilon - 1/2} + M^{\varepsilon - 3/2} \sum_{m_1 \neq m_2} \left(\frac{N^{3/2} M^{1/2}}{|m_1 - m_2| K_0} + \left(\frac{K_0 M^{1/2}}{N^{3/2}} \right)^{\kappa} N^{\lambda} \right) \\ &\ll N M^{\varepsilon - 1/2} + N^{3/2} K_0^{\varepsilon - 1} + M^{\frac{1}{2} + \frac{\kappa}{2}} K_0^{\kappa} N^{\lambda - \frac{3}{2}\kappa}. \end{aligned}$$

Hence in view of (8.4) the expression in (8.2) is bounded by

$$\begin{aligned} &G K_0^{\frac{5}{2}+\varepsilon} \left(N^{\frac{3}{4}} M^{-\frac{1}{4}} + N K_0^{-\frac{1}{2}} + K_0^{\frac{\kappa}{2}} M^{\frac{1}{4} + \frac{\kappa}{4}} N^{\frac{\lambda}{2} + \frac{1}{4} - \frac{3}{4}\kappa} \right) \\ &\quad \ll G K_0^{\frac{5}{2}+\varepsilon} N^{\frac{3}{4}} M^{-\frac{1}{4}} + G K_0^{3+\varepsilon} + G K_0^{\frac{11}{4}+\varepsilon} M^{\frac{3}{8}} N^{\frac{1}{8}} \\ &\quad \ll G K_0^{\frac{5}{2}+\varepsilon} N^{\frac{3}{4}} M^{-\frac{1}{4}} + G K_0^{3+\varepsilon} + G K_0^{\frac{5}{2}+\varepsilon} K_0^{\frac{1}{4}} (N G^{-2})^{\frac{3}{8}} N^{\frac{1}{8}} \\ &\quad \ll G K_0^{\frac{5}{2}+\varepsilon} N^{\frac{3}{4}} M^{-\frac{1}{4}} + G K_0^{3+\varepsilon} + G K_0^{\frac{13}{4}+\varepsilon} G^{-\frac{3}{4}} \end{aligned} \tag{8.5}$$

with $(\kappa, \lambda) = (\frac{1}{2}, \frac{1}{2})$. The bound in (8.5) will be used for large M. For small M we shall transform the sum

$$S(N) := \sum_{\frac{1}{2}N \leqslant n \leqslant \frac{5}{2}N} \varphi(n) d(n) n^{-1/4} \exp(iF(m, n))$$

by Voronoi's summation formula (see e.g., [3, Chapter 3]), treating the real and imaginary part separately. Here $\varphi(x) \geqslant 0$ is a smooth function supported in $[\frac{1}{2}N, \frac{5}{2}N]$ such that it equals unity in $[N, 2N]$ and $\varphi^{(r)}(x) \ll_r N^{-r}$ $(r = 0, 1, \dots)$. Then we have

$$\begin{aligned} S(N) &= \int_{\frac{1}{2}N}^{\frac{5}{2}N} (\log x + 2\gamma) x^{-1/4} \varphi(x) \exp(iF(m, x))\, dx \\ &\quad + \sum_{n=1}^{\infty} \int_{\frac{1}{2}N}^{\frac{5}{2}N} \varphi(x) x^{-1/4} \alpha(nx) \exp(iF(m, x))\, dx, \end{aligned} \tag{8.6}$$

where $\alpha(nx)$ admits an asymptotic expansion whose first term is

$$-2^{1/2}(xn)^{-1/4} \sin(4\pi\sqrt{nx} - \pi/4).$$

By the first derivative test the first integral in (8.6) is

$$\ll \frac{N^{5/4} \log N}{M^{1/2} K_0},$$

hence it contributes to (8.2)

$$\ll GK_0^{3/2}N^{5/4}M^{-1/4}\log^2 K_0 = GK_0^{11/4}\log^2 K_0.$$

Further consider the main contribution of the terms in (8.6), which is a multiple of

$$\int_{\frac{1}{2}N}^{\frac{5}{2}N} \varphi(x)x^{-1/2}n^{-1/4}\exp\left(4\pi i\sqrt{nx} \pm iK_0\sum_{j=1}^{\infty} b_j\left(\frac{m}{x}\right)^{\frac{1}{2}j}\right)dx. \tag{8.7}$$

