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To Professor Włodzimierz Staś on his 75^{th} birthday

ON SUMS OF THREE UNIT FRACTIONS WITH POLYNOMIAL DENOMINATORS

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Abstract: The equation $m/(ax+b) = 1/F_1(x) + 1/F_2(x) + 1/F_3(x)$ is shown to be impossible under some conditions on polynomials $ax+b$ and F_1, F_2, F_3.
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A well known conjecture of Erdős and Straus [2] asserts that for every integer $n > 1$ the equation

$$\frac{4}{n} = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}$$

is solvable in positive integers x_1, x_2, x_3. Sierpiński [10] has made an analogous conjecture concerning $5/n$ and the writer has conjectured that for every positive integer m the equation

$$\frac{m}{n} = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} \tag{1}$$

is solvable in positive integers x_1, x_2, x_3 for all integers $n > n_0(m)$ (see [10], p. 25). For $m \le 12$ one knows many identities

$$\frac{m}{ax+b} = \frac{1}{F_1(x)} + \frac{1}{F_2(x)} + \frac{1}{F_3(x)}, \tag{2}$$

where a, b are integers, $a > 0$ and F_i are polynomials with integral coefficients and the leading coefficients positive, see [1], [5], [7], [8], [11], Section 28.5. It could seem that a proof of solvability of (2) for a fixed m and $n > n_0(m)$ could be obtained by producing a finite set of identities of the form (2) with a fixed a and b running through the set of all residues $\bmod a$. The theorems given below show that this is impossible.

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Theorem 1. *Let a, b be integers, $a > 0$, $(a, b) = 1$. If b is a quadratic residue* $\bmod a$, *then there are no polynomials F_1, F_2, F_3 in $\mathbb{Z}[x]$ with the leading coefficients positive, satisfying* (2) *with $m \equiv 0 \bmod 4$.*

Theorem 2. *Let m, a, b be integers, $a > 0$, $m > 3b > 0$. There are no polynomials F_1, F_2, F_3 in $\mathbb{Z}[x]$ with the leading coefficients positive, satisfying* (2).

Theorem 1 in the crucial case $m = 4$ has been quoted in the book [4] (earlier inaccurately in [3]), but the proof has not been published before. The theorem is closely related to a result of Yamamoto [12] and the crucial lemma is a consequence of his work. Possibly, Theorem 2 can be generalized as follows. Let k, m, a, b be positive integers, $m > kb$. There are no polynomials $F_1, F_2, \ldots, F_k$ in $\mathbb{Z}[x]$ with the leading coefficients positive such that

$$\frac{m}{ax+b} = \sum_{i=1}^{k} \frac{1}{F_i(x)}.$$

Note that by a theorem of Sander [9] the above equation has only finitely many solutions in polynomials F_i for fixed a, b, m and k.

Notation. For $\Omega \subset \mathbb{R}[x]$ we shall denote by Ω^+ the set of polynomials in Ω with the leading coefficient positive.

For two polynomials A, B in $\mathbb{Z}[x]$, not both zero, we shall denote by (A, B) the polynomial $D \in \mathbb{Z}[x]^+$ with the greatest possible degree and the greatest possible leading coefficient such that $A/D \in \mathbb{Z}[x]$ and $B/D \in \mathbb{Z}[x]$.

Lemma 1. *If A, B, C, D are in $\mathbb{Z}[x]$, $(A, B) = 1$ and $A/B = C/D$, then $C = HA$, $D = HB$ for an $H \in \mathbb{Z}[x]$. If $(C, D) = 1$ then $H = \pm 1$.*

Proof. This follows from Theorem 44 in [6], the so called Gauss's lemma. ∎

Lemma 2. *The equations*

$$n^2 = 4(cs - b^*)b^*r - s \tag{3}$$

and

$$n^2 s = 4(cs - b^*)b^*r - 1 \tag{4}$$

have no solutions in positive integers b^*, c, n, r, s.

Proof. This is a consequence of Theorem 2 in [12]: according to this theorem n^2 does not satisfy either of the two congruences

$$n^2 \equiv -s (\bmod 4a^*b^*), \tag{5}$$

$$n^2 s \equiv -1 (\bmod 4a^*b^*), \tag{6}$$

where a^*, b^*, s are positive integers and $s \mid a^* + b^*$, while just such congruences follow from (3) and (4) with $a^* = cs - b^*$. The impossibility of the congruences

(5) and (6) is established in [12] by evaluation of the Kronecker symbol $(-s/ab)$; instead one can use the Jacobi symbol as follows.

