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To Professor Włodzimierz Staś on his 75th birthday

SOME ADDITIVE PROBLEMS OF GOLDBACH'S TYPE

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1. Introduction

1.1 Problems and results. One of the long-standing conjectures in the theory of numbers dates from 1742 when Goldbach asked whether any even integer exceeding 4 is the sum of two primes. Exceptions, if any, are sparse. This is the essence of a celebrated theorem of Montgomery and Vaughan (1975). Refining earlier work of Estermann (1938), van der Corput (1938) and Chudakov (1938), they showed that the number $E(N)$ of even integers not exceeding N, and not the sum of two primes, satisfies an inequality $E(N) \ll N^{1-\delta}$ with some $\delta > 0$. The best numerical value for δ is currently due to Hongze Li (1999) who obtained the bound $E(N) \ll N^{0.921}$.

Numerous variants and generalisations of Goldbach's original question have been studied, and it is hopeless to survey these developments. We shall mainly be concerned with linear forms. Let p, p_1, p_2, p_3 denote prime variables and consider a binary linear form $\lambda_1 p_1 + \lambda_2 p_2$ with real coefficients λ_1, λ_2. When λ_1, λ_2 are integers, one obtains results very similar to those on Goldbach's problem. Very recently, Arkhipov, Buriev and Chubarikov (1999) considered the equation

$$p_1 + [\lambda p_2] = n \tag{1.1}$$

where $\lambda > 0$ is a given algebraic irrational number, and as usual, $[\,]$ denotes the integer part functional. Arkhipov, Buriev and Chudarikov estimated the number $E_\lambda(N)$ of natural numbers $n \le N$ for which (1.1) has no solution, and obtained $E_\lambda(N) \ll N^{7/9+\varepsilon}$. The uninitiated reader may find it surprising that the exponent 7/9 in this seemingly harder problem is better than in the corresponding result for Goldbach's problem. However, some years prior to Arkhipov, Buriev and

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Chudarikov, in work on the distribution of binary linear forms at prime argument, Cook, Perelli and the author (1997) have implicitly obtained the superior estimate

$$E_\lambda(N) \ll N^{2/3+\varepsilon}. \tag{1.2}$$

More generally, the following theorem was established.

Theorem 1.1. *Let $0 < \tau < 1$. Let $V(N)$ be a set of real numbers contained in $[\frac{1}{2}N, N]$ such that $|v_1 - v_2| > 2\tau$ whenever $v_1, v_2 \in V(N)$ are distinct. Let $\lambda_1, \lambda_2 \geq 1$ be real numbers and λ_1/λ_2 be algebraic and irrational. Then, the number of $v \in V(N)$ for which the inequality*

$$|\lambda_1 p_1 + \lambda_2 p_2 - v| < \tau$$

has no solution, does not exceed $O(N^{2/3+\varepsilon}\tau^{-2})$.

In this result, take $\lambda_1 = 1, \lambda_2 = \lambda > 0$ algebraic and irrational. Furthermore, take $\tau = \frac{1}{4}$ and $V(N)$ as the set of all numbers $n + \frac{1}{2}$ with $n \in \mathbb{N}$ and $\frac{1}{2}N \leq n < N$. If

$$\left|p_1 + \lambda p_2 - \left(n + \frac{1}{2}\right)\right| < \frac{1}{4} \tag{1.3}$$

has a solution, then

$$n - \frac{3}{4} < p_1 + [\lambda p_2] < n + \frac{3}{4},$$

and hence, (1.1) is also soluble. Thus, at least when $\lambda > 1$, Theorem 1.1 implies (1.2), but an inspection of the work of Brüdern, Cook and Perelli (1997) shows that the conditions $\lambda_i \geq 1$ have been introduced merely for technical convenience, and Theorem 1.1 remains valid for any fixed positive λ_1, λ_2.

With the simple derivation of a bound for $E_\lambda(N)$ in mind, which in a sense is preborn to the result on which it improves, one is tempted to predict a more general result concerning the values of $[\lambda_1 p_1] + [\lambda_2 p_2]$ which contains our estimate for $E_\lambda(N)$ as a special case. However, the elementary transition from (1.3) to (1.1) fails if λ_1 and λ_2 are both non-integral. Yet, it is still possible to draw a successful conclusion.

Theorem 1.2. *Let λ_1, λ_2 be positive real algebraic numbers such that $1, \lambda_1, \lambda_1/\lambda_2$ are linearly independent over the rationals. Let $E_{\lambda_1,\lambda_2}(N)$ denote the number of natural numbers $n \leq N$ for which there is no solution of the equation*

$$[\lambda_1 p_1] + [\lambda_2 p_2] = n.$$

Then,

$$E_{\lambda_1,\lambda_2}(N) \ll N^{5/6+\varepsilon}.$$

When three or more variables are present, the fundamental questions concerning sums of primes have been answered affirmatively by Vinogradov (1937). We recall his epoch-making asymptotic formula in weighted form. The sum

$$r(n) = \sum_{p_1+p_2+p_3=n} (\log p_1)(\log p_2)(\log p_3)$$

satisfies

$$r(n) = \frac{1}{2}n^2 \prod_{p|n} (1 - (p-1)^{-2}) \prod_{p\nmid n} (1 - (1-p)^{-3}) + O_A\left(\frac{n^2}{(\log n)^A}\right) \tag{1.4}$$

for any $A > 0$. In this formula, the size of the error term is directly linked with the distribution of primes in arithmetic progressions, and the horizontal distribution of the zeros of Dirichlet L-series. In this spirit, Montgomery and Vaughan (1973) showed that if it were possible to replace the error term in (1.4) by $O(n^{1+\theta})$ with some $\theta \in (\frac{1}{2}, 1)$, then the Riemann zeta function has no zeros in $\Re(s) > \theta$. In the opposite direction, Friedlander and Goldston (1997) showed that if all Dirichlet L-functions have no zeros in $\Re(s) > \theta \geq \frac{4}{5}$, then the error term in (1.4) reduces to $O(n^{1+\theta+\varepsilon})$. If coefficients are introduced in the same manner as in Theorem 1.2, it suffices to assume the Riemann hypothesis for the Riemann zeta function alone to verify an analogous statement.

Theorem 1.3. *Let $\lambda_1, \lambda_2, \lambda_3$ be positive algebraic numbers such that the sets $1, \lambda_1, \lambda_1/\lambda_2$ and $1, \lambda_1, \lambda_1/\lambda_3$ are linearly independent over the rationals. Let*

$$r_\lambda(n) = \sum_{[\lambda_1 p_1]+[\lambda_2 p_2]+[\lambda_3 p_3]=n} (\log p_1)(\log p_2)(\log p_3).$$

Then

$$r_\lambda(n) = \frac{1}{2}n^2(\lambda_1\lambda_2\lambda_3)^{-1} + O(n^2 e^{-\sqrt{\log n}}).$$

If the Riemann hypothesis holds for the Riemann zeta function, then

$$r_\lambda(n) = \frac{1}{2}n^2(\lambda_1\lambda_2\lambda_3)^{-1} + O(n^{75/38+\varepsilon}).$$

The results in Theorem 1.3 should be of a fairly predictable nature to an expert in additive prime number theory. We have included them here for illustrational purposes, to serve as a model for an approach to asymptotic formulae which differs from the work of earlier writers in various aspects.

Our last theorem is of a slightly different flavour. Questions concerning integers representable as the sum of a prime and a perfect square also have their root in correspondence of Goldbach and Euler. Davenport and Heilbronn (1937) showed that for any fixed $k \geq 2$, almost all natural numbers are of the shape $p+x^k$, and more recently, a bound for the exceptional set of the same quality as for Goldbach's problem was obtained by various writers; the best account is Zaccagnini (1992). Attempts to replace x^k by a sequence increasing more rapidly than any polynomial have had limited success. Romanov (1934) studied the integers of the form $p + 2^k$, and showed that these form a set of positive density. Then, also the sequence of numbers representable as

$$p + 2^{k_1} + 2^{k_2} + \ldots + 2^{k_r}$$

must have positive density, depending on r. Gallagher (1975) established that this density tends to 1 as r grows, but so far no "natural" sequence seems to be known where the density is actually 1. We prove

Theorem 1.4. *Let $1 < \gamma < \frac{3}{2}$. Let $D_\gamma(N)$ denote the number of natural numbers $n \le N$ for which the equation*

$$p + \left[\exp\left((\log k)^\gamma\right)\right] = n$$

has no solution. Then

$$D_\gamma(N) \ll N \exp\Big(-\frac{1}{2}(\log N)^{\theta-\varepsilon}\Big)$$

where $\theta = \min(\frac{3}{\gamma} - 2, \frac{1}{3})$.

1.2. Methods. A common feature in our theorems is the occurrence of an integer part. Traditional approaches to additive problems of this type make use of the Hardy-Littlewood method. One then encounters exponential sums like

$$\sum_{p \le N} e(\alpha[\lambda p]) \log p. \tag{1.5}$$

and the aforementioned work of Arkhipov, Buriev and Chubarikov is no exception. Bounds of Weyl's type for (1.5) may be deduced from estimates for the more familiar sum

$$S(\alpha) = \sum_{p \le N} e(\alpha p) \log p \tag{1.6}$$

by writing

$$[\lambda p] = \lambda p - \{\lambda p\}, \tag{1.7}$$

and removing $\{\lambda p\}$ by a Fourier analysis. The disadvantage is that not only $S(\lambda\alpha)$ occurs, but also $S(\lambda(\alpha + h))$ for many integers h. The elementary argument which links (1.1) and (1.3) avoids such a Fourier analysis altogether. Instead, we directly solve the diophantine inequality (1.3) by the Davenport-Heilbronn Fourier transform method which in many ways is simpler than the classical Hardy-Littlewood method. This simplicistic idea also suffices to establish Theorem 1.4. We present the details as a warm-up in §2.

