*Functiones et Approximatio*
*XXVIII (2000), 105–116*

To Professor Włodzimierz Staś on his 75th birthday

# ON THE INTEGRAL OF THE ERROR TERM IN THE FOURTH MOMENT OF THE RIEMANN ZETA-FUNCTION

ALEKSANDAR IVIĆ

## 1. Introduction

The aim of this note is to provide an asymptotic formula for $\int_0^T E_2(t)\,dt$, where $E_2(T)$ is the error term in the asymptotic formula for the fourth moment of $|\zeta(\frac{1}{2}+it)|$. The asymptotic formula for the fourth moment of the Riemann zeta-function $\zeta(s)$ on the critical line is customarily written as

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^4\,dt \;=\; TP_4(\log T) \;+\; E_2(T), \tag{1.1}$$

where

$$P_4(x) \;=\; \sum_{j=0}^{4} a_j x^j. \tag{1.2}$$

It is classically known that $a_4 = 1/(2\pi^2)$, and it was proved by D. R. Heath-Brown [1] that

$$a_3 \;=\; 2(4\gamma - 1 - \log(2\pi) - 12\zeta'(2)\pi^{-2})\pi^{-2}.$$

He also produced more complicated expressions for $a_0, a_1$ and $a_2$ in (1.2) ($\gamma = 0.577\ldots$ is Euler's constant). For an explicit evaluation of the $a_j$'s the reader is referred to [4].

In recent years, due primarily to the application of powerful methods of spectral theory (see Y. Motohashi's monograph [13] for a comprehensive account), much advance has been made in connection with $E_2(T)$. We refer the reader to the works [5]–[9], [11]–[13] and [16]. It is known now that

$$E_2(T) \;=\; O(T^{2/3}\log^{C_1} T), \quad E_2(T) \;=\; \Omega(T^{1/2}), \tag{1.3}$$

$$\int_0^T E_2(t)\,dt \;=\; O(T^{3/2}), \quad \int_0^T E_2^2(t)\,dt \;=\; O(T^2\log^{C_2} T), \tag{1.4}$$

**1991 Mathematics Subject Classification:** 11M06, 11F72

with effective constants $C_1$, $C_2 > 0$ (the values $C_1 = 8, C_2 = 22$ are worked out in [13]). The above results were proved by Y. Motohashi and the author: (1.3) and the first bound in (1.4) in [3], [8], [13] and the second upper bound in (1.4) in [7]. The omega-result in (1.3) ($f = \Omega(g)$ means that $f = o(g)$ does not hold, $f = \Omega_\pm(g)$ means that $\limsup f/g > 0$ and that $\liminf f/g < 0$) was improved to $E_2(T) = \Omega_\pm(T^{1/2})$ by Y. Motohashi [12]. Recently the author [6] made further progress in this problem by proving the following quantitative omega-result: there exist two constants $A > 0$, $B > 1$ such that for $T \ge T_0 > 0$ every interval $[T, BT]$ contains points $T_1, T_2$ for which

$$E_2(T_1) \; > \; AT_1^{1/2}, \quad E_2(T_2) \; < \; -AT_2^{1/2}. \tag{1.5}$$

There is an obvious discrepancy between the $O$-result and $\Omega$-result in (1.3), and it may be well conjectured that $E_2(T) = O_\varepsilon(T^{1/2+\varepsilon})$ for any given $\varepsilon > 0$ ($\varepsilon$ will denote arbitrarily small constants, not necessarily the same ones at each occurrence). This bound, if true, is very strong, since it would imply (e.g., by Lemma 7.1 of [3]) the hitherto unproved bound $\zeta(\frac{1}{2} + it) \ll_\varepsilon t^{1/8+\varepsilon}$. The upper bound in (1.3) seems to be the limit of the existing methods, since the only way to estimate the relevant exponential sum in this problem, namely (see [3],[8] and [13])

$$\sum_{K<\kappa_j\le 2K} \alpha_j H_j^3(\tfrac{1}{2}) \exp\left(i\kappa_j \log\left(\frac{T}{\kappa_j}\right)\right) \qquad (1 \ll K \le T^{1/2}) \tag{1.6}$$

appears to be trivial estimation, coming from the bound

$$\sum_{K<\kappa_j\le 2K} \alpha_j \left|H_j^3(\tfrac{1}{2})\right| \; \ll \; K^2 \log^C K \qquad (C > 0). \tag{1.7}$$