(3) gives $n^2 = (4b^*cr-1)s - 4b^{*2}r$, (4) gives $(ns)^2 = (4b^*crs-1)s - 4b^{*2}rs$, while for $e = 2^\alpha e_0 > 0$, e_0 odd, we have by the reciprocity law ([6], Section 42)

$$\left(\frac{-4b^{*2}e}{4b^*es-1}\right) = -\left(\frac{e_0}{4b^*es-1}\right) = -(-1)^{(e_0-1)/2}\left(\frac{4b^*es-1}{e_0}\right)$$
$$= -(-1)^{(e_0-1)/2}\left(\frac{-1}{e_0}\right) = -1.$$

■

Proof of Theorem 1. It is clearly sufficient to prove the theorem for $m = 4$. Assume that we have (2) with $m = 4$. Thus

$$4F_1(x)F_2(x)F_3(x) = (ax+b)\big(F_2(x)F_3(x) + F_1(x)F_3(x) + F_1(x)F_2(x)\big),$$

hence

$$F_1(-b/a)F_2(-b/a)F_3(-b/a) = 0.$$

If we had $F_i(-b/a) = 0$ for each $i \le 3$, then there would exist polynomials $G_i \in \mathbb{Q}[x]^+$ such that $F_i(x) = (ax+b)G_i(x)$. Since $(a,b) = 1$ it follows from Gauss's lemma that $G_i \in \mathbb{Z}[x]^+$. Choosing an integer k such that $(ak+b)G_1(k)G_2(k)G_3(k) \ne 0$ we should obtain

$$4 = \frac{1}{G_1(k)} + \frac{1}{G_2(k)} + \frac{1}{G_3(k)} \le 3, \qquad \text{a contradiction.}$$

Hence, up to a permutation of F_1, F_2, F_3 there are two possibilities

$$F_1(-b/a) = F_2(-b/a) = 0 \ne F_3(-b/a), \tag{7}$$

$$F_1(-b/a) = 0 \ne F_2(-b/a)F_3(-b/a). \tag{8}$$

In the case (7) $F_i(x) = (ax+b)G_i(x)$ $(i = 1,2)$, $(F_3(x), ax+b) = 1$, where $G_i \in \mathbb{Z}[x]^+$. Let us put

$$\begin{aligned} &D = (G_1, G_2), \quad G_i = DH_i \ (i = 1,2),\\ &C = (4DH_1H_2 - H_1 - H_2, DH_1H_2) = (H_1 + H_2, D),\\ &D = CR, \quad H_1 + H_2 = CS. \end{aligned}$$

H_i, C, R, S are in $\mathbb{Z}[x]^+$ and we have $(H_1, H_2) = 1$, $(RH_1H_2, S) = 1$. By (2) with $m = 4$

$$\frac{ax+b}{F_3} = \frac{4DH_1H_2 - H_1 - H_2}{DH_1H_2} = \frac{4RH_1H_2 - S}{RH_1H_2}.$$

Since $(ax+b, F_3) = 1 = (4RH_1H_2 - S, RH_1H_2)$ and both F_3 and RH_1H_2 are in $\mathbb{Z}[x]^+$, it follows by Lemma 1 that

$$ax + b = 4RH_1H_2 - S = 4(CS - H_2)H_2R - S. \tag{9}$$

Since b is a quadratic residue for a and C, H_2, R, S are in $\mathbb{Z}[x]^+$ there exist integers k and n such that

$$ak + b = n^2 \quad \text{and} \quad b^* = H_2(k),\ c = C(k),\ r = R(k),\ s = S(k) \quad \text{are in } \mathbb{Z}^+,$$

which in view of (9) contradicts Lemma 2.

Consider now the case (8). We have here

$$F_1(x) = (ax+b)G_1(x),\ F_i = DH_i\ (i = 2, 3)$$

where $G_1 \in \mathbb{Z}[x]^+$, $D = (F_2, F_3)$, $(H_2, H_3) = 1$ and $(DH_i, ax+b) = 1$ $(i = 2, 3)$, $H_i \in \mathbb{Z}[x]^+$. Hence, by (2) with $m = 4$

$$\begin{aligned} \frac{4}{ax+b} &= \frac{1}{(ax+b)G_1} + \frac{H_2 + H_3}{DH_2H_3}, \\ \frac{DH_2H_3}{ax+b} &= \frac{G_1(H_2+H_3)}{4G_1 - 1}. \end{aligned} \tag{10}$$