More care is required when the equation in question contains two or more variables with integer parts. Rather than applying a Fourier analysis to the term $\alpha\{\lambda p\}$ implicit in (1.5), one may use (1.7) in the equation

$$[\lambda_1 p_1] + [\lambda_2 p_2] = n \tag{1.8}$$

and then dissolve $\{\lambda p\}$ by a Fourier analysis, performed either on the level of the diophantine equation, or again within an exponential sum. To fix ideas, consider a solution of the inequality

$$\Big|\lambda_1 p_1 + \lambda_2 p_2 - \Big(n + \frac{3}{4}\Big)\Big| < \frac{1}{4} \tag{1.9}$$

subject to the additional constraint

$$\{\lambda_2 p_2\} < \frac{1}{4}. \tag{1.10}$$

Then $\{\lambda_1 p_1\} + \{\lambda_2 p_2\} \in [0, \frac{5}{4})$, and consequently, (1.8) holds. Thus, in order to solve (1.8), it suffices to find solutions to (1.9) satisfying (1.10). This can be done by the Davenport-Heilbronn method, in much the same way as Brüdern, Cook and Perelli (1997). The analysis will involve $S(\lambda_1 \alpha)$ and, according to (1.10), an exponential sum like

$$\sum_{\substack{p \le N \\ \{\lambda_2 p\} < \frac{1}{4}}} e(\alpha p) \log p. \tag{1.11}$$

The condition $\{\lambda_2 p\} < \frac{1}{4}$ can be removed from the summation condition by developing it into a Fourier series. It turns out that the interplay of this series and the Fourier transform underlying the Davenport-Heilbronn approach is smooth and does not cause much extra trouble. In §3 we describe a weighted version of the above argument, and use it to establish Theorem 1.3.

Asymptotic formulae for numbers of solutions as in Theorem 1.4 are not available by naive tricks such as introducing a harmless condition like (1.10) into a diophantine inequality (1.8). A further elaboration of this idea is now necessary. One may sort the primes p_2, p_3 in the sum $r_\lambda(n)$ into subsets where the extra conditions

$$X_j \le \{\lambda_j p_j\} < X_j + Y \tag{1.12}$$

are satisfied, and then perform a transition to a diophantine inequality as before. The recombination of the main terms entails some technical complication because we will have to "smooth" the constraints (1.12). Fortunately, certain unpleasant error terms in this process have an interpretation as a diophantine counting problem, and may be readily dismissed. The work in §4 will be devoted to this most elaborate variant of the simple observation which links (1.1) and (1.3).

The methods of this paper should be applicable whenever an integer part occurs in a diophantine problem, and promise success provided the cognate diophantine inequality can be treated. The moral is that diophantine inequalities are easier to deal with by the analytic machinery, but related integer part equations are not genuinely harder. It appears to this writer that this point has been overlooked. Not only is there a noticeable difference between Theorem 1.1 and the work of Arkhipov, Buriev and Chudarikov, but also in other similar situations. To mention just one example, we turn our attention briefly to the ongoing chase for the largest real number $c > 1$ such that the inequality

$$\left|p_1^c + p_2^c + p_3^c - n\right| < (\log n)^{-1} \tag{1.13}$$

has prime solutions for all large n, and that

$$[p_1^c] + [p_2^c] + [p_3^c] = n \tag{1.14}$$

is soluble. In recent years, the largest admissible value for (1.14) has always been inferior to the record for (1.13); see Kumchev and Nedeva (1998) and Cao and Zhai (1999, 2000), for example. Although we have not checked any details, the arguments in §§3–4 should suffices to close the gap between (1.13) and (1.14).

1.3. Notation. Most of our notation is standard or otherwise explained at the appropriate stage of the argument. The letter p, with or without subscript, denotes prime numbers. Whenever ε occurs in a statement, it is asserted that the statement is true for any fixed value of $\varepsilon > 0$, with implicit constants depending on ε. All constants implicit in Vinogradov or Landau symbols also depend on $\lambda, \lambda_1, \lambda_2, \lambda_3$, but on no other parameter.

2. Adding primes to a rapidly increasing sequence

2.1. Some useful functions. We recall some basic facts from elementary Fourier analysis before we move on to establish Theorem 1.4 which is the main concern in this section.

For any function $f \in L_1(\mathbb{R})$, define the Fourier transform

$$\hat{f}(\alpha) = \int_{-\infty}^{\infty} f(\beta)e(-\alpha\beta)d\beta.$$

For functions $f, g \in L_1(\mathbb{R})$, the convolution is defined by

$$f * g(\alpha) = \int_{-\infty}^{\infty} f(\alpha - \beta)g(\beta)d\beta,$$

provided that this integral exists. One has $\widehat{f * g} = \hat{f} \cdot \hat{g}$.

Let $\eta > 0$ and define

$$k_\eta(\alpha) = \left(\frac{\sin \pi\eta\alpha}{\pi\alpha}\right)^2; \quad \Upsilon_\eta(\alpha) = \max(0, \eta - |\alpha|). \tag{2.1}$$

Then $\widehat{k_\eta} = \Upsilon_\eta$ and $\widehat{\Upsilon}_\eta = k_\eta$. Because these functions are all even, the interplay formula for Fourier transforms and convolutions gives

$$\widehat{k_\eta^m} = \Upsilon_\eta * \ldots * \Upsilon_\eta \tag{2.2}$$

where m is any natural number, and the convolution on the right is m-fold.

Lemma 2.1. *Let m be any natural number. The function $K : \mathbb{R} \to \mathbb{R}$, defined for $\alpha \neq 0$ by*

$$K(\alpha) = K_m(\alpha) = \left(\frac{\sin(\pi\alpha/8m)}{\pi\alpha}\right)^{4m}. \tag{2.3}$$

has the properties $\widehat{K}(\alpha) \geq 0$ *for all* $\alpha \in \mathbb{R}, \widehat{K}(\alpha) = 0$ *for* $|\alpha| \geq \frac{1}{4}$, *and* $\widehat{K}(\alpha) \geq c_m > 0$ *for* $|\alpha| \leq \frac{1}{8}$ *and some constant* c_m *depending only on* m.

Proof. Note that $K_m = k_{1/(8m)}^{2m}$. The lemma is now immediate from (2.1) and (2.2).

2.2. The Fourier transform method initiated. We now begin our approach to Theorem 1.4. Fix $\gamma \in (1, \frac{3}{2})$ and a constant C with $0 < C < \gamma - 1$. Let N denote a large real parameter, and define M by the equation

$$\exp\big((\log M)^{\gamma}\big) = N. \tag{2.4}$$

In addition to the exponential sum $S(\alpha)$ defined in (1.6), we require the sum

$$T(\alpha) = \sum_{\frac{1}{2}M < l \leq M} e\big(\alpha \exp((\log l)^{\gamma})\big). \tag{2.5}$$

Our next lemma is merely a special case of Theorem 1.3 in Brüdern and Perelli (1998).

Lemma 2.2. *For* $N^{-11/12} \leq |\alpha| \leq N^C$ *one has*

$$|T(\alpha)| \ll M \exp\big(-\kappa(\log M)^{3-2\gamma}\big)$$

where κ *is a suitable positive constant.*

Fix an integer m with $3Cm > 2$, define $K(\alpha)$ by (2.3), and introduce the integral

$$R(n) = \int_{-\infty}^{\infty} S(\alpha)T(\alpha)e\Big(-\alpha\Big(n + \frac{1}{2}\Big)\Big)K(\alpha)d\alpha. \tag{2.6}$$

By (1.6), (2.5) and Lemma 2.1, we see that $R(n) > 0$ implies that there is a solution to the inequality

$$\Big|p + e^{(\log l)^{\gamma}} - n - \frac{1}{2}\Big| < \frac{1}{4},$$

and for any such solution, we necessarily have

$$p + [e^{(\log l)^{\gamma}}] = n.$$

We now proceed to show that *for all but* $O(Ne^{-(\log N)^{\theta}})$ (with θ as in Theorem 1.4) *values of* $n \in [\frac{1}{2}N, N]$, *one has* $R(n) > 0$. By summing over dyadic intervals, this implies Theorem 1.4.

2.3. A mean square estimate. A formal application of the methods of Davenport and Heilbronn (1946) leads one to expect that for $n \in [\frac{1}{2}N, N]$ one has

$R(n) \gg M$, and indeed this is true for most values n. We begin by dissecting the real line into the *major arc*

$$\mathfrak{M} = [-N^{-11/12}, N^{-11/12}], \tag{2.7}$$

the pair of *minor arcs*

$$\mathfrak{m} = \{\alpha : N^{-11/12} < |\alpha| < N^C\}, \tag{2.8}$$

and the *tail*

$$\mathfrak{t} = \{\alpha : |\alpha| \geq N^C\}. \tag{2.9}$$

The treatment of the tail is straightforward. The bounds $|S(\alpha)| \ll N$ (which follows from Chebychev's upper bound) and $|T(\alpha)| \leq M$ together with (2.3) suffice to confirm the inequalities

$$\int_{\mathfrak{t}} |S(\alpha)T(\alpha)K(\alpha)|d\alpha \ll NM \int_{N^C}^{\infty} \alpha^{-4m} d\alpha \ll N^{-1}M, \tag{2.10}$$

on recalling the choice for m.