The case of the "minus" sign is less difficult, and in the case of the "plus" sign, let

$$f(x) = f(x; m, n, K_0) := 4\pi\sqrt{nx} + K_0\sum_{j=1}^{\infty} b_j\left(\frac{m}{x}\right)^{\frac{1}{2}j},$$

so that

$$\frac{\partial f}{\partial x} = 2\pi\sqrt{\frac{n}{x}} - K_0\sum_{j=1}^{\infty}\tfrac{1}{2}jb_jm^{j/2}x^{-j/2-1}.$$

If $n > CK_0^2MN^{-2}$ with sufficiently large $C > 0$, then $\frac{\partial f}{\partial x} \asymp \sqrt{\frac{n}{x}}$. Therefore the above integral becomes, on integrating by parts,

$$in^{-1/4}\int_{\frac{1}{2}N}^{\frac{5}{2}N}\left(\frac{\varphi(x)x^{-1/2}}{\frac{\partial f}{\partial x}}\right)'\exp(if(x))\,dx.$$

But as

$$\left(\frac{\varphi(x)x^{-1/2}}{\frac{\partial f}{\partial x}}\right)' \ll \frac{1}{\sqrt{nxN}},$$

it follows by repeated integration by parts that the contribution of $n > CK_0^2MN^{-2}$ is negligible. If $n \leqslant CK_0^2MN^{-2}$, then the exponential integral in question may have a saddle point x_0, namely the solution of $\frac{\partial f}{\partial x} = 0$. Hence

$$2\pi\sqrt{\frac{n}{x_0}} = K_0\sum_{j=1}^{\infty}\tfrac{1}{2}jb_jm^{j/2}x_0^{-j/2-1},$$

giving (since $b_1 = 1$)

$$x_0 \sim \frac{K_0}{2\pi}\sqrt{\frac{m}{n}},$$

and $x_0 \in [\frac{1}{2}N, \frac{5}{2}N]$ for $n \asymp K_0^2MN^{-2}$. By the saddle point method (see [3, Chapter 2]) the main contribution comes from the saddle point and is

$$\ll \left|\frac{\partial^2 f}{\partial x^2}\Big|_{x=x_0}\right|^{-1/2} \ll \left(\frac{1}{N}\sqrt{\frac{n}{N}}\right)^{-1/2} \ll N^{3/4}n^{-1/4}.$$

Thus the integral in (8.7) is $\ll N^{1/4}n^{-1/2}$, and consequently the sum in (8.6) is

$$\ll N^{1/4} \sum_{n \leqslant CK_0^2MN^{-2}} d(n)n^{-1/2} \ll K_0M^{1/2}N^{-3/4}\log K_0,$$

and the total contribution is therefore

$$\ll GK_0^{11/4}\log^2 K_0 + GK_0^{7/2}M^{3/4}N^{-3/4}\log^2 K_0. \tag{8.8}$$

Hence for $M \geqslant N^{3/2}/K_0$ we use (8.5) and otherwise we apply (8.8); if $N \leqslant K_0^{2/3}$ then $N^{3/2}/K_0 \leqslant 1$, but then we can simply use (8.5). We obtain, in view of (7.11) and (7.12) and the discussion thereafter, that the total contribution of the error terms in Theorem 1 will be

$$\ll K_0^{5/4}\log^{37/4} K_0 + GK_0^{1+\varepsilon} + K_0^{5/4+\varepsilon}G^{-3/4} \ll K_0^{5/4}\log^{37/4} K_0$$

for

$$G = K_0^{1/7}. \tag{8.9}$$

This completes the proof of Theorem 1. Note that, apart from the contribution of the integral with six zeta values (cf. (7.3)), the remaining terms are of the order $K_0^{8/7+\varepsilon}$ with the choice $G = K_0^{1/7}$, and more refined exponential sum techniques could yield even smaller values of G. From (7.12) it follows that the leading coefficient of $P_3(x)$ in (1.10) is $4/(3\pi^2)$.