This follows by the Cauchy-Schwarz inequality from the bounds (see [13])

$$\sum_{\kappa_j\le K} \alpha_j H_j^2(\tfrac{1}{2}) \ll K^2 \log K, \quad \sum_{\kappa_j\le K} \alpha_j H_j^4(\tfrac{1}{2}) \ll K^2 \log^{15} K \tag{1.8}$$

with $C = 8$ in (1.7). Here as usual $\{\lambda_j = \kappa_j^2 + \frac{1}{4}\} \cup \{0\}$ denotes the discrete spectrum of the non-Euclidean Laplacian acting on $SL(2,\mathbb{Z})$-automorphic forms, and $\alpha_j = |\rho_j(1)|^2(\cosh \pi\kappa_j)^{-1}$, where $\rho_j(1)$ is the first Fourier coefficient of the Maass wave form corresponding to the eigenvalue $\lambda_j$ to which the Hecke series $H_j(s)$ is attached. It is precisely the presence of $H_j^3(\frac{1}{2})$ in (1.6) which makes the sum in question very hard to deal with, and any decrease of the exponent 2/3 in the upper bound for $E_2(T)$ in (1.3) will likely involve the application of genuine new ideas.

In [6] the author proved that there exist constants $A > 0$ and $B > 1$ such that, for $T \ge T_0 > 0$, every interval $[T, BT]$ contains points $t_1, t_2$ for which

$$\int_0^{t_1} E_2(t)\,dt \; > \; At_1^{3/2}, \quad \int_0^{t_2} E_2(t)\,dt \; < \; -At_2^{3/2}. \tag{1.9}$$

This result, of course, implies that $\int_0^T E_2(t)\,dt = \Omega_\pm(T^{3/2})$. It was also used in [6] to prove a lower bound result, whose special case $a = 2$ gives

$$\int_0^T E_2^2(t)\,dt \gg T^2, \tag{1.10}$$

thus sharpening (1.8) and showing that the upper bound in (1.4) is very close to the true order of magnitude of the mean square integral of $E_2(T)$.

The main aim of this paper is to prove a result, which gives an asymptotic formula for the integral of $E_2(t)$, thereby sharpening the first bound in (1.4). This is the following

**Theorem 1.1.** *Let*

$$\eta(T) := (\log T)^{3/5}(\log\log T)^{-1/5}, \tag{1.11}$$

$$R_1(\kappa_h) := \sqrt{\frac{\pi}{2}}\left(2^{-i\kappa_h}\frac{\Gamma(\frac14 - \frac12 i\kappa_h)}{\Gamma(\frac14 + \frac12 i\kappa_h)}\right)^3 \Gamma(2i\kappa_h)\cosh(\pi\kappa_h). \tag{1.12}$$

*Then there exists a constant $C > 0$ such that*

$$\int_0^T E_2(t)\,dt = 2T^{\frac32}\Re\mathrm{e}\left\{\sum_{j=1}^{\infty}\alpha_j H_j^3(\tfrac12)\frac{T^{i\kappa_j}}{(\frac12 + i\kappa_j)(\frac32 + i\kappa_j)}R_1(\kappa_j)\right\} + O(T^{\frac32}e^{-C\eta(T)}). \tag{1.13}$$

From Stirling's formula for the gamma-function it follows that $R_1(\kappa_j) \ll \kappa_j^{-1/2}$, hence by (1.7) and partial summation it follows that the series on the right-hand side of (1.13) is absolutely convergent, and it can be also shown (see [3], [5], [6]) that $\Re\mathrm{e}\,\{\ldots\}$ is also $\Omega_\pm(1)$. Thus from Theorem 1.1 we can easily deduce all previously known $\Omega$-results for $E_2(T)$. The error term in (1.13) is similar to the error term in the strongest known form of the prime number theorem (see e.g., [2, Chapter 12]). This is by no means a coincidence, and the reason for such a shape of the error term in (1.13) will transpire from the proof of Theorem 1.1, which will be given in Section 3.

## 2. A mean square result

We shall deduce the proof of Theorem 1.1 from a mean square result for the function

$$\mathcal{Z}_2(s) := \int_1^\infty |\zeta(\tfrac12 + ix)|^4 x^{-s}\,dx \qquad (\Re\mathrm{e}\, s = \sigma > 1). \tag{2.1}$$