The minor arcs require a mean square approach for which we borrow an idea from Brüdern, Cook and Perelli (1997). In $L_2(\mathbb{R})$, equipped with the standard inner product

$$(f, g) = \int_{-\infty}^{\infty} f(\alpha)\overline{g(\alpha)}d\alpha,$$

the functions

$$\Phi_n(\alpha) = K(\alpha)^{1/2} e\Big(-\alpha\Big(n + \frac{1}{2}\Big)\Big) \tag{2.11}$$

satisfy $(\Phi_n, \Phi_j) = 0$ for $n \neq j$, whence $\Phi_n/(\Phi_n, \Phi_n)^{1/2}$ is an orthonormal family. But $(\Phi_n, \Phi_n) = \int_{-\infty}^{\infty} K(\alpha)d\alpha$ is independent of n. The function

$$F(\alpha) = \begin{cases} S(\alpha)T(\alpha)K(\alpha)^{1/2} & \text{for } \alpha \in \mathfrak{m}, \\ 0 & \text{otherwise}, \end{cases}$$

is in $L_2(\mathbb{R})$, and we have

$$\int_{\mathfrak{m}} S(\alpha)T(\alpha)K(\alpha)e\Big(-\alpha\Big(n + \frac{1}{2}\Big)\Big)d\alpha = (F, \Phi_n). \tag{2.12}$$

By Bessel's inequality,

$$\sum_{n=-\infty}^{\infty} |(F, \Phi_n)|^2 \ll (F, F) = \int_{\mathfrak{m}} |S(\alpha)T(\alpha)|^2 K(\alpha)d\alpha. \tag{2.13}$$

We observe that Lemma 2.1 and elementary prime number theory suffice to establish the bound

$$\int_{-\infty}^{\infty} |S(\alpha)|^2 K(\alpha) d\alpha \ll \sum_{p \le N} (\log p)^2 \ll N \log N, \tag{2.14}$$

and therefore, from Lemma 2.2, (2.12) and (2.13), we may conclude that

$$\sum_{n=-\infty}^{\infty} \Big| \int_{\mathfrak{m}} S(\alpha)T(\alpha)K(\alpha)e\Big(- \alpha\Big(n + \frac{1}{2}\Big)\Big) d\alpha \Big|^2 \ll NM^2 (\log N) \exp\big(- 2\kappa (\log M)^{3-2\gamma}\big).$$

By a standard argument, this implies an upper bound for the number of n for which the contribution of $\mathfrak{m}$ can be large. Since the tail makes no contribution by (2.10), we may conclude as follows.

Lemma 2.3. *The number of integers* $n \in [\frac{1}{2}N, N]$ *for which the inequality*

$$\Big| \int_{|\alpha| > N^{-11/12}} S(\alpha)T(\alpha)K(\alpha)e\Big(- \alpha\Big(n + \frac{1}{2}\Big)\Big) d\alpha \Big| > \frac{M}{\log N}$$

holds, does not exceed

$$\ll N(\log N)^3 \exp(-2\kappa(\log M)^{3-2\gamma}).$$

2.4. The approximation of the major arc contribution. Although our work on the major arcs is standard in principle, some care has to be exercised due to the "slow" convergence of the singular integral. The early steps follow the pattern laid down by Vaughan (1974). Let

$$I(\alpha) = \int_0^N e(\beta\alpha) d\beta, \quad J(\alpha) = \sum_{l \le N} \sum_{\varrho} l^{\varrho - 1} e(\alpha l), \tag{2.15}$$

where on the right the sum over ϱ is over all zeros of the Riemann zeta function with $\Re(\varrho) \ge \frac{2}{3}$, and $|\Im(\varrho)| < N^{1/3}$. Now define the function Δ by

$$S(\alpha) = I(\alpha) - J(\alpha) + \Delta(\alpha). \tag{2.16}$$

We then recall from Vaughan (1974), Lemma 5 and Lemma 8, the estimates

$$\Delta(\alpha) \ll N^{\frac{2}{3}+\varepsilon}(1 + N|\alpha|) \tag{2.17}$$

uniformly for $\alpha \in \mathbb{R}$, and

$$\int_{-1/2}^{1/2} |J(\alpha)|^2 d\alpha \ll N \exp\big(- (\log N)^{\frac{1}{3}-\varepsilon}\big). \tag{2.18}$$

Note that Vaughan (1974) has the exponent $\frac{1}{3}-\varepsilon$ in (2.18) replaced by the weaker value $\frac{1}{5}$, but an inspection of his proof clearly yields (2.18).

By (2.16), we can write

$$\int_{\mathfrak{M}} S(\alpha)T(\alpha)K(\alpha)e\Big(-\alpha\Big(n+\frac{1}{2}\Big)\Big)d\alpha = R_I(n) - R_J(n) + R_\Delta(n) \tag{2.19}$$

where

$$R_I(n) = \int_{\mathfrak{M}} I(\alpha)T(\alpha)K(\alpha)e\Big(-\alpha\Big(n+\frac{1}{2}\Big)\Big)d\alpha,$$

and likewise defined numbers $R_J(n), R_\Delta(n)$. The term $R_\Delta(n)$ is always small. To see this, note that (2.17) yields $\Delta(\alpha) \ll N^{3/4+\varepsilon}$ for $\alpha \in \mathfrak{M}$. The trivial bounds for $T(\alpha)$ and $K(\alpha)$ together with $\int_{\mathfrak{M}} d\alpha \ll N^{-11/12}$ give

$$R_\Delta(n) \ll N^{\frac{3}{4}-\frac{11}{12}+\varepsilon} M \ll 1. \tag{2.20}$$

By Bessel's inequality, used in the same manner as in the previous section, we have

$$\sum_{n=-\infty}^{\infty} |R_J(n)|^2 \ll \int_{\mathfrak{M}} |J(\alpha)T(\alpha)|^2 K(\alpha)d\alpha.$$

The trivial bound for T and (2.18) yield

$$\sum_{n=-\infty}^{\infty} |R_J(n)|^2 \ll M^2 N \exp(-(\log N)^{\frac{1}{3}-\varepsilon}),$$

and by a familiar argument, we may now conclude as follows.

Lemma 2.4. *The number of natural numbers $n \le N$ for which $|R_J(n)| > M/\log N$ holds does not exceed $\ll N\exp(-(\log N)^{\frac{1}{3}-\varepsilon})$.*

Finally, we compare $R_I(n)$ with the integral

$$R^*(n) = \int_{-\infty}^{\infty} I(\alpha)T(\alpha)K(\alpha)e\Big(-\alpha\Big(n+\frac{1}{2}\Big)\Big)d\alpha. \tag{2.21}$$

The argument used to establish (2.10) also shows that the contribution to (2.21) arising from the tail $\mathfrak{t}$ does not exceed $O(N^{-1}M) = O(N^{-1/2})$, whence

$$R_I(n) - R^*(n) \ll N^{-\frac{1}{2}} + \int_{\mathfrak{m}} |I(\alpha)T(\alpha)|K(\alpha)d\alpha.$$

By (2.15) and a partial integration, $I(\alpha) \ll |\alpha|^{-1}$ for $\alpha \ne 0$. The trivial bound $K(\alpha) \ll 1$ and Lemma 2.2 now show that

$$\begin{aligned}\int_{\mathfrak{m}} |I(\alpha)T(\alpha)|K(\alpha)d\alpha &\ll M\exp\big(-\kappa(\log M)^{3-2\gamma}\big)\int_{N^{-11/12}}^{N^{C}} \frac{d\alpha}{\alpha} \\ &\ll M(\log N)\exp\big(-\kappa(\log M)^{3-2\gamma}\big)\end{aligned}$$

so that we also have $|R_I(n) - R^*(n)| \le M/\log N$ when N is large. We combine this with (2.19), (2.20) and Lemma 2.4, and summarize the result in the next lemma.

Lemma 2.5. *The number of integers n with $1 \le n \le N$ and*

$$\left|R^*(n) - \int_{\mathfrak{M}} S(\alpha)T(\alpha)K(\alpha)e\Big(-\alpha\Big(n+\frac{1}{2}\Big)\Big)d\alpha\right| > \frac{2M}{\log N}$$

does not exceed $O\big(N\exp\big(-(\log N)^{\frac{1}{3}-\varepsilon}\big)\big)$.

2.5. The endgame. It remains to evaluate $R^*(n)$. By (2.5), (2.15), and (2.21),

$$R^*(n) = \int_0^N \sum_{\frac{1}{2}M<l\le M} \widehat{K}\Big(\beta + e^{(\log l)^\gamma} - n - \frac{1}{2}\Big)d\beta.$$

Now suppose that $\frac{1}{2}N \le n \le N$, and that $\frac{1}{2}M < l \le \frac{5}{8}M$. Then, a short calculation shows that the interval of all real numbers β satisfying the inequality

$$\left|\beta + e^{(\log l)^\gamma} - n - \frac{1}{2}\right| \le \frac{1}{8}$$

is a subset of $[0, N]$. From Lemma 2.1, we know that $\widehat{K}(\alpha) \ge 0$ for all α, and $\widehat{K}(\alpha) \gg 1$ for $|\alpha| \le \frac{1}{8}$. It follows that

$$\int_0^N \widehat{K}\Big(\beta + e^{(\log l)^\gamma} - n - \frac{1}{2}\Big)d\beta \gg 1$$

uniformly for $\frac{1}{2}M < l \le \frac{5}{8}M$ and $\frac{1}{2}N \le n \le N$. Summing over l yields

$$R^*(n) \gg M \tag{2.22}$$

for all $n \in [\frac{1}{2}N, N]$. If we now recall (2.6) and combine (2.22) with the results in Lemmata 2.3 and 2.5, we arrive at the conclusion that $R(n) \gg M$ must hold for all but

$$O\Big(N\exp\big(-(\log N)^{\frac{3}{\gamma}-2-\varepsilon}\big) + N\exp\big(-(\log N)^{\frac{1}{3}-\varepsilon}\big)\Big)$$

values of n in the range $\frac{1}{2}N \le n \le N$. As we have remarked earlier, this suffices to establish Theorem 1.4.