9. Another proof of Theorem 2

We shall sketch now another proof of Theorem 2 (cf. (4.1)), namely

$$\begin{aligned}&\sum_{\kappa_j \leqslant K} \alpha_j H_j^4(\tfrac{1}{2}) + O\left(\log^2 K \int_0^K |\zeta(\tfrac{1}{2}+it)|^8\,dt\right)\\ &= K^2P_6(\log K) + O(K^{4/3+\varepsilon}).\end{aligned} \tag{9.1}$$

The argument is based on M. Jutila's proof [7] of (1.9), and will be outlined below. Similarly as in the proof of Theorem 1, it is the contribution of $\mathcal{H}_2(f;h)$ (see (3.6)) that is the essential one. To introduce $H_j^2(\frac{1}{2})$ in Motohashi's transformation formula for sums of $H_j^2(\frac{1}{2})$ ([15, Lemma 3.8]) and obtain the formula for sums of $H_j^4(\frac{1}{2})$, one uses [7, Lemma 1]. This formula says that

$$\begin{aligned}H_j^2(\tfrac{1}{2}) = &\sum_{mn \leqslant 3K^2} t_j(m)t_j(n)(mn)^{-1/2}\exp(-(mn/K^2)^\lambda)\\ &- \sum_{mn \leqslant 3K^2} t_j(m)t_j(n)(mn)^{-1/2}R_j(mnK^2) + O(1),\end{aligned} \tag{9.2}$$

for $|\kappa_j - K| \leqslant G\log K$ with $\log^2 K < G < K^{1-\delta}$ for $0 < \delta < 1$, $\lambda = C\log K$ with sufficiently large $C > 0$. The function R_j in (9.2) comes from the squaring of the functional equation for $H_j(\frac{1}{2} + w)$, namely

$$R_j(x) = \frac{1}{2\pi^4 i\lambda}\int_{-\lambda^{-1}-i\lambda^2}^{-\lambda^{-1}+i\lambda^2} (16\pi^4 x)^w \Gamma^2(\tfrac{1}{2} - w + i\kappa_j)\Gamma^2(\tfrac{1}{2} - w - i\kappa_j)\times$$
$$\times (\cosh(\pi\kappa_j) + \sin(\pi w))^2\Gamma(w/\lambda)\, dw.$$

In the context of [7] the error term $O(1)$ in (9.2) suffices, but similarly to [15, Lemma 3.9] this error term can be considerably sharpened. The main term (i.e., $K^2P_6(\log K)$ in (9.1)) is derived analogously as was done in the proof of Theorem 1; it is obtained in terms of the expressions resembling the functions C_j^* $(j = 1, 2)$ in (7.8), only in this case they will be somewhat more complicated. Namely to obtain the asymptotic formula for the sum

$$\sum_{j=1}^{\infty} \alpha_j H_j^4(\tfrac{1}{2}) h_0(\kappa_j) \tag{9.3}$$

with h_0 given by (1.6), we use the Mellin relation

$$\exp(-x^\lambda) = \frac{1}{2\pi i\lambda}\int_{(1)} \Gamma(z/\lambda)x^{-z}\, dz \qquad (x, \lambda > 0)$$

in conjunction with (9.2) and [15, Lemma 3.8]. We use the identity (3.3) to transform the product of two t_j-functions into one, and extend summation over all values of m, n, producing a negligible error. Then we obtain two divisor functions, and we use the classical identity

$$\sum_{n=1}^{\infty} d^2(n)n^{-s} = \frac{\zeta^4(s)}{\zeta(2s)} \quad (\Re e\, s > 1).$$