It was introduced and studied in [12], [13, Chapter 5], and then further used and studied in [5], [6] and [9]. Y. Motohashi [12] has shown that $\mathcal{Z}_2(s)$ has meromorphic continuation over $\mathbb{C}$. In the half-plane $\Re\mathrm{e}\, s > 0$ it has the following singularities: the pole $s = 1$ of order five, simple poles at $s = \frac{1}{2} \pm i\kappa_j$ ($\kappa_j = \sqrt{\lambda_j - 1/4}$) and poles at $s = \frac{1}{2}\rho$, where $\rho$ denotes complex zeros of $\zeta(s)$. The residue of $\mathcal{Z}_2(s)$ at $s = \frac{1}{2} + i\kappa_h$ equals

$$R(\kappa_h) := \sqrt{\frac{\pi}{2}}\left(2^{-i\kappa_h}\frac{\Gamma(\frac{1}{4} - \frac{1}{2}i\kappa_h)}{\Gamma(\frac{1}{4} + \frac{1}{2}i\kappa_h)}\right)^3 \Gamma(2i\kappa_h)\cosh(\pi\kappa_h)\sum_{\kappa_j = \kappa_h} \alpha_j H_j^3(\tfrac{1}{2}),$$

and the residue at $s = \frac{1}{2} - i\kappa_h$ equals $\overline{R(\kappa_h)}$. The function $\mathcal{Z}_2(s)$ is a natural tool for investigations involving $E_2(T)$ (see (3.3) and (3.4)). Its spectral decomposition (see [12] and [13, Chapter 5]) enables one to connect problems with $E_2(T)$ to results from spectral theory. We shall prove the following

**Theorem 2.1.** *Let*

$$\sigma = \frac{1}{2} - C\delta(V), \quad \delta(V) := (\log V)^{-2/3}(\log\log V)^{-1/3}, \tag{2.2}$$

*where $C > 0$ is a suitable constant. Then*

$$\int_V^{2V} |\mathcal{Z}_2(\sigma + iv)|^2\, dv \ll_\varepsilon V^{2+\varepsilon}. \tag{2.3}$$

**Proof.** We note that in [9] the bound (2.3) was shown to hold for $\frac{1}{2} < \sigma < 1$, but it is the region $\sigma < \frac{1}{2}$ that is more difficult to deal with. As in [9] we write

$$\begin{aligned}\mathcal{Z}_2(s) &= \int_1^\infty I(T,\Delta)T^{-s}\, dT + \int_1^\infty (|\zeta(\tfrac{1}{2} + iT)|^4 - I(T,\Delta))T^{-s}\, dT \\ &= \mathcal{Z}_{21}(s) + \mathcal{Z}_{22}(s),\end{aligned} \tag{2.4}$$

say, where

$$I(T,\Delta) = \frac{1}{\sqrt{\pi}\Delta}\int_{-\infty}^\infty |\zeta(\tfrac{1}{2} + i(T+t))|^4 \exp\Big(-\big(\tfrac{t}{\Delta}\big)^2\Big)\, dt \quad (\Delta = T^\xi,\ \tfrac{1}{3} \le \xi \le \tfrac{1}{2}). \tag{2.5}$$

Before we pass to specific bounds, we shall discuss the method that will be used. Let us suppose that we want to obtain an upper bound for

$$I := \int_T^{2T}\left|\int_a^b g(x)x^{-s}\, dx\right|^2 dt \qquad (s = \sigma + it,\ T \ge T_0 > 0), \tag{2.6}$$

where $g(x)$ is a real-valued, integrable function on $[a, b]$, a subinterval of $[1, \infty)$ (which is not necessarily finite), and which satisfies $g(x) \ll x^C$ for some $C > 0$. Let $\varphi(x) \in C^\infty(0,\infty)$ be a test function such that $\varphi(x) \ge 0$, $\varphi(x) = 1$ for

$T \le x \le 2T$, $\varphi(x) = 0$ for $x < \frac{1}{2}T$ or $x > \frac{5}{2}T$ $(T \ge T_0 > 0)$, $\varphi(x)$ is increasing in $[\frac{1}{2}T, T]$ and decreasing in $[2T, \frac{5}{2}T]$. Then we have, by $r$ integrations by parts,

$$\begin{aligned}\int_{T/2}^{5T/2} \varphi(t)\left(\frac{y}{x}\right)^{it} dt &= (-1)^r \int_{T/2}^{5T/2} \varphi^{(r)}(t)\frac{(y/x)^{it}}{(i\log(y/x))^r}\,dt \\ &\ll_r T^{1-r}\left|\log\frac{y}{x}\right|^{-r} \ll T^{-A}\end{aligned} \tag{2.7}$$

for any fixed $A > 0$ and any given $\varepsilon > 0$, provided that $|y - x| \ge xT^{\varepsilon-1}$ and $r = r(A, \varepsilon)$ is large enough. Recalling that $g(x) \ll x^C$ and using (2.7) it follows that