3. Binary linear forms at prime arguments

3.1. Another useful function. Let $\Upsilon = \Upsilon_\eta$ be defined by (2.1). Then the function

$$\Psi_\eta(\alpha) = \sum_{h=-\infty}^{\infty} \Upsilon_\eta(\alpha+h) \tag{3.1}$$

is of period 1, and the Fourier transform of the function $h \mapsto \Upsilon(\alpha+h)$ is $h \mapsto e(\alpha h)\widehat{\Upsilon}(h)$. The Poisson summation formula, applied to (3.1), now yields the Fourier series

$$\Psi_\eta(\alpha) = \sum_{h=-\infty}^{\infty} k_\eta(h)e(\alpha h). \tag{3.2}$$

3.2. Counting solutions of (1.9). In this section, we use the function $\Psi = \Psi_{1/8}$ only, and form the sum

$$S_1(\alpha,\lambda) = \sum_{p\le N} \Psi\Big(\lambda p + \frac{1}{8}\Big)e(\alpha p)\log p. \tag{3.3}$$

Note that $\Psi(\lambda p + \frac{1}{8}) \neq 0$ if and only if $0 < \{\lambda p\} < \frac{1}{4}$. With K given by (2.3), we now consider the integral

$$\mathcal{I}(n) = \int_{-\infty}^{\infty} S(\lambda_1\alpha)S_1(\lambda_2\alpha.\lambda_2)K(\alpha)e\Big(-\alpha\Big(n+\frac{3}{4}\Big)\Big)d\alpha \tag{3.4}$$

and observe that

$$\mathcal{I}(n) = \sum_{p_1.p_2\le N} \widehat{K}\Big(\lambda_1 p + \lambda_2 p - n - \frac{3}{4}\Big)\Psi\Big(\lambda_2 p_2 + \frac{1}{8}\Big).$$

In particular, $\mathcal{I}(n) > 0$ implies that there is at least one solution of (1.9) satisfying (1.10), and hence also of (1.8). We now proceed to show that indeed $\mathcal{I}(n) > 0$ holds for all but $O(N^{5/6+\varepsilon})$ values of n in the interval

$$\frac{1}{4}(\lambda_1+\lambda_2)N < n \le \frac{1}{2}(\lambda_1+\lambda_2)N. \tag{3.5}$$

Theorem 1.3 follows from this by a dyadic splitting up argument.

By (3.2) and (3.3),

$$S_1(\lambda_2\alpha.\lambda_2) = \sum_{h=-\infty}^{\infty} k_{1/8}(h)e\Big(\frac{h}{8}\Big)S\big(\lambda_2(\alpha+h)\big),$$

and accordingly, we deduce from (3.4) that

$$\mathcal{I}(n) = \sum_{h=-\infty}^{\infty} k_{1/8}(h)e\Big(\frac{h}{8}\Big)\mathcal{I}_h(n) \tag{3.6}$$

where

$$\mathcal{I}_h(n) = \int_{-\infty}^{\infty} S(\lambda_1\alpha)S\big(\lambda_2(\alpha+h)\big)K(\alpha)e\Big(-\alpha\Big(n+\frac{3}{4}\Big)\Big)d\alpha.$$

This identity may be viewed as a Fourier expansion of the weighted count $\mathcal{I}(n)$ for the number of solutions of the simultaneous conditions (1.9), (1.10). One expects that only $\mathcal{I}_0(n)$ is large and contributes roughly N for all n satisfying (3.5). We proceed to confirm this for almost all n.

3.3. Some simple terms. We dismiss from (3.6) large values of h. For this purpose we note that Lemma 2.1 and elementary prime number theory show that

$$\int_{-\infty}^{\infty} |S(\lambda_1\alpha)|^2 K(\alpha)d\alpha = \sum_{p_1,p_2\le N} \widehat{K}\big(\lambda(p_1-p_2)\big)(\log p_1)\log p_2$$

whence

$$\int_{-\infty}^{\infty} |S(\lambda_1\alpha)|^2 K(\alpha)d\alpha \ll \sum_{p\le N}(\log p)^2 \ll N\log N, \tag{3.7}$$

and likewise,

$$\int_{-\infty}^{\infty} \big|S\big(\lambda_2(\alpha+h)\big)\big|^2 K(\alpha)d\alpha \ll N\log N \tag{3.8}$$

uniformly in $h\in\mathbb{Z}$. Since $k(h)\ll h^{-2}$, it now follows from (3.7), (3.8) and Schwarz's inequality that

$$\sum_{|h|>H} k_{1/8}(h)|\mathcal{I}_h(n)| \ll N(\log N)H^{-1}.$$

We take $H=(\log N)^2$ and then infer from (3.6) that

$$\mathcal{I}(n) = \sum_{|h|\le\log^2 N} k_{1/8}(h)e\Big(\frac{h}{8}\Big)\mathcal{I}_h(n) + O\Big(\frac{N}{\log N}\Big). \tag{3.9}$$

Next, we remove from the remaining integrals $\mathcal{I}_h(n)$ a tail $|\alpha|>\log N$. By a simple change of variable and an argument similar to, but easier than the deduction of (3.7), one readily confirms that the chain of inequalities

$$\int_A^{A+1} |S(\lambda_i\alpha)|^2 d\alpha \ll \int_0^1 |S(\alpha)|^2 d\alpha \ll N\log N \tag{3.10}$$

is valid uniformly for $A\in\mathbb{R}$. Since $K(\alpha)\ll|\alpha|^{-4}$ holds for all α with $|\alpha|\ge 1$, it follow that for $X\ge 1$ one has

$$\int_X^{\infty} |S(\lambda_1\alpha)|^2 K(\alpha)d\alpha \ll X^{-3}N\log N,$$

and similarly, uniformly for $h\in\mathbb{Z}$,

$$\int_X^{\infty} |S(\lambda_2(\alpha+h))|^2 K(\alpha)d\alpha \ll X^{-3}N\log N.$$

We choose $X = \log N$ and then infer from Schwarz's inequality that

$$\int_{\log N}^{\infty} |S(\lambda_1\alpha)S(\lambda_2(\alpha+h))|K(\alpha)d\alpha \ll N(\log N)^{-2}$$

also holds uniformly in h. Consequently, the integral

$$\mathcal{I}_h^*(n) = \int_{-\log N}^{\log N} S(\lambda_1\alpha)S(\lambda_2(\alpha+h))K(\alpha)e\Big(-\alpha\Big(n+\frac{3}{4}\Big)\Big)d\alpha$$

differs from $\mathcal{I}_h(n)$ by at most $O(N(\log N)^{-2})$, and the inequality $k_{1/8}(h) \ll h^{-2}$ (when $h \neq 0$) suffices to deduce from (3.9) that

$$\mathcal{I}(n) = \sum_{|h|\leq \log^2 N} k_{1/8}(h)e\Big(\frac{h}{8}\Big)\mathcal{I}_h^*(n) + O\Big(\frac{N}{\log N}\Big). \tag{3.11}$$

3.4. Another mean square estimate. The initial treatment now completed, we turn our attention to a prospective main term and bound the remaining terms in (3.11) on average over n. In a detour from previous practise, the major and minor arcs are redefined as

$$\mathfrak{M} = [-N^{-1/3}, N^{-1/3}], \quad \mathfrak{m} = \{\alpha : N^{-1/3} < |\alpha| < \log N\}.$$

We formulate the key result as a lemma, but postpone its proof to §3.5 below.

Lemma 3.1. *Suppose λ_1, λ_2 are real algebraic numbers such that $1, \lambda_1, \lambda_1/\lambda_2$ span a 3-dimensional $\mathbb{Q}$-vectorspace in $\mathbb{R}$. Let $\delta > 0$. Then, for sufficiently large N, for any $\alpha \in \mathbb{R}, h \in \mathbb{Z}$ with $|\alpha| \leq \log N$ and $|h| \leq \log^2 N$, the inequality*

$$\big|S(\lambda_1\alpha)S\big(\lambda_2(\alpha+h)\big)\big| > N^{\frac{11}{6}+\delta}$$

implies that $h = 0$ and that $|\alpha| \leq N^{-1/2}$.

Define a function V on $\mathbb{R}$ by writing

$$V_0(\alpha) = \sum_{0<|h|\leq \log^2 N} k_{1/8}(h)e\Big(\frac{h}{8}\Big)S(\lambda_1\alpha)S\big(\lambda_2(\alpha+h)\big),$$

and

$$V(\alpha) = \begin{cases} k_{1/8}(0)S(\lambda_1\alpha)S(\lambda_2\alpha) + V_0(\alpha) & \text{for } \alpha \in \mathfrak{m}, \\ V_0(\alpha) & \text{for } \alpha \in \mathfrak{M}, \\ 0 & \text{for } |\alpha| > \log N. \end{cases}$$

For later reference, we note that Lemma 3.1 yields

$$V(\alpha) \ll N^{11/6+\varepsilon}. \tag{3.12}$$

Furthermore, we have from (3.11) that

$$\mathcal{I}(n) = k_{1/8}(0)\mathcal{J}(n) + \mathcal{V}(n) + O\Big(\frac{N}{\log N}\Big) \tag{3.13}$$

where

$$\mathcal{J}(n) = \int_{\mathfrak{M}} S(\lambda_1\alpha)S(\lambda_2\alpha)K(\alpha)e\Big(-\alpha\Big(n+\frac{3}{4}\Big)\Big)d\alpha,$$

$$\mathcal{V}(n) = \int_{-\infty}^{\infty} V(\alpha)K(\alpha)e\Big(-\alpha\Big(n+\frac{3}{4}\Big)\Big)d\alpha.$$

From Lemma 5 of Brüdern, Cook and Perelli (1997) we invoke the lower bound[1]

$$\mathcal{J}(n) \gg N \tag{3.14}$$

for all integers n with $\frac{1}{4}(\lambda_1+\lambda_2)N \le n \le \frac{1}{2}(\lambda_1+\lambda_2)N$.