It follows that, similarly to the case of Theorem 1, the main term for (9.3) will be of the form

$$4\pi^{-3/2}K^3G(\mathcal{D}_1^*(K,G) + \mathcal{D}_2^*(K,G)),$$

where $\mathcal{D}_1^*(K,G)$ comes from the first sum on the right-hand side of (9.2). We have (γ is Euler's constant)

$$\mathcal{D}_1^*(K,G) = \frac{1}{2\pi i\lambda}\int_{(1)}\Big\{(\log K + \gamma - \log(2\pi))\frac{\zeta^4(w+1)}{\zeta(2w+2)}$$
$$+ \frac{1}{2}\Big(\frac{\zeta^4(w+1)}{\zeta(2w+2)}\Big)'\Big\}\zeta(2w+1)K^{2w}\Gamma(w/\lambda)\, dw, \tag{9.4}$$

and analogously $\mathcal{D}_2^*(K,G)$ comes from the second sum on the right-hand side of (9.2). The integrand in (9.4) has a pole of order six at $w = 0$. We shift the line of integration to $\Re e\, w = -1$, developing the integrand into power series to calculate the residue. The coefficient of $\log^6 K$ is found to be $4/(15\pi^2)$, and clearly the coefficients of lower powers of the logarithm can be also evaluated explicitly. This is the analogue of $A_3 = 1/3$ in (7.9). The coefficient of $\log^6 K$ coming from $\mathcal{D}_2^*(K,G)$ will be the same. Proceeding as was done in Section 7, we see then that the leading coefficient of $P_6(x)$ in (1.11) is $16/(15\pi^4)$, as claimed.

We continue now the second proof of Theorem 2. From the discussion above it is seen that the relevant sum to be estimated (this corresponds to [7, eq. (16)]) is, up to a constant factor,

$$GK^{5/2}\sum_{f\ll K^2} v(f)d(f)f^{-3/4} \sum_{m\leqslant fG^{-2}\log^2 K} m^{-1/4}d(m)d(m+f)$$
$$\times\left(\sqrt{\frac{m}{f}}+\sqrt{1+\frac{m}{f}}\right)^{-2iK}\exp\left(-G^2\log^2\left(\sqrt{\frac{m}{f}}+\sqrt{1+\frac{m}{f}}\right)\right)$$
$$\times\left(\log\left(\sqrt{\frac{m}{f}}+\sqrt{1+\frac{m}{f}}\right)\right)^{-1},$$

where v is a smooth weight function supported in $[F,\,2F]$ with $F \ll K_0^2$, and $K_0 \leqslant K \leqslant 2K_0$. A new ingredient is the last log-factor (coming from integration), which is of the order $\ll \sqrt{f/m}$. Consider now the sum over $f \asymp F$ and $m \asymp M$. Then, by the above remarks, the final estimate in [7], namely

$$\ll GK^{\varepsilon}(F^{-1/2}KM^{1/2})^{3/2},$$

should be modified by cancelling the factor G and multiplying by $\sqrt{F/M}$. Therefore the contribution coming from $\mathcal{H}_2(f;h)$ will be

$$\ll K_0^{3/2+\varepsilon}(M/F)^{1/4} \ll K_0^{4/3+\varepsilon},$$

since $M/F \ll G^{-2}\log^2 K_0$ and $G = K_0^{1/3}$ ($\asymp Q$ of Section 4). This finishes the discussion concerning the second proof of Theorem 2.