Let us put $C = (D, H_2 + H_3)$, $D = CR$, $H_2 + H_3 = CS$, so that C, R, S are in $\mathbb{Z}[x]^+$. Since $(DH_2H_3, ax+b) = 1$ we infer from Lemma 1 that $4G_1 - 1 = (ax+b)H_1$, where $H_1 \in \mathbb{Z}[x]^+$. Hence, by (10),

$$\frac{RH_2H_3}{S} = \frac{G_1}{H_1}.$$

Since $(RH_2H_3, S) = 1 = (G_1, H_1)$ and S and H_1 are in $\mathbb{Z}[x]^+$ it follows from Lemma 1 that $H_1 = S$, $G_1 = RH_2H_3$ and

$$(ax+b)S = 4G_1 - 1 = 4RH_2H_3 - 1 = 4(CS - H_2)H_2R - 1. \tag{11}$$

Since b is a quadratic residue $\bmod a$ and C, H_2, R, S are in $\mathbb{Z}[x]^+$ there exist integers k and n such that

$$ak + b = n^2 \quad \text{and} \quad b^* = H_2(k),\ c = C(k),\ r = R(k),\ s = S(k) \quad \text{are in } \mathbb{Z}^+,$$

which in view of (11) contradicts Lemma 2. ■

Proof of Theorem 2. If $F_i(0) \neq 0$ for all i it follows from (2) on substituting $x = 0$ that

$$\frac{m}{b} = \sum_{i=1}^{3} \frac{1}{F_i(0)} \leq 3,$$

contrary to the assumption $m > 3b$.

If $F_i(0) \neq 0$ for all but one i, it follows from (2) on taking the limit for $x \to 0$

$$\frac{m}{b} = \pm\infty,$$

a contradiction.

If $F_i(0) = 0$ for all i, it follows $F_i(x) = xG_i(x)$, $G_i \in \mathbb{Z}[x]^+$ and by (2)

$$\frac{mx}{ax+b} = \sum_{i=1}^{3} \frac{1}{G_i(x)}.$$

When $x \to \infty$ the terms on the left hand side are less than the limit m/a, the terms on the right hand side are greater or equal to the limit, which contradicts the equality.

Thus $F_i(0) = 0$ for exactly two $i \leq 3$ and we may assume without loss of generality that

$$F_i(0) = 0 \ (i = 1, 2), \qquad F_3(0) \neq 0.$$

Arguing as in the proof of Theorem 1 we infer that $F_i(-b/a) = 0$ for at least one i. Hence up to a permutation of F_1, F_2 there are the following possibilities:

(12) $\quad F_i(-b/a) = 0 \, (i = 1, 2, 3)$;

(13) $\quad F_i(-b/a) = 0 \, (i = 1, 2), \quad F_3(-b/a) \neq 0$;

(14) $\quad F_i(-b/a) = 0 \, (i = 1, 3), \quad F_2(-b/a) \neq 0$;

(15) $\quad F_i(-b/a) \neq 0 \, (i = 1, 2), \quad F_3(-b/a) = 0$;

(16) $\quad F_i(-b/a) \neq 0 \, (i = 1, 3), \quad F_2(-b/a) = 0$.

We shall consider these cases successively.

Case (12). Here $F_i(x) = (ax+b)G_i(x)$, $G_i \in \mathbb{Q}[x]^+$ $(i = 1, 2, 3)$ and by Gauss's lemma $(a, b)G_i \in \mathbb{Z}[x]^+$. Taking an integer k such that $G_i(k) \neq 0$ we obtain from (2)

$$m = \sum_{i=1}^{3} \frac{1}{G_i(k)} \leq 3(a, b) \leq 3b,$$

contrary to the assumption.

Case (13). Here $F_i(x) = x(ax+b)G_i(x)$, $G_i \in \mathbb{Q}[x]^+$ $(i = 1, 2)$

$$m = \frac{1}{xG_1(x)} + \frac{1}{xG_2(x)} + \frac{ax+b}{F_3}$$

and taking the limit for $x \to \infty$ we infer that $F_3 = cx + d$, where $c = a/m$. Hence

$$0 = \frac{1}{xG_1} + \frac{1}{xG_2} + \frac{b-md}{cx+d}.$$

For x large enough the first two terms are positive, hence $b - md < 0$ and $d > 0$.

Without loss of generality $G_2(-d/c) = 0$, hence $G_2 = (cx+d)H_2(x)$, $H_2 \in \mathbb{Q}[x]^+$,

$$0 = \lim_{x\to\infty} \frac{cx+d}{xG_1(x)} + b - md,$$

thus $G_1(x) = c/(md-b)$ and

$$0 = \frac{md-b}{cx} + \frac{1}{x(cx+d)H_2} + \frac{b-md}{cx+d} = \frac{(md-b)d}{x(cx+d)} + \frac{1}{x(cx+d)H_2}.$$

This is impossible, since for x large enough both terms on the right hand side are positive.

Case (14). Here $F_1 = x(ax+b)G_1$, $F_2 = xG_2$, $F_3 = (ax+b)G_3$, where $G_i \in \mathbb{Q}[x]^+$ $(i = 1,2,3)$ and

$$m = \frac{1}{xG_1} + \frac{ax+b}{xG_2} + \frac{1}{G_3}.$$

The first and the second term on the right hand side are greater than their limits for $x \to \infty$, the third term is greater or equal, while the left hand side is constant: this gives a contradiction.