We now imitate the principle underlying the proof of Lemma 2.3 to estimate $\mathcal{V}(n)$ in mean square. With

$$\Phi_n(\alpha) = K(\alpha)^{1/2}e\Big(-\alpha\Big(n+\frac{3}{4}\Big)\Big), \quad F(\alpha) = V(\alpha)K(\alpha)^{1/2}$$

and the identity $\mathcal{V}(n) = (F, \Phi_n)$, Bessel's inequality shows just as in (2.13) that

$$\sum_{n=-\infty}^{\infty} |\mathcal{V}(n)|^2 \ll \int_{-\infty}^{\infty} |V(\alpha)|^2 K(\alpha)d\alpha. \tag{3.15}$$

By (3.7), (3.8) and Schwarz's inequality,

$$\int_{-\infty}^{\infty} |V(\alpha)|K(\alpha)d\alpha \ll N\log N,$$

so that from (3.12) and (3.15) we infer the bound

$$\sum_{n=-\infty}^{\infty} |\mathcal{V}(n)|^2 \ll N^{17/6+\varepsilon}. \tag{3.16}$$

Theorem 1.3 is now available. Suppose that the integer n satisfies $|\mathcal{V}(n)| \le N/\log N$. Then, for $\frac{1}{4}(\lambda_1+\lambda_2)N \le n \le \frac{1}{2}(\lambda_1+\lambda_2)N$, we conclude from (3.11) and (3.14) that $\mathcal{I}(n) \gg N$ holds. It follows that $\mathcal{I}(n) = 0$ for n in the above range implies that $|\mathcal{V}(n)| > N/\log N$. By (3.16) this is possible for at most $O(N^{5/6+\varepsilon})$ integers n. We have remarked already in §3.2 that this suffices to establish Theorem 1.3.

3.5. A method from diophantine approximation. Our main tool in this section, which is solely devoted to the proof of Lemma 3.1, is Schmidt's famous generalisation of Roth's theorem to linear forms. We record only the special case we require, as a lemma. It is an immediate corollary of Schmidt (1980), Corollary 1E.

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1 Lemma 5 of the reference given is subject to the conditions $\lambda_1, \lambda_2 \ge 1$, and claims (3.14) for $\frac{1}{2}N \le n \le N$, but an inspection of the proof shows that (3.14) holds in the relevant range for n provided only that $\lambda_1 > 0, \lambda_2 > 0$.

Lemma 3.2. *Let μ_1, μ_2 be real algebraic numbers such that $1, \mu_1, \mu_2$ are linearly independent over the rationals. Let $\delta > 0$. Then there is a constant $C = C(\mu_1, \mu_2, \delta) > 0$ such that for all $(m_1, m_2, m_3) \in \mathbb{Z}^3 \setminus \{(0,0,0)\}$ one has*

$$|\mu_1 m_1 + \mu_2 m_2 + m_3| > C(\max |m_j|)^{-2-\delta}.$$

We also require Vinogradov's classical estimate for exponential sums over primes. A convenient version occurs as Lemma 3 in Brüdern, Cook and Perelli (1997), which we restate here.

Lemma 3.3. *Let $1 \le A \le N^{1/5}(\log N)^{-5}$ and suppose that $|S(\alpha)| \ge \frac{N}{A}$. Then there are coprime integers a, q with*

$$1 \le q \ll A^2(\log N)^8, \quad |q\alpha - a| \ll A^2 N^{-1}(\log N)^8.$$

To launch our approach to Lemma 3.1, we investigate the consequences of the hypotheses therein in the light of Lemma 3.3. We trivially have $|S(\alpha)| \ll N$. Hence, the inequality $|S(\lambda_1\alpha)S(\lambda_2(\alpha+h))| \ge N^{11/6+\delta}$ implies that there are numbers A_1, A_2 with $A_1 \ge 1, A_2 \ge 1$ and

$$|S(\lambda_1\alpha)| \ge N/A_1, \quad \big|S\big(\lambda_2(\alpha+h)\big)\big| \ge N/A_2, \quad A_1A_2 \ll N^{\frac{1}{6}-\delta}.$$

Two applications of Lemma 3.3 now yield integers a_i, q_i with

$$\begin{aligned} 1 \le q_i &\ll A_i^2(\log N)^8, \\ \Big|\lambda_1\alpha - \frac{a_1}{q_1}\Big| &\ll \frac{A_1^2(\log N)^8}{q_1 N}, \\ \Big|\lambda_2(\alpha+h) - \frac{a_2}{q_2}\Big| &\ll \frac{A_2^2(\log N)^8}{q_2 N}. \end{aligned}$$

The last two inequalities we divide by λ_j and then eliminate α to obtain

$$\Big|\frac{a_1}{\lambda_1 q_1} - \frac{a_2}{\lambda_2 q_2} + h\Big| \ll \frac{(\log N)^8}{N}\Big(\frac{A_1^2}{q_1} + \frac{A_2^2}{q_2}\Big).$$

This we rewrite, after multiplication by q_1, q_2, as

$$\big|\lambda_1^{-1}a_1q_2 - \lambda_2^{-1}a_2q_1 + hq_1q_2\big| \ll (A_1A_2)^2N^{-1}(\log N)^{16} \ll N^{-\frac{2}{3}-\delta}. \tag{3.17}$$

The numbers $1, \lambda_1, \lambda_1/\lambda_2$ are linearly independent over $\mathbb{Q}$, and thus, this is also true for $1, \mu_1, \mu_2$ with $\mu_j = \lambda_j^{-1}$. We now apply Lemma 3.2 with $m_1 = a_1q_2$, $m_2 = -a_2q_1$, $m_3 = hq_1q_2$. Since $|h| \le (\log N)^2$, we must have $|m_3| \ll A_1^2A_2^2(\log N)^{18}$. To bound m_1, note that $|\alpha| \le \log N$ gives

$$|a_1| = |\lambda_1 q_1 \alpha| + O(A_1^2 N^{-1}\log^8 N) \ll A_1^2(\log N)^{10}$$

so that we also have $|m_1| \ll A_1^2 A_2^2 (\log N)^{18}$. A similar argument confirms the same bound for $|m_2|$ so that we now conclude that

$$\max\big(|a_1 q_2|, |a_2 q_1|, |h q_1 q_2|\big) \ll A_1^2 A_2^2 (\log N)^{18} \ll N^{\frac{1}{3}}.$$

By (3.17) and Lemma 3.2, it follows that for sufficiently large N we must have $a_1 q_2 = a_2 q_1 = h q_1 q_2 = 0$. This is possible only if $a_1 = a_2 = h = 0$. But $a_1 = 0$ implies that

$$|\alpha| \ll A_1^2 N^{-1} (\log N)^8 \ll N^{-2/3},$$

and the Lemma is proved.

4. Ternary linear forms at prime arguments

4.1 Partitions of unity and envelopes. If one attempts to prove an asymptotic formula for the number of solutions of an equation involving fractional parts, the ideas of Chapter 3 are still applicable when combined with a partition of unity, as we shall now explain. This makes it possible to localize the fractional parts of some terms, and one can then link the original equation with an associated inequality as in earlier descriptions of the method. Rather than proceeding in undue generality, we concentrate on the special case considered in Theorem 1.3.

Let $L \in \mathbb{N}, L \geq 10$ and write $\eta = L^{-1}$. Then, by (2.1) and (3.1),

$$\sum_{l=0}^{L-1} \Psi_\eta(\alpha + l\eta) = \eta \tag{4.1}$$

holds for all $\alpha \in \mathbb{R}$. Suppressing dependence on L, we write

$$\psi_l(\alpha) = \Psi_\eta(\alpha + l\eta). \tag{4.2}$$

Then (4.1) reads

$$L\big(\psi_0(\alpha) + \psi_1(\alpha) + \ldots + \psi_{L-1}(\alpha)\big) = 1 \tag{4.3}$$

for any $\alpha \in \mathbb{R}$. We use this with $\alpha = \lambda_i p_i \; (i = 1, 2)$ in the definition of $r_\lambda(n)$ (in Theorem 1.3) to sort the solutions of

$$[\lambda_1 p_1] + [\lambda_2 p_2] + [\lambda_3 p_3] = n \tag{4.4}$$

according to the values of $\{\lambda_1 p_1\}$ and $\{\lambda_2 p_2\}$. One then obtains an identity

$$r_\lambda(n) = L^2 \sum_{l_1, l_2 = 0}^{L-1} r(n; l_1, l_2) \tag{4.5}$$

where

$$r(n;l_1,l_2) = \sum_{\substack{p_1,p_2,p_3 \\ (4.4)}} \psi_{l_1}(\lambda_1 p_1)\psi_{l_2}(\lambda_2 p_2)\log \mathbf{p}; \tag{4.6}$$

here the summation is restricted to all triples $\mathbf{p} = (p_1, p_2, p_3)$ of primes satisfying (4.4), and

$$\log \mathbf{p} = (\log p_1)(\log p_2)(\log p_3) \tag{4.7}$$

is used as a shorthand.

With the decomposition (4.5) we have reached the final formula for $r_\lambda(n)$ based on partitions of unity. In the next step we provide an envelope for $r(n;l_1,l_1)$, that is, upper and lower bounds for $r(n;l_1,l_2)$ in which the enveloping functions are more suitable for the analytic treatment, and are defined in terms of solutions of diophantine inequalities.

To put these ideas into effect, let W^*, W_* be the continuous functions on $\mathbb{R}$ defined by

$$W^*(\alpha) = \begin{cases} 1 & \text{for } |\alpha| \le \frac{1}{2} + 2\eta, \\ 0 & \text{for } |\alpha| \ge \frac{1}{2} + 4\eta, \end{cases}$$
$$W_*(\alpha) = \begin{cases} 1 & \text{for } |\alpha| \le \frac{1}{2} - 4\eta, \\ 0 & \text{for } |\alpha| \ge \frac{1}{2} - 2\eta, \end{cases} \tag{4.8}$$

and which are linear on the pairs of intervals $\frac{1}{2} + 2\eta < |\alpha| < \frac{1}{2} + 4\eta$ and $\frac{1}{2} - 4\eta < |\alpha| < \frac{1}{2} - 2\eta$, respectively. Furthermore, for any pair l_1, l_2 of integers with $1 \le l_i \le L - 1$, put

$$g = g(l_1, l_2) = n + \frac{1}{2} + (l_1 + l_2)\eta, \tag{4.9}$$

and then define the counting functions

$$r^*(n;l_1,l_2) = \sum_{\mathbf{p}} W^*(\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 - g)\psi_{l_1}(\lambda_1 p_1)\psi_{l_2}(\lambda_2 p_2)\log \mathbf{p}, \tag{4.10}$$

and analogously, $r_*(n;l_1,l_2)$ with W^* replaced by W_*. Their enveloping effect is exhibited in

Lemma 4.1. *Let $1 \le l_i \le L - 1$. Then*

$$r_*(n;l_1,l_2) \le r(n;l_1,l_2) \le r^*(n;l_1,l_2).$$

Proof. We begin with the second inequality. Let $\mathbf{p}$ be a solution of (4.4), which makes a non-zero contribution to $r(n;l_1,l_2)$ in (4.6). Then, by (4.2), (3.1) and (2.1), one has

$$(l_i - 1)\eta < \{\lambda_i p_i\} < (l_i + 1)\eta. \qquad (i = 1, 2) \tag{4.11}$$

The identity

$$\sum_{i=1}^{3} \lambda_i p_i = \sum_{i=1}^{3} [\lambda_i p_i] + \sum_{i=1}^{3} \{\lambda_i p_i\} \tag{4.12}$$

now shows that

$$n + (l_1 + l_2 - 2)\eta < \sum_{i=1}^{3} \lambda_i p_i < n + 1 + (l_1 + l_2 + 2)\eta,$$

and by (4.8) and (4.9) we conclude that $W^*(\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 - g) = 1$. The inequality $r(n; l_1, l_2) \le r^*(n; l_1, l_2)$ is now immediate from (4.6) and (4.10).