10. The first moment of $H_j(\frac12)$

As promised in the Introduction, we shall say a few words at the end on the sum

$$\sum_{\kappa_j\leqslant T}\alpha_j H_j(\tfrac12). \tag{10.1}$$

In conjunction with the conjecture (1.12) I expect the sum in (10.1) to be equal to

$$AT^2 + O(T\log^3 T) \qquad (A = \frac{1}{\pi^2}), \tag{10.2}$$

where the error term in (10.2) comes from the integral with $|\zeta(\frac{1}{2}+it)|^2$ in (1.12), and the value of A is provided by Random matrix theory (see the discussion at the end of Section 1). However obtaining (10.2) is rather difficult. Namely, simple specialization (simplification) of the procedure used by Y. Motohashi [14] for sums of $H_j^2(\frac{1}{2})$ does not work directly. In any case it can be shown that

$$T^2(\log T)^{-7/2} \ll \sum_{\kappa_j\leqslant T} \alpha_j H_j(\tfrac{1}{2}) \ll T^2(\log T)^{1/2}. \tag{10.3}$$

The upper bound in (10.3) follows from the Cauchy-Schwarz inequality and (1.2). To derive the lower bound, let

$$S(T) := \sum_{T\leqslant\kappa_j\leqslant 2T} \alpha_j H_j(\tfrac{1}{2}).$$

For a given $V > 0$ we have (since $H_j(\frac{1}{2}) \geqslant 0$)

$$S(T) \geqslant V \sum_{T\leqslant\kappa_j\leqslant 2T, H_j(\frac{1}{2})\geqslant V} \alpha_j,$$

and we obtain

$$\begin{aligned} T^2\log T &\ll \sum_{T\leqslant\kappa_j\leqslant 2T} \alpha_j H_j^2(\tfrac{1}{2}) = \sum_{H_j(\frac{1}{2})\geqslant V} + \sum_{H_j(\frac{1}{2})<V} \\ &\ll \left(\sum_{T\leqslant\kappa_j\leqslant 2T, H_j(\frac{1}{2})\geqslant V} \alpha_j \sum_{T\leqslant\kappa_j\leqslant 2T} \alpha_j H_j^4(\tfrac{1}{2})\right)^{1/2} + V^2 \sum_{T\leqslant\kappa_j\leqslant 2T} \alpha_j \\ &\ll \left(V^{-1}S(T)T^2\log^6 T\right)^{1/2} + T^2V^2. \end{aligned}$$

Here we used the best possible bounds (cf. [4, eq. (5.48)] and (1.11))

$$\sum_{\kappa_j\leqslant T} \alpha_j \ll T^2, \quad \sum_{\kappa_j\leqslant T} \alpha_j H_j^4(\tfrac{1}{2}) \ll T^2\log^6 T.$$

The choice $V = \delta\sqrt{\log T}$ for sufficiently small $\delta > 0$ yields then

$$T^4\log^2 T \ll V^{-1}S(T)T^2\log^6 T,$$

giving the lower bound in (10.3).

One way to tackle the sum in (10.1) is to take $n = 1$ in Kuznetsov's trace formula ([14, eq. (2.5)]) and multiply by m^{-u} to obtain

$$\begin{aligned} &\sum_{j=1}^{\infty} \varepsilon_j\alpha_j t_j(m)m^{-u}h(\kappa_j) \\ &= -\frac{1}{\pi}\int_{-\infty}^{\infty} \sigma_{2ir}(m)m^{-u-ir}\frac{h(r)}{|\zeta(1+2ir)|^2}\,dr \\ &\quad + \sum_{\ell=1}^{\infty} m^{-u}\ell^{-1}S(m,-1;\ell)\psi\Big(4\pi\frac{\sqrt{m}}{\ell}\Big), \end{aligned} \tag{10.4}$$

where $S(m,n;\ell)$ is the Kloosterman sum, $h(r)$ is given by (2.3), while with $h^*(s)$ given by (2.6) we set

$$\psi(x) = \frac{1}{\pi^2}\int_{(\alpha)} \frac{(x/2)^{-2s}}{\cos(\pi s)} h^*(s)\,ds \qquad (-3/2 < \alpha < 3/2). \tag{10.5}$$