$$\begin{aligned} I &\le \int_{T/2}^{5T/2} \varphi(t)\left|\int_a^b g(x)x^{-s}\,dx\right|^2 dt \\ &= \int_a^b\int_a^b g(x)g(y)(xy)^{-\sigma}\int_{T/2}^{5T/2}\varphi(t)\left(\frac{y}{x}\right)^{it} dt\,dx\,dy \\ &\ll 1 + \int_{T/2}^{5T/2}\varphi(t)\int_a^b |g(x)|x^{-\sigma}\int_{x-xT^{\varepsilon-1}}^{x-xT^{\varepsilon-1}} |g(y)|y^{-\sigma}\,dy\,dx\,dt. \end{aligned} \tag{2.8}$$

and the problem is reduced to the estimation of the integral of $g(x)$ over short intervals; here actually $g(x)$ does not have to be real-valued. In (2.8) we may further use the elementary inequality $|g(x)g(y)| \le \frac{1}{2}(g^2(x) + g^2(y))$, and thus reduce the problem to mean square estimates.

In the expression for $\mathcal{Z}_{22}(s)$ in (2.4) we denote by $I_1(s, X)$ the integral in which $T \le X$, and by $I_2(s, X)$ the remaining integral, where $X\,(\ll V^C)$ is a parameter to be chosen later. We have $(s = \sigma + it)$

$$\begin{aligned}\int_V^{2V} |I_1(s, X)|^2\,dt \ll &\int_V^{2V}\left|\int_1^X |\zeta(\tfrac{1}{2} + iT)|^4 T^{-s}\,dT\right|^2 dt \\ &+ \int_V^{2V}\left|\int_{-\log V}^{\log V}\int_1^X |\zeta(\tfrac{1}{2} + iT + iu)|^4 T^{-s}\,dT\,e^{-u^2}\,du\right|^2 dt \\ &+ 1.\end{aligned}$$

Both mean square integrals above are estimated analogously. The first one is, by using (2.8),

$$\begin{aligned} &\ll_\varepsilon 1 + \int_V^{2V}\int_1^X |\zeta(\tfrac{1}{2} + ix)|^4 x^{-\sigma}\int_{x-xV^{\varepsilon-1}}^{x+xV^{\varepsilon-1}} |\zeta(\tfrac{1}{2} + iy)|^4 y^{-\sigma}\,dy\,dx \\ &\ll_\varepsilon 1 + \int_V^{2V}\int_1^X |\zeta(\tfrac{1}{2} + ix)|^4 x^{-2\sigma}(xV^{\varepsilon-1} + x^{c+\varepsilon})\,dx\,dt \\ &\ll_\varepsilon V^\varepsilon(X^{2-2\sigma} + V + VX^{1+c-2\sigma}) \ll_\varepsilon V^\varepsilon(X + VX^c). \end{aligned} \tag{2.9}$$

Here we used (1.1), (2.2) the weak form of the fourth moment of $|\zeta(\frac{1}{2}+ix)|$ and the bound (see (1.3))

$$E_2(T) \ll_\varepsilon T^{c+\varepsilon} \qquad (\tfrac{1}{2} \le c \le \tfrac{2}{3}). \tag{2.10}$$

To estimate the contribution of $I_2(s,X)$, note that from [9, (4.10)] we have that the relevant part of $I_2(s,X)$ is, on integrating by parts,

$$\int_0^b \int_X^\infty E_2'(\tau) f(\tau,\alpha)\, d\tau\, d\alpha$$
$$= O\left(\sup_\alpha |E_2(X) f(X,\alpha)|\right) - \int_0^b \int_X^\infty E_2(\tau) \frac{\partial f(\tau,\alpha)}{\partial \tau}\, d\tau\, d\alpha.$$

where $b > 0$ is a small constant, and $f(\tau,\alpha)$ is precisely defined in [9]. It was shown there that, for $0 < \sigma < \frac{1}{2}, t \ll V$, we have the estimates

$$f(\tau,\alpha) \ll \tau^{2\xi-2-\sigma}(\log^2\tau + V\log\tau + V^2)\log^3\tau$$

and

$$\frac{\partial f(\tau,\alpha)}{\partial \tau} \ll \tau^{2\xi-3-\sigma} V \log^3\tau (\log^2\tau + V\log\tau + V^2).$$

We use (2.5), (2.9), (2.10) and the above estimates to obtain, if $\sigma$ satisfies (2.2),

$$\int_V^{2V} |I_1(s,X)|^2\, dt \ll_\varepsilon V^5 X^{2c+4\xi-4-2\sigma} + V^6 \int_X^\infty E_2^2(\tau)\tau^{4\xi-5-2\sigma}\, d\tau$$
$$\ll_\varepsilon V^\varepsilon (V^5 X^{2c+4\xi-5} + V^6 X^{4\xi-4}).$$