Case (15). Here $F_i = xG_i$, $(i = 1,2)$, $F_3 = (ax+b)G_3$, where $G_i \in \mathbb{Z}[x]^+$, $G_i(-b/a) \neq 0$ $(i = 1,2)$, $G_3 \in \mathbb{Q}[x]^+$ and

$$\frac{mx}{ax+b} = \frac{1}{G_1(x)} + \frac{1}{G_2(x)} + \frac{x}{(ax+b)G_3(x)}.$$

If $G_3 \notin \mathbb{Q}^+$ all three terms on the right hand side are greater than or equal to their limits for $x \to \infty$, while the left hand side is less than the limit, a contradiction. Hence $G_3 = g \in \mathbb{Q}^+$ and

$$\frac{(m-1/g)x}{ax+b} = \frac{1}{G_1} + \frac{1}{G_2},$$

which contradicts $G_1G_2(-b/a) \neq 0$.

Case (16). Here $F_1 = xG_1$, $F_2 = x(ax+b)G_2$, where $G_1 \in \mathbb{Z}[x]^+$, $G_2 \in \mathbb{Q}[x]^+$ and

$$\frac{mx}{ax+b} = \frac{1}{G_1} + \frac{1}{(ax+b)G_2} + \frac{x}{F_3}. \tag{17}$$

If $\deg F_3 = 0$ we take the limit for $x \to \infty$ and obtain $m/a = \infty$, a contradiction.

If $\deg F_3 > 1$, when $x \to \infty$ the left hand side of (17) is less than its limit, while all three terms on the right hand side are greater than or equal to their limits, which gives a contradiction. Thus

$$\deg F_3 = 1,\ F_3 = cx+d, \qquad \text{where } c \in \mathbb{Z}^+,\ d/c \neq b/a. \tag{18}$$

We consider four subcases:

(i) $\deg G_1 > 1$;

(ii) $\deg G_1 = 1$, $G_1/F_3 \notin \mathbb{Q}$;

(iii) $\deg G_1 = 1$, $G_1/F_3 \in \mathbb{Q}$;

(iv) $\deg G_1 = 0$.

Subcase (i). Taking the limit for $x \to \infty$ we infer from (17) and (18) that $a = cm$ and

$$\begin{aligned} \frac{mx}{cmx+b} &= \frac{1}{G_1} + \frac{1}{(cmx+b)G_2} + \frac{x}{cx+d}; \\ \frac{x(md-b)}{cx+d} &= \frac{cmx+b}{G_1} + \frac{1}{G_2}, \end{aligned} \tag{19}$$

hence $md - b > 0$, $d > 0$. When $x \to \infty$ the left hand side of (18) is less than its limit, while both terms on the right hand side are greater than or equal to their limits, which gives a contradiction.

Subcase (ii). As in the subcase (i) we have $md - b > 0$, $d > 0$. Let $G_1 = ex + f$, $e > 0$, $f/e \neq b/a, d/c$. It follows from (19) that

$$G_2 = g^{-1}(cx+d)(ex+f), \ g \in \mathbb{Q}^+$$

and substituting $x = 0$ we obtain

$$0 = \frac{b}{f} + \frac{g}{df}; \qquad g = -bd < 0,$$

a contradiction.

Subcase (iii). Let $G_1 = e^{-1}(cx+d)$, $e \in \mathbb{Q}^+$. We obtain from (17) and (18)

$$\frac{mx}{ax+b} = \frac{1}{(ax+b)G_2} + \frac{x+e}{cx+d},$$

hence $G_2 = f^{-1}(cx+d)$, $f \in \mathbb{Q}^+$ and substituting $x = 0$

$$0 = \frac{f}{bd} + \frac{e}{d}; \quad f = -be < 0,$$

a contradiction.

Subcase (iv). Let $G_1 = g$. It follows from (17) and (18) that $G_2 = e^{-1}(cx+d)$, $e \in \mathbb{Q}^+$,

$$\frac{mx}{ax+b} = \frac{1}{g} + \frac{e}{(ax+b)(cx+d)} + \frac{x}{cx+d}$$

and multiplying both sides by $(ax+b)(cx+d)$

$$(cgm - ac - ag)x^2 + (dgm - bg - ad - bc)x - bd - e = 0.$$

Hence

(20) $$cgm - ac - ag = 0,$$

(21) $$dgm - bg - ad - bc = 0,$$

(22) $$bd + e = 0,$$

which is impossible, since (20) gives $gm - a = ag/c > 0$, (21) gives $d = (bg + bc)/(gm - a) > 0$, contrary to (22). ■

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