We proceed now, assuming that $\Re e\, u > 2$ and $\alpha = -2/3$ in (10.5). Using the trivial bound $|S(m,-1;\ell)| \leqslant \ell$, we note that summation over m in (10.4) yields, by absolute convergence,

$$\begin{aligned}&\sum_{j=1}^{\infty} \varepsilon_j \alpha_j H_j(u) h(\kappa_j) + \frac{1}{\pi}\int_{-\infty}^{\infty} \zeta(u+ir)\zeta(u-ir)\frac{h(r)}{|\zeta(1+2ir)|^2}\,dr \\ &= \sum_{m=1}^{\infty} m^{-u} \sum_{\ell=1}^{\infty} \ell^{-1} S(m,-1;\ell)\psi(4\pi\frac{\sqrt{m}}{\ell}).\end{aligned} \tag{10.6}$$

By deforming suitably the contour and applying the residue theorem, we see that the integrated term admits analytic continuation to the region $\Re e\, u < 1$ which is of the form

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \zeta(u+ir)\zeta(u-ir)\frac{h(r)}{|\zeta(1+2ir)|^2}\,dr + 4\frac{h(i(u-1))}{\zeta(3-2u)}.$$

Since $H_j(\frac{1}{2}) = 0$ if $\varepsilon_j = -1$ and $h(\pm\frac{1}{2}i) = 0$, (10.6) reduces to (compare with (1.12) when $k = 1$)

$$\sum_{j=1}^{\infty} \alpha_j H_j(\tfrac{1}{2}) h(\kappa_j) + \frac{1}{\pi}\int_{-\infty}^{\infty} |\zeta(\tfrac{1}{2}+ir)|^2 \frac{h(r)}{|\zeta(1+2ir)|^2}\,dr = L(\tfrac{1}{2}), \tag{10.7}$$

where $L(u)$ is the analytic continuation of the function

$$\sum_{m=1}^{\infty} m^{-u} \sum_{\ell=1}^{\infty} \ell^{-1} S(m,-1;\ell)\psi(4\pi\frac{\sqrt{m}}{\ell}) \qquad (\Re e\, u > 2). \tag{10.8}$$

One can try to transform the expression for $L(u)$ by using the properties of the Kloosterman–Selberg zeta-function

$$Z_{m,n}(s) := (2\pi\sqrt{mn})^{2s-1} \sum_{\ell=1}^{\infty} S(m,n;\ell)\ell^{-2s} \qquad (\Re e\, s > 1).$$

Namely one has the spectral decomposition (see [4, eqs. (5.65)–(5.68)] of $Z_{m,n}(s)$. This can be used in (10.8), and one expects that the main contribution will come from the discrete spectrum (i.e. [4, (5.66)]). However this will lead eventually to the same type of sum as the one we started from.

One can follow the approach of [14] and write $(-3/2 < \alpha < -1/4)$

$$
\begin{aligned}
L(u) &= \pi^{-2}\sum_{\ell=1}^{\infty}\ell^{-1}P(u;\ell),\\
P(u;\ell) &= \int_{(\alpha)}(2\pi/\ell)^{-2s}\frac{h^*(s)}{\cos(\pi s)}Q(s;u,\ell)\,ds,\\
Q(s;u,\ell) &= \sum_{m=1}^{\infty}m^{-u-s}S(m,-1;\ell)\\
&= \sum_{(a,\ell)=1,a\bar a\equiv 1(\mathrm{mod}\,\ell)} \mathrm{e}(-a/\ell)E(u+s;\mathrm{e}(\bar a/\ell)),
\end{aligned}
\tag{10.9}
$$

where E is the Lerch zeta-function $(1 \leqslant h \leqslant k,\ k \geqslant 2,\ h,k \in \mathbb{N},\ \mathrm{e}(z) = \mathrm{e}^{2\pi i z})$

$$
E\left(s;\mathrm{e}\left(\frac{h}{k}\right)\right) := \sum_{m=1}^{\infty}\mathrm{e}\left(\frac{mh}{k}\right)m^{-s} = \sum_{j=1}^{k}\mathrm{e}\left(\frac{jh}{k}\right)k^{-s}\zeta(s,\frac{j}{k}),
$$