It follows that

$$\begin{aligned}
&\int_V^{2V} |\mathcal{Z}_{22}(\sigma+it)|^2\, dt \\
&\quad \ll_\varepsilon V^\varepsilon (VX^c + X + V^5 X^{2c+4\xi-5} + V^6 X^{4\xi-4}) \\
&\quad \ll_\varepsilon V^\varepsilon \left(V^{5/(4+c-4\xi)} + V^{(4+6c-4\xi)/(4+c-4\xi)} + V^{(15c-5)/(4+c-4\xi)}\right) \\
&\quad \ll_\varepsilon V^{(4+6c-4\xi)/(4+c-4\xi)+\varepsilon}
\end{aligned} \tag{2.11}$$

with $X = V^{5/(4+c-4\xi)}$, since in view of $\xi \le \frac{1}{2}$, $\frac{1}{2} \le c \le \frac{2}{3}$ we have

$$5 \le 4 + 6c - 4\xi, \quad 15c - 5 \le 4 + 6c - 4\xi.$$

Then with $\xi = \frac{1}{3}$, which we henceforth assume, we obtain

$$\int_V^{2V} |\mathcal{Z}_{22}(\sigma+iv)|^2\, dv \ll_\varepsilon V^{2+\varepsilon},$$

so that (2.3) will follow from

$$\int_V^{2V} |\mathcal{Z}_{21}(\sigma+iv)|^2\,dv \ll_\varepsilon V^{2+\varepsilon}. \tag{2.12}$$

It was shown in [9] that the major contribution to $\mathcal{Z}_{22}(s)$ comes from ($s = \sigma + it$, $V \le t \le 2V$ and $\sigma$ satisfies (2.2))

$$\sum_{t-V^\varepsilon\le\kappa_j\le t+V^\varepsilon} \alpha_j H_j^3(\tfrac12)|\tfrac12 + i\kappa_j - s|^{-1}\kappa_j^{-\frac12}\left|\int_{T(\kappa_j)}^\infty M^*(\kappa_j;T)T^{\frac12+i\kappa_j-s}\,dT\right|, \tag{2.13}$$

where

$$T(r) := r^{\frac{1}{1-\xi}}\log^{-D} r = r^{\frac32}\log^{-D} r \qquad (D>0). \tag{2.14}$$

and $M^*(r;T)$ is a precisely defined function from spectral theory which satisfies, for $T \ge T(r)$ (cf. [9, (4.28)]), the bound

$$M^*(r;T) \ll_\varepsilon rT^{-2} + r^{2+\varepsilon}T^{2\xi-3}. \tag{2.15}$$

Thus the major contribution to the integral in (2.13) will therefore be, since $H_j(\frac12) \ge 0$ (see Katok–Sarnak [10]),

$$\begin{aligned}\int_V^{2V}\Bigg|\sum_{t-V^\varepsilon\le\kappa_j\le t+V^\varepsilon} &\alpha_j H_j^3(\tfrac12)|\tfrac12 + i\kappa_j - s|^{-1}V^{-\frac12}\times\\ &\times\left|\int_{T(V)}^\infty M^*(\kappa_j;T)T^{\frac12+i\kappa_j-s}\,dT\right|\Bigg|^2 dt.\end{aligned} \tag{2.16}$$

Recall that $\sigma$ is given by (2.2), and that by the zero-free region for $\zeta(s)$ we have the bound (see [2, Lemma 12.3] and (2.2))

$$\frac{1}{\zeta(\alpha+it)} \ll (\log t)^{2/3}(\log\log t)^{1/3} \qquad (\alpha \ge 1-\delta(t),\ t \ge t_0 > 0).$$