The proof of the lower bound is similar. Any triple $\mathbf{p}$ with a non-zero contribution to $r_*(n; l_1, l_2)$ satisfies (4.11) and

$$n + (l_1 + l_2 + 2)\eta < \sum_{i=1}^{3} \lambda_i p_i < n + 1 + (l_1 + l_2 - 2)\eta.$$

By (4.12) and (4.11) it follows that

$$n - 1 < \sum_{i=1}^{3} [\lambda_i p_i] < n + 1,$$

whence (4.4) holds. Therefore, $\mathbf{p}$ is counted by $r(n; l_1, l_2)$, and this establishes the first inequality in the Lemma.

Note that the above argument does not apply when $l_1 l_2 = 0$. For example, if $l_1 = 0$, then $\psi_0(\lambda_1 p_1) \ne 0$ does no longer imply (4.11), but the inequalities $0 \le \{\lambda_1 p_1\} < \eta$ or $1 - \eta < \{\lambda_1 p_1\} < 1$. Although a more elaborate transition from $r(n; 0, l_2)$ to a diophantine inequality is still possible, it is easier to consider such terms in (4.5) as error terms.

Lemma 4.2. *Let $10 \le L < n^{1/2}$. Then, subject to the conditions in Theorem 1.3, one has*

$$\sum_{\substack{l_1, l_2 = 0 \\ l_1 l_2 = 0}}^{L-1} r(n; l_1, l_2) \ll n^2 (\log n)^3 L^{-3}.$$

We postpone the proof of this lemma to §4.6 below; it is a fairly straightforward consequence of well-known techniques.

4.2. Fourier integrals for the envelope. The functions W^* and W_* can be expressed as a linear combination of Υ-functions. By (2.1) and (4.8), one has

$$W^*(\alpha) = \frac{8 + L}{2 + 16\eta} \Upsilon_{\frac{1}{2} + 4\eta}(\alpha) - \frac{4 + L}{2 + 8\eta} \Upsilon_{\frac{1}{2} + 2\eta}(\alpha)$$

whence

$$\widehat{W}^*(\alpha) = \frac{8 + L}{2 + 16\eta} k_{\frac{1}{2} + 4\eta}(\alpha) - \frac{4 + L}{2 + 8\eta} k_{\frac{1}{2} + 2\eta}(\alpha).$$

The upper bound $|\widehat{W}^*(\alpha)| \le \int |W^*(\alpha)| d\alpha \le 1+8\eta$ is trivial, and since $0 < \eta < \frac{1}{2}$ certainly holds, we deduce from (2.1) that

$$\widehat{W}^*(\alpha) \ll \min(1, |\alpha|^{-1}, L|\alpha|^{-2}). \tag{4.13}$$

A similar analysis is possible for $\widehat{W}_*(\alpha)$, and the analogue of (4.13) again holds. Since W^* and W_* are even functions, we also have

$$\widehat{\widehat{W}}{}^* = W^*, \qquad \widehat{\widehat{W}}_* = W_*.$$

We are now in a position to express $r^*(n; l_1, l_2)$ and $r_*(n; l_1, l_2)$ in terms of a Fourier integral. Let

$$N = 2\max(n/\lambda_1, n/\lambda_2, n/\lambda_3)$$

and write, for $i = 1$ or 2,

$$T_i(\alpha, l) = \sum_{p \le N} \psi_l(\lambda_i p) e(\lambda_i \alpha p) \log p.$$

Then, for any $0 \le l_i \le L-1$, one finds from (4.10) and the properties of W^* that

$$r^*(n; l_1, l_2) = \int_{-\infty}^{\infty} T_1(\alpha, l_1) T_2(\alpha, l_2) S(\lambda_3 \alpha) e(-\alpha g) \widehat{W}^*(\alpha) d\alpha, \tag{4.14}$$

and likewise, with r_* and $\widehat{W}_*$ in place of $r^*, \widehat{W}^*$. The Fourier series (3.2) and (4.2) transform T_i into

$$T_i(\alpha, l) = \sum_{h=-\infty}^{\infty} k_\eta(h) e(lh\eta) S\big(\lambda_j(\alpha + h)\big),$$

and therefore,

$$r^*(n; l_1, l_2) = \sum_{h_1, h_2 \in \mathbb{Z}} k_\eta(h_1) k_\eta(h_2) e\big((l_1 h_2 + l_2 h_2)\eta\big) s^*(n; \mathbf{l}, \mathbf{h}) \tag{4.15}$$

where

$$s^*(n; \mathbf{l}, \mathbf{h}) = \int_{-\infty}^{\infty} S\big(\lambda_1(\alpha + h_1)\big) S\big(\lambda_2(\alpha + h_2)\big) S(\lambda_3 \alpha) e(-g\alpha) \widehat{W}^*(\alpha) d\alpha.$$

A similar formula is available for r_*, with $\widehat{W}^*$ replaced by $\widehat{W}_*$. The double series in (4.15) is the final decomposition of the enveloping functions on which the later analysis rests. One expects that only the summand with $\mathbf{h} = (0,0)$ contributes a main term, and this is indeed the case. The next two lemmata make this precise.

Lemma 4.3. *For* $1 \leq l_1, l_2 \leq L-1$ *one has, subject to the hypotheses in Theorem* 1.3,

$$s^*\big(n; \mathbf{l}, (0,0)\big) = \frac{n^2}{2\lambda_1\lambda_2\lambda_3} + O\Big(n^2 e^{-\sqrt{\log n}} + n^2 L^{-1}\Big),$$

and if furthermore the Riemann hypothesis for the Riemann zeta function is true, then

$$s^*\big(n; \mathbf{l}, (0,0)\big) = \frac{n^2}{2\lambda_1\lambda_2\lambda_3} + O\big(n^{2+\varepsilon}\eta + n^{19/10+\varepsilon}\big).$$

Moreover, the same asymptotic formulae are valid for $s_*\big(n; \mathbf{l}, (0,0)\big)$.

All terms with $\mathbf{h} \neq (0,0)$ are treated as error terms. We summarize the result in

Lemma 4.4. *For* $1 \leq l_1 \leq l_2 \leq L-1$, *one has, subject to the hypotheses in Theorem* 1.3,

$$\sum_{0 \neq \mathbf{h} \in \mathbb{Z}^2} k_\eta(h_1)k_\eta(h_2)|s^*(n; \mathbf{l}, \mathbf{h})| \ll n^{2+\varepsilon}\eta + n^{75/38+\varepsilon},$$

and the same bound holds with s_* *in place of* s^*.

We deduce Theorem 1.3 from these lemmata in the next section, and then prove Lemma 4.3 in §4.4 and Lemma 4.4 in §4.5.

4.3. Proof of Theorem 1.3. Assume the Riemann hypothesis, and choose $L = [n^{1/38}]$. Then, by (4.15), Lemmata 4.3 and 4.4,

$$r^*(n; l_1, l_2) = \frac{n^2\eta^2}{2\lambda_1\lambda_2\lambda_3} + O\big(n^{2+\varepsilon}\eta^3\big)$$

for all pairs l_1, l_2 with $1 \leq l_1, l_2 \leq L-1$. By (4.5), Lemma 4.1 and Lemma 4.2, we deduce that

$$\begin{aligned} r_\lambda(n) &\leq L^2 \sum_{l_1, l_2 = 1}^{L-1} r^*(n; l_1, l_2) + O\big(n^{2+\varepsilon}\eta\big) \\ &\leq \frac{n^2}{2\lambda_1\lambda_2\lambda_3} + O\big(n^{2+\varepsilon}\eta\big), \end{aligned}$$

and the opposite inequality

$$r_\lambda(n) \geq \frac{n^2}{2\lambda_1\lambda_2\lambda_3} + O\big(n^{2+\varepsilon}\eta\big)$$

is obtained likewise, using r_* in place of r^*. This establishes the conditional part of Theorem 1.3, and the unconditional part follows in like manner.

4.4. An asymptotic formula. We now discuss Lemma 4.3. A detailed proof is given subject to the Riemann hypothesis. At the end of the section, a guideline for the necessary adjustments in the unconditional case is provided. The argument is essentially standard, so we shall be brief.