initially defined for $\Re e\, s > 1$. It can be expressed in terms of the Hurwitz zeta-function, defined for $0 < a \leqslant 1,\ \sigma > 1$ by $\zeta(s,a) = \sum_{n=0}^{\infty}(n+a)^{-s}$. Since $\zeta(s,\frac{j}{k})$ has a only the simple pole at $s = 1$ with residue 1, it follows that E is entire, and satisfies the functional equation

$$
\begin{aligned}
&E\left(s;\mathrm{e}\left(\frac{h}{k}\right)\right)\\
&= \frac{\Gamma(1-s)}{(2\pi)^{1-s}}\left\{\mathrm{e}^{\frac{\pi i}{2}(1-s)}\zeta(1-s,\frac{h}{k}) + \mathrm{e}^{\frac{\pi i}{2}(s-1)}\zeta(1-s,1-\frac{h}{k})\right\}.
\end{aligned}
\tag{10.10}
$$

This means that the second expression in (10.9) provides the analytic continuation of $Q(s;u,\ell)$ as an entire function of both u and s, of polynomial growth in $|u|+|s|$.

This, however, differs from Motohashi's situation [14], where he obtained the Estermann zeta-function D, represented in the region of absolute convergence by the series

$$
D(s,\xi;\mathrm{e}(b/\ell)) := \sum_{n=1}^{\infty}n^{-s}\sigma_\xi(n)\mathrm{e}(nb/\ell) \qquad (1 \leqslant b \leqslant \ell,\ b,\ell \in \mathbb{N}).
$$

This function has two simple poles (at $s = 1$ and $1+\xi$) which are (in part) responsible for the main term $(2.34)_1$ in [14]. But we do not have such a term here! What we get is simply, since E is entire,

$$
L(\tfrac{1}{2}) = \frac{1}{\pi^2}\int_{(\alpha)}(2\pi)^{-2s}\frac{h^*(s)}{\cos(\pi s)}\sum_{\ell=1}^{\infty}\sum_{(a,\ell)=1}\mathrm{e}(-a/\ell)\ell^{2s-1}E(s+\tfrac{1}{2};\mathrm{e}(\bar a/\ell))\,ds. \tag{10.11}
$$

In (10.11) we have $-3/2 < \alpha < -1/2$. To transform further $L(\frac{1}{2})$ we make the change of variable $s = \frac{1}{2} - w$ in (10.11) and use the functional equation (10.10). It follows that $L(\frac{1}{2})$ is a linear combination of

$$I_+ := \int_{(\beta)} (2\pi)^w h^*(\tfrac{1}{2} - w) \frac{\Gamma(w)}{\sin(\frac{1}{2}\pi w)} M_+(w)\, dw \quad (1 < \beta < 2)$$

and

$$I_- := \int_{(\beta)} (2\pi)^w h^*(\tfrac{1}{2} - w) \frac{\Gamma(w)}{\cos(\frac{1}{2}\pi w)} M_-(w)\, dw \quad (1 < \beta < 2),$$

where for $\Re e\, w > 1$

$$M_+(w) := \sum_{\ell=1}^{\infty} \sum_{(a,\ell)=1, a\bar{a}\equiv 1 (\mathrm{mod}\, \ell)} e(-a/\ell) l^{-2w} \left((\zeta(w, \bar{a}/\ell) + \zeta(w, 1 - \bar{a}/\ell) \right),$$

$$M_-(w) := \sum_{\ell=1}^{\infty} \sum_{(a,\ell)=1, a\bar{a}\equiv 1 (\mathrm{mod}\, \ell)} e(-a/\ell) l^{-2w} \left((\zeta(w, \bar{a}/\ell) - \zeta(w, 1 - \bar{a}/\ell) \right).$$

The problem is to obtain analytic continuation of the functions $M_\pm(w)$ to the left of the line $\Re e\, w = 1$, since one would like to move the contour of integration in I_+ and I_- to the left.

It transpires that in any case it seems difficult to show that the sum in (10.1) equals the expression in (10.2).

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