This gives $|\frac12 + i\kappa_j - s|^{-1} \ll \log V$ in (2.16). We use the Cauchy-Schwarz inequality, (1.8) and the asymptotic formula (see [13])

$$\sum_{\kappa_j\le K} \alpha_j H_j^2(\tfrac12) = (A\log K + B)K^2 + O(K\log^6 K) \qquad (A>0)$$

to estimate sums of $\alpha_j H_j^2(\frac12)$ in short intervals. We obtain then that the expression in (2.16) is, on using (2.9) and the inequality $|g(x)g(y)| \le \frac12(g^2(x)+g^2(y))$, (2.14) and (2.15),

$$\begin{aligned}
&\ll V^{-1}\log^2 V\int_V^{2V}\sum_{t-V^\varepsilon\le\kappa_j\le t+V^\varepsilon}\alpha_j H_j^2(\tfrac12)\sum_{t-V^\varepsilon\le\kappa_j\le t+V^\varepsilon}\alpha_j H_j^4(\tfrac12)\times\\
&\quad\times\left|\int_{T(V)}^\infty M^*(\kappa_j;T)T^{\frac12+i\kappa_j-s}\,dT\right|^2 dt\\
&\ll_\varepsilon V^\varepsilon\sum_{V-V^\varepsilon\le\kappa_j\le 2V+V^\varepsilon}\alpha_j H_j^4(\tfrac12)\int_{T(V)}^\infty |M^*(\kappa_j;T)|^2T^{2-2\sigma}\,dT\\
&\ll_\varepsilon V^\varepsilon\sum_{V-V^\varepsilon\le\kappa_j\le 2V+V^\varepsilon}\alpha_j H_j^4(\tfrac12)\int_{T(V)}^\infty (V^2T^{-4}+V^4T^{4\xi-6})T\,dT\\
&\ll_\varepsilon V^\varepsilon\sum_{V-V^\varepsilon\le\kappa_j\le 2V+V^\varepsilon}\alpha_j H_j^4(\tfrac12)\Big(V^2T^{-2}(V)+V^4T^{4\xi-4}(V)\Big)\\
&\ll V^\varepsilon\sum_{V-V^\varepsilon\le\kappa_j\le 2V+V^\varepsilon}\alpha_j H_j^4(\tfrac12)\ll_\varepsilon V^{2+\varepsilon}.
\end{aligned}$$

This establishes (2.12) and thus finishes the proof of Theorem 2.1. ■

## 3. The proof of Theorem 1.1

In this section we shall prove Theorem 1.1. The starting point is the inversion formula

$$|\zeta(\tfrac12+ix)|^4 = \frac{1}{2\pi i}\int_{(1+\varepsilon)}\mathcal{Z}_2(s)x^{s-1}\,ds, \tag{3.1}$$

where as usual $\int_{(c)} = \lim_{T\to\infty}\int_{c-iT}^{c+iT}$. Namely, if $F(s)=\int_0^\infty f(x)x^{s-1}\,dx$ is the Mellin transform of $f(x)$, $y^{\sigma-1}f(y)\in L^1(0,\infty)$ and $f(y)$ is of bounded variation in a neighbourhood of $y=x$, then one has the Mellin inversion formula (see [14])

$$\frac{f(x+0)+f(x-0)}{2} = \frac{1}{2\pi i}\int_{(\sigma)}F(s)x^{-s}\,ds.$$

We use this formula with $f(x)=\frac1x|\zeta(\frac12+\frac ix)|^4$ for $0<x\le1$ and $f(x)=0$ for $x>1$, and then change $x$ to $1/x$ to obtain (3.1).

Now we replace the line of integration in (3.1) by the contour $\mathcal{L}$, consisting of the same straight line from which the segment $[1+\varepsilon-i,\,1+\varepsilon+i]$ is removed and replaced by a circular arc of unit radius, lying to the left of the line, which passes over the pole $s=1$ of the integrand. By the residue theorem we have

$$|\zeta(\tfrac12+ix)|^4 = \frac{1}{2\pi i}\int_{\mathcal{L}}\mathcal{Z}_2(s)x^{s-1}\,ds + Q_4(\log x)\qquad (x>1), \tag{3.2}$$

where we have, since the coefficients of $P_4(z)$ are naturally connected to the principal part of the Laurent expansion of $\mathcal{Z}_2(s)$ at $s = 1$ (see [3] and [13]),

$$Q_4(\log x) = P_4(\log x) + P_4'(\log x)$$

and $P_4(y)$ is given by (1.1) and (1.2). If we integrate (3.2) from $x = 1$ to $x = T$ and take into account the defining relation (1.1) of $E_2(T)$, we shall obtain

$$E_2(T) = \frac{1}{2\pi i}\int_{\mathcal{L}} \mathcal{Z}_2(s)\frac{T^s}{s}\,ds + O(1) \qquad (T > 1). \tag{3.3}$$