Before we embark on the main argument we provide an estimate for frequent use later. From (3.10) and (4.13) we have

$$\int_{-L}^{L} |S(\lambda_j\alpha)|^2|\widehat{W}^*(\alpha)|d\alpha \ll N(\log N)^4.$$

and for any $Y \geq L$

$$\int_{Y}^{\infty} |S(\lambda_j\alpha)|^2|\widehat{W}^*(\alpha)|d\alpha \ll NLY^{-1}(\log N)^2. \tag{4.16}$$

In particular, it follows that

$$\int_{-\infty}^{\infty} |S(\lambda_j\alpha)^2\widehat{W}^*(\alpha)|d\alpha \ll N(\log N)^4. \tag{4.17}$$

Now recall from (4.15) that

$$s^*(n;\mathbf{1},\mathbf{0}) = \int_{-\infty}^{\infty} S(\lambda_1\alpha)S(\lambda_2\alpha)S(\lambda_3\alpha)e(-\alpha g)\widehat{W}^*(\alpha)d\alpha = s^*,$$

say. As usual, we begin by removing the tail $\mathfrak{t} = \{\alpha : |\alpha| > N^{\frac{1}{5}}\}$. We use (4.16) with $j = 1, 2$ and $Y = N^{1/5}$. A trivial estimate for $S(\lambda_3\alpha)$ and Schwarz's inequality then confirm that

$$\int_{\mathfrak{t}} |S(\lambda_1\alpha)S(\lambda_2\alpha)S(\lambda_3\alpha)\widehat{W}^*(\alpha)|d\alpha \ll N^{19/10+\varepsilon}. \tag{4.18}$$

The major and minor arcs are defined by

$$\mathfrak{M} = [-N^{-3/4}, N^{-3/4}]; \quad \mathfrak{m} = \{\alpha : N^{-3/4} < |\alpha| < N^{1/5}\},$$

and we proceed by estimating the contribution of the minor arcs. For $i = 1$ or 2, let

$$\mathfrak{m}_i = \{\alpha \in \mathfrak{m} : |S(\lambda_i\alpha)| \leq N^{\frac{9}{10}+\delta}\}$$

where $0 < \delta < \frac{1}{100}$ is still at our disposal. By (4.17) and an obvious use of Schwarz's inequality we infer that

$$\int_{\mathfrak{m}_i} |S(\lambda_1\alpha)S(\lambda_2\alpha)S(\lambda_3\alpha)\widehat{W}^*(\alpha)|d\alpha \ll N^{\frac{19}{10}+2\delta}. \tag{4.19}$$

It remains to show that the set of all $\alpha \in \mathfrak{m}$ with

$$|S(\lambda_1\alpha)| > N^{\frac{9}{10}+\delta}, \quad |S(\lambda_2\alpha)| > N^{\frac{9}{10}+\delta} \tag{4.20}$$

is empty to conclude from (4.19) that

$$\int_{\mathfrak{m}} |S(\lambda_1\alpha)S(\lambda_2\alpha)S(\lambda_3\alpha)\widehat{W}^*(\alpha)|d\alpha \ll N^{\frac{19}{10}+2\delta}. \tag{4.21}$$

To confirm this claim, note that from (4.20) and Lemma 3.2 one finds that there are coprime integers a_i, q_i $(i = 1, 2)$ with

$$1 \le q_i \le N^{\frac{1}{5}-\delta} \quad , \quad \left|\lambda_i\alpha - \frac{a_i}{q_i}\right| \le q_i^{-1}N^{-\frac{4}{5}-\delta}.$$

As in §3.3, it follows that

$$\left|\lambda_2 a_1 q_2 - \lambda_1 a_2 q_1\right| \le (\lambda_1 + \lambda_2)N^{-\frac{3}{5}-\delta}, \tag{4.22}$$

but that $|a_1| \ll q_1|\alpha| + 1 \ll q_1 N^{1/5}$, whence

$$|a_1 q_2| \ll N^{\frac{3}{5}-2\delta} \quad , \quad |a_2 q_2| \ll N^{\frac{3}{5}-2\delta}. \tag{4.23}$$

By Roth's theorem in diophantine approximation, recalling that λ_1/λ_2 is algebraic and irrational, (4.22) and (4.23) can hold, for N sufficiently large, only if $a_1a_2 = 0$. But then $|\alpha| \ll N^{-\frac{4}{5}-\delta}$, whence $\alpha \in \mathfrak{M}$. Hence the set of all $\alpha \in \mathfrak{m}$ with (4.20) is indeed empty.

The treatment of the major arc is very routine. If the Riemann hypothesis holds, an argument very similar to the proof of Lemma 5 of Vaughan (1974) will show that

$$S(\alpha) = I(\alpha) + O\Big(N^{\frac{1}{2}+\varepsilon}(1 + N|\alpha|)\Big),$$

and one then readily establishes the formulae

$$\begin{aligned}
&\int_{\mathfrak{M}} S(\lambda_1\alpha)S(\lambda_2\alpha)S(\lambda_3\alpha)e(-g\alpha)\widehat{W}^*(\alpha)d\alpha \\
&\quad = \int_{\mathfrak{M}} I(\lambda_1\alpha)I(\lambda_2\alpha)I(\lambda_3\alpha)e(-g\alpha)\widehat{W}^*(\alpha)d\alpha + O\big(N^{\frac{7}{4}+\varepsilon}\big) \\
&\quad = \int_{-\infty}^{\infty} I(\lambda_1\alpha)I(\lambda_2\alpha)I(\lambda_3\alpha)e(-g\alpha)\widehat{W}^*(\alpha)d\alpha + O\big(N^{\frac{7}{4}+\varepsilon}\big) \\
&\quad = \int_{[0,N]^3} W^*\big(\lambda_1\alpha_1 + \lambda_2\alpha_2 + \lambda_3\alpha_3 - g\big)d(\alpha_1, \alpha_2, \alpha_3) + O\big(N^{\frac{7}{4}+\varepsilon}\big)
\end{aligned}$$

By (4.9), we have $n < g < n + 2$. From the definitions of W^* and N it is now immediate that the final integral equals $\frac{n^2}{2\lambda_1\lambda_2\lambda_3} + O(n^2\eta)$, and it follows that

$$\int_{\mathfrak{M}} S(\lambda_1\alpha)S(\lambda_2\alpha)S(\lambda_3\alpha)e(-g\alpha)\widehat{W}^*(\alpha)d\alpha = \frac{n^2}{2\lambda_1\lambda_2\lambda_3} + O(n^2\eta) + O\big(n^{\frac{7}{4}+\varepsilon}\big).$$

and from (4.18) and (4.21) we finally deduce that

$$s^* = \frac{n^2}{2\lambda_1\lambda_2\lambda_3} + O(n^2\eta + n^{19/10+\varepsilon}),$$

because $\delta > 0$ in (4.21) can be made arbitrarily small. If W^* is replaced by W_* throughout the argument, one obtains the same asymptotic formula for $s_*(n; \mathbf{l}, \mathbf{0})$.

The necessary modifications in the unconditional version of Lemma 4.3 are mainly in the choice of major arcs. One now chooses

$$\mathfrak{M} = \{\alpha : |\alpha| < N^{-1}e^{\sqrt{\log N}}\}, \quad \mathfrak{m} = \{\alpha : N^{-1}e^{\sqrt{\log N}} < |\alpha| < N^{\frac{1}{5}}\}$$

and

$$\mathfrak{m}_i = \{\alpha \in \mathfrak{m} : \ |S(\lambda_i\alpha)| \le Ne^{-\sqrt{\log N}}\}.$$

The above argument then remains valid with only modest adjustment of detail which we may leave to the reader.

4.5. Proof of Lemma 4.4. The proof of Lemma 4.4 is not dissimilar to the ideas underlying the argument used to establish Lemma 3.1. However, a reasonably efficient treatment of the double sum over $\mathbf{h}$ in (4.15) results into a tedious discussion of cases.

We begin by removing dependence on $\mathbf{l}$. By (4.15) one has

$$\sum_{\mathbf{h}\ne\mathbf{0}} k_\eta(h_1)k_\eta(h_2)|s^*(n;\mathbf{l},\mathbf{h})| \le \sum_{\mathbf{h}\ne\mathbf{0}} k_\eta(h_1)k_\eta(h_2)\mathcal{S}(\mathbf{h}) \tag{4.24}$$

where

$$\mathcal{S}(\mathbf{h}) = \int_{-\infty}^{\infty} |S(\lambda_1(\alpha+h_1))S(\lambda_2(\alpha+h_2))S(\lambda_3\alpha)\widehat{W}^*(\alpha)|\,d\alpha.$$

We split the sum in (4.24) into $|h_1| \le |h_2|$ and $|h_1| > h_2$, and from now on concentrate on the former case, the latter can be handled likewise by observing symmetry.

By Schwarz's inequality and (4.17), one has $\mathcal{S}(\mathbf{h}) \ll N^2(\log N)^4$ uniformly in $\mathbf{h}$. Recalling the simple bound $k_\eta(h) \ll h^{-2}$ we now see that the contribution to (4.24) of terms with $|h_2| \ge |h_1| \ge L^{5/2}$ does not exceed

$$\ll N^2(\log N)^4 \sum_{h_1 > L^{5/2}} h_1^{-2} \sum_{h_2 \ge h_1} h_2^{-2} \ll N^2(\log N)^4 L^{-5}$$

which is acceptable in view of the claim of the Lemma. We are left with terms satisfying $|h_1| \le L^{5/2}$, and these we split further into the special case $h_1 = 0$ and dyadic ranges $H \le h_1 < 2H$ with $1 \le H \le L^{5/2}$ running over powers of 2.

We also split the integral $\mathcal{S}(\mathbf{h})$ into the center

$$\mathcal{S}_0(\mathbf{h}) = \int_{-L}^{L} |S(\lambda_1(\alpha+h_1))S(\lambda_2(\alpha+h_2))S(\lambda_3\alpha)\widehat{W}^*(\alpha)|\,d\alpha, \tag{4.25}$$

the integral $\mathcal{S}_Y(\mathbf{h})$, defined as in (4.25) but with integration over $Y < |\alpha| \le 2Y$, and the tail $\mathcal{S}_\infty(\mathbf{h})$ where in (4.25) integration is over $|\alpha| > L^4$. By (4.16) we have $\mathcal{S}_\infty(\mathbf{h}) \ll N^2(\log N)^4 L^{-3}$, and since (3.1) shows that

$$\sum_{h\in\mathbb{Z}} k_\eta(h) = \eta, \tag{4.26}$$

we conclude that

$$\sum_{\mathbf{h}\neq\mathbf{0}} k_\eta(h_1)k_\eta(h_2)\mathcal{S}_\infty(\mathbf{h}) \ll N^2(\log N)^4\eta^5$$

which is again acceptable. By a standard dyadic splitting up argument, we now see that (4.24) is bounded by $N^{2+\varepsilon}\eta^5$ plus $O(\log^2 N)$ terms

$$E_1(H,Y) = \sum_{H\le h_1<2H}\ \sum_{h_2>h_1} k_\eta(h_1)k_\eta(h_2)\mathcal{S}_Y(\mathbf{h}), \tag{4.27}$$

$$E_0(Y) = \eta^2 \sum_{h\neq 0} k_\eta(h)\mathcal{S}_Y(0,h), \tag{4.28}$$

where $1 \le H \le L^{5/2}$ and $Y = 0$ or satisfies $L \le Y \le L^4$. We proceed by verifying the bounds

$$E_1(H,Y) \ll N^{2+\varepsilon}\eta^5, \quad E_0(Y) \ll N^{2+\varepsilon}\eta^5 \tag{4.29}$$

for the above ranges of H and Y. It follows that the contribution to (4.24) of terms with $h_1 \le h_2$ does not exceed $O(N^{2+\varepsilon}\eta^5)$, and the aforementioned symmetry argument is then sufficient to complete the proof of Lemma 4.4.

We begin with $E_1(H,Y)$. Let $1 \le B \le N^{1/6}$ be a parameter to be determined later. In the integral defining $\mathcal{S}_Y(\mathbf{h})$ the subset of all α where $|S(\lambda_3\alpha)| \le NB^{-1}$ contributes

$$NB^{-1}\int_{|\alpha|>Y} \big|S\big(\lambda_1(\alpha+h_1)\big)S\big(\lambda_2(\alpha+h_2)\big)\widehat{W}^*(\alpha)\big|\,d\alpha$$

to $\mathcal{S}_Y(\mathbf{h})$ (this is true also if $Y=0$), and by (4.16) and Schwarz's inequality, this does not exceed $NB^{-1}N(\log N)^4LY_0^{-1}$ where

$$Y_0 = Y \quad (Y \ge L), \quad Y_0 = L \quad (Y=0) \tag{4.30}$$

By exchanging the roles of $S(\lambda_3\alpha)$ and $S\big(\lambda_1(\alpha+h_1)\big)$, the same upper bound is valid for the contribution to $\mathcal{S}_Y(\mathbf{h})$ arising from the set of all α where $\big|S\big(\lambda_1(\alpha+h_1)\big)\big| \le NB^{-1}$ holds for all $H \le h_1 < 2H$. We shall choose B such that the set of α with the simultaneous conditions

$$\big|S(\lambda_3\alpha)\big| > NB^{-1}, \quad \big|S\big(\lambda_1(\alpha+h_1)\big)\big| \ge NB^{-1} \tag{4.31}$$

for some $H \le h_1 < 2H$ is empty. It then follows from the above discussion that

$$\mathcal{S}_Y(\mathbf{h}) \ll N^2(\log N)^4 L(BY_0)^{-1}, \tag{4.32}$$

uniformly for all $\mathbf{h}$ with $H \le h_1 < 2H$. The estimation of $E_1(H,Y)$ is now readily completed by verifying the bound

$$\sum_{H \le h_1 < 2H} \sum_{h_2 \ge h_1} k_\eta(h_1)k_\eta(h_2) \ll \min(\eta^2, H^{-2}). \tag{4.33}$$

In fact, if $H \le L$, two application of (4.26) show that the left hand side of (4.33) does not exceed η^2, and for $H > L$ the inequality $k_\eta(h) \ll h^{-2}$ bounds the sum in question by H^{-2}, as required.

By (4.27), (4.32) and (4.33),

$$E_1(H,Y) \ll N^2(\log N)^4 L(BY_0)^{-1} \min(\eta^2, H^{-2}),$$

and (4.29) follows by choosing

$$B = L^6 Y_0^{-1} \min(\eta^2, H^{-2}). \tag{4.34}$$

It remains to verify that $B \le N^{1/6}$, and that no α can satisfy (4.31). Note that $Y_0 \ge L$ whence by (4.34), we have $B \le L^5\eta^2 = L^3 \le N^{1/6}$, as required. Next, suppose that (4.31) holds for some α relevant for $\mathcal{S}_Y(\mathbf{h})$. By Lemma 3.3, we then find integers a_i, q_i with

$$1 \le q_i \ll B^2 \log^8 N,$$
$$\left|\lambda_1(\alpha + h_1) - \frac{a_1}{q_1}\right| \ll \frac{B^2 \log^8 N}{q_1 N},$$
$$\left|\lambda_3\alpha - \frac{a_3}{q_3}\right| \ll \frac{B^2 \log^8 N}{q_3 N}.$$

As in §3.5, the argument leading to (3.17) now shows that

$$\left|\lambda_1^{-1}a_1q_3 - \lambda_3^{-1}a_3q_1 - hq_1q_2\right| \ll B^4N^{-1}(\log N)^{16}, \tag{4.35}$$

and that

$$|h_1q_1q_2| \ll HB^4\log^{16} N,\ |a_3q_1| \ll Y_0B^4\log^{16} N,\ |a_1q_3| \ll (H+Y_0)B^4\log^{16} N. \tag{4.36}$$

If, for some $\delta > 0$, we would have

$$B^4N^{-1} < \left((H+Y_0)B^4\right)^{-2}N^{-\delta}, \tag{4.37}$$

then from (4.35), (4.36) and Lemma 3.2 we could conclude that, in particular, $hq_1q_2 = 0$ which is a contradiction. Thus, it remains to verify (4.37). A simple verication based on (4.34) shows that

$$B^{12}(H+Y_0)^2 \ll L^{72}Y_0^{-12}\min(\eta^2, H^{-2})^{12} \ll L^{38},$$

and (4.37) follows for $L \le n^\theta$ with $\theta < \frac{1}{38}$, as required.

The estimation of $E_0(Y)$ is very similar. By (2.1) and (4.17), one has

$$\sum_{|h|>L^3} k_\eta(h)\mathcal{S}_Y(0,h) \ll N^2(\log N)^4 L^{-3},$$

so that the contribution of terms with $|h| > L^3$ to $E_0(Y)$ is $O(N^{2+\varepsilon}\eta^5)$, which is acceptable. The remaining summands in (4.28) are split into $O(\log N)$ subsums $E_0(H,Y)$ stemming from ranges $H \le h < 2H$ with $1 \le H \le L^3$. The estimation of $E_0(H,Y)$ can be performed exactly as we did with $E_1(H,Y)$, but with $S(\lambda_2(\alpha+h))$ in the role of $S(\lambda_1(\alpha+h_1))$. The factor η^2 in (4.28) makes it possible to choose

$$B = L^5 Y_0 \min(\eta^2, H^{-2})$$

which makes the overall treatment even simpler, and (4.29) follows for $E_0(Y)$ also.

In all arguments we may replace $\widehat{W}^*$ by $\widehat{W}_*$, and then deduce the bound for s_* in place of s^*. This completes the proof of Lemma 4.4.

4.6. Proof of Lemma 4.2. The next Lemma is well-known, and a proof is included only for completeness.

Lemma 4.5. *Let λ be a real algebraic irrational number. Then, for $N^{-1/2} < \eta < 1$ one has*

$$\sum_{m\le N} \Psi(\lambda m) \ll \eta^2 N.$$

Proof. By (3.1),

$$\sum_{m\le N} \Psi(\lambda m) = \sum_{h=-\infty}^{\infty} k_\eta(h) \sum_{m\le N} e(\lambda h m).$$

The term with $h = 0$ contributes $\eta^2[N]$. Trivial estimates and $k_\eta(h) \ll h^{-2}$ show that the tail $|h| > \eta^{-2}$ also contributes $O(\eta^2 N)$. The remaining range $1 \le |h| \le \eta^{-2}$ we split into $|h| \le \eta^{-1}$ and dyadic ranges $H \le h < 2H$ with $\eta^{-1} \le H < \eta^{-2}$. Since $k_\eta(h) \ll \eta^2$ for $|h| \le \eta^{-1}$, we see that this range contributes to the sum in question at most

$$\eta^2 \sum_{h\le\eta^{-1}} \min(N, \|\lambda h\|^{-1}), \tag{4.38}$$

where $\|\beta\| = \min_{m\in\mathbb{Z}} |\beta - m|$. For any coprime rational numbers a, q with $|\lambda - \frac{a}{q}| \leq q^{-2}$, Lemma 2.1 of Vaughan (1997) bounds (4.38) by

$$\eta^2\Big(\frac{N\eta^{-1}}{q} + N + q\log q\Big),$$

and by Roth's theorem on diophantine approximation, one can always find a q with the required property, in the range $N^{5/8} \ll q \ll N^{3/4}$, so that (4.38) is indeed $O(\eta^2 N)$. For $\eta^{-1} \leq H \leq \eta^{-2}$, an interval $H \leq |h| < 2H$ will contribute to the sum in question (on using $k_\eta(h) \ll h^{-2} \ll H^{-2}$) at most

$$H^{-2} \sum_{H\leq h<2H} \min(N, \|\lambda h\|^{-1}) \ll H^{-2}\Big(\frac{NH}{q} + N + q\log q\Big).$$

This does not exceed $\eta^2 N^{7/8}$, and the Lemma follows.

To prove Lemma 4.2, we first observe that by (4.3), (4.6) and (4.7), one finds that

$$\sum_{l_2=0}^{L-1} r(n; 0, l_2) = \eta \sum_{\substack{\mathbf{p} \\ (4.4)}} \Psi_\eta(\lambda_1 p_1) \log \mathbf{p}. \tag{4.39}$$

Note that $\log \mathbf{p} \ll \log^3 N$. For any fixed prime $p_1 \leq n/\lambda_1 < N$, the number of solutions (4.4) in primes p_2, p_3 does not exceed $O(n - \lambda_1 p_1) = O(n)$. Hence