A further integration, coupled with the deformation of the contour, enables one to deduce from (3.3) the formula

$$\int_0^T E_2(t)\,dt = \frac{1}{2\pi i}\int_{(c)} \mathcal{Z}_2(s)\frac{T^{s+1}}{s(s+1)}\,ds + O(T) \qquad (\tfrac12 < c < 1,\, T > 1), \tag{3.4}$$

since in view of the bound (see [9])

$$\int_0^T |\mathcal{Z}_2(\sigma + it)|^2\,dt \ll_\varepsilon T^{2+\varepsilon} \qquad (\tfrac12 < \sigma < 1) \tag{3.5}$$

we may take $\frac12 < c < 1$ as the range for $c$ in (3.4). The formula (3.4) is the key one in the proof of Theorem 1.1. We replace the line of integration in the integral on the right-hand side of (3.4) by the contour consisting of the segment $[\sigma_0 - it_0, \sigma_0 + it_0]$, and the curve

$$\sigma = \tfrac12 - C\delta(|t|),\ \delta(x) := (\log x)^{-2/3}(\log\log x)^{-1/3},\ |t| \ge t_0,\ \sigma_0 = \tfrac12 - C\delta(t_0), \tag{3.6}$$

where $C$ denotes positive, possibly different constants. Since $\mathcal{Z}_2(s)$ has poles at complex zeros of $\zeta(2s)$ it follows, by the strongest known zero-free region for $\zeta(s)$ (see [6, Chapter 6]), that the function $\mathcal{Z}_2(s)$ is regular on the new contour. The residue theorem yields

$$\begin{aligned}\int_0^T E_2(t)\,dt &= 2\Re\mathrm{e}\left\{\sum_{j=1}^{\infty}\frac{T^{\frac32+i\kappa_j}}{(\frac12+i\kappa_j)(\frac32+i\kappa_j)}\alpha_j H_j^3(\tfrac12)R_1(\kappa_j)\right\}\\ &\quad + O(T^{\sigma_0+1})\\ &\quad + O\left(\int_{t_0}^{\infty} T^{\frac32 - C\delta(t)}t^{-2}|\mathcal{Z}_2(\tfrac12 - C\delta(t) + it)|\,dt\right)\end{aligned} \tag{3.7}$$

with $R_1(\kappa_j)$ given by (1.12). Let $\eta(T)$ be defined by (1.11) and put

$$U = U(T) := e^{C\eta(T)} = e^{C\log^{3/5}T(\log\log T)^{-1/5}}.$$

Then

$$\int_{t_0}^{\infty} = \int_{t_0}^{U} + \int_{U}^{\infty} \ll T^{3/2}e^{-C\delta(U)\log T} + T^{3/2}U^{\varepsilon-\frac12} \ll T^{3/2}e^{-C\eta(T)}, \tag{3.8}$$

since by Theorem 2.1 we have

$$\int_{V}^{2V} |\mathcal{Z}_2(\tfrac12 - C\delta(v) + iv)|^2\,dv \ll_\varepsilon V^{2+\varepsilon}. \tag{3.9}$$

Namely we split the integral in the $O$–term in (3.7) into subintegrals over $[V,\,2V]$. The contour $\sigma = \frac12 - C\delta(v)$ is replaced by $\sigma = \frac12 - C\delta(V)$, which is technically easier. In this process we obtain integrals over horizontal segments whose contributions will be $\ll_\varepsilon V^{2+\varepsilon}$, since by (5.10) and (5.24) of [9] (with $\xi = \frac13$) we have the bound

$$\mathcal{Z}_2(\tfrac12 - C\delta(v) + iv) \ll_\varepsilon v^{1+\varepsilon}.$$

Finally by the Cauchy-Schwarz inequality for integrals and (3.9) we obtain

$$\int_{1}^{\infty} |\mathcal{Z}_2(\tfrac12 - C\delta(v) + iv)|v^{-2}\,dv \ll 1,$$
$$\int_{V}^{\infty} |\mathcal{Z}_2(\tfrac12 - C\delta(v) + iv)|v^{-2}\,dv \ll_\varepsilon V^{\varepsilon-\frac12},$$

thereby establishing (3.8) and completing the proof of Theorem 1.1.

In concluding it may be remarked that, similarly as in [5], one may obtain quickly from (3.3) the bound (see (1.3))

$$E_2(T) \ll_\varepsilon T^{\frac23+\varepsilon}, \tag{3.10}$$

which is (up to "$\varepsilon$") the stongest one known. Namely by [5, (5.3)] we have

$$E_2(T) \le C_1 H^{-1}\int_{T}^{T+H} E_2(x)f(x)\,dx + C_2 H\log^4 T \qquad (C_1, C_2 > 0,\ 1 \ll H \le \tfrac14 T), \tag{3.11}$$

where $f(x)$ $(> 0)$ is a smooth function supported in $[T,\,T+H]$, such that $f(x) = 1$ for $T + \frac14 H \le x \le T + \frac34 H$. Then from (3.3) we have ($\frac12 < c < 1$)

$$E_2(T) \le \frac{C_1}{2\pi i H}\int_{(c)} \frac{\mathcal{Z}_2(s)}{s}\int_{T}^{T+H} f(x)x^s\,dx\,ds + C_2 H\log^4 T.$$

We take $c = \frac12 + \varepsilon$, use (3.5), the Cauchy-Schwarz inequality, and the fact that by $r$ integrations by parts it follows that

$$\int_{T}^{T+H} f(x)x^s\,dx = (-1)^r\int_{T}^{T+H} \frac{x^{s+r}}{(s+1)\dots(s+r)} f^{(r)}(x)\,dx$$
$$\ll_{\sigma,r} T^{\sigma+r}H^{1-r}|t|^{-r}.$$

Hence the above integral over $s$ may be truncated at $|\Im m\, s| = T^{1+\varepsilon}H^{-1}$ with a negligible error, and we obtain

$$\begin{aligned} E_2(T) &\ll_\varepsilon T^{\frac{1}{2}+\varepsilon}\int_1^{T^{1+\varepsilon}H^{-1}} |\mathcal{Z}_2(\tfrac{1}{2}+\varepsilon+it)|\frac{dt}{t} + H\log^4 T \\ &\ll_\varepsilon T^\varepsilon(TH^{-\frac{1}{2}}+H) \ll T^{\frac{2}{3}+\varepsilon} \end{aligned}$$

with $H = T^{2/3}$. A lower bound for $E_2(T)$, similar to (3.11), also holds, and therefore (3.10) follows as asserted.

## References

[1] D. R. Heath-Brown, *The fourth moment of the Riemann zeta-function*, Proc. London Math. Soc. **(3)38** (1979), 385–422.

[2] A. Ivić, *The Riemann zeta-function*, John Wiley and Sons, New York, 1985.

[3] A. Ivić, *Mean values of the Riemann zeta-function*, LN's **82**, Tata Institute of Fundamental Research, Bombay, 1991 (distr. by Springer Verlag, Berlin etc.).

[4] A. Ivić, *On the fourth moment of the Riemann zeta-function*, Publs. Inst. Math. (Belgrade) **57(71)** (1995), 101–110.

[5] A. Ivić, *The Mellin transform and the Riemann zeta-function*, Proceedings of the Conference on Elementary and Analytic Number Theory (Vienna, July 18–20, 1996), Universität Wien & Universität für Bodenkultur, Eds. W. G. Nowak and J. Schoißengeier, Vienna 1996, 112–127.

[6] A. Ivić, *On the error term for the fourth moment of the Riemann zeta-function*, J. London Math. Soc., (2)**60** (1999), 21–32.

[7] A. Ivić and Y. Motohashi, *The mean square of the error term for the fourth moment of the zeta-function*, Proc. London Math. Soc. (3)**66** (1994), 309–329.

[8] A. Ivić and Y. Motohashi, *The fourth moment of the Riemann zeta-function*, J. Number Theory **51** (1995), 16–45.

[9] A. Ivić, M. Jutila and Y. Motohashi, *The Mellin transform of powers of the Riemann zeta-function*, to appear in *Acta Arith.*

[10] S. Katok and P. Sarnak, *Heegner points, cycles and Maass forms*, Israel J. Math. **84** (1993), 193–227.

[11] Y. Motohashi, *An explicit formula for the fourth power mean of the Riemann zeta-function*, Acta Math. **170** (1993), 181–220.

[12] Y. Motohashi, *A relation between the Riemann zeta-function and the hyperbolic Laplacian*, Annali Scuola Norm. Sup. Pisa, Cl. Sci. IV ser. **22** (1995), 299–313.

[13] Y. Motohashi, *Spectral theory of the Riemann zeta-function*, Cambridge University Press, Cambridge, 1997.

[14] E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Clarendon Press, Oxford, 1948.

[15] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, Clarendon Press, Oxford, 1951.

[16] N. I. Zavorotnyi, *On the fourth moment of the Riemann zeta-function* (in Russian), in "Automorphic functions and number theory I", Coll. Sci. Works, Vladivostok, 1989, 69–125.

**Address:** Aleksandar Ivić, Katedra Matematike RGF-a, Universiteta u Beogradu, Djušina 7, 11000 Beograd, Serbia (Yugoslavia)
**E-mail:** `aleks@ivic.matf.bg.ac.yu`, `aivic@rgf.rgf.bg.ac.yu`
**Received:** 14 September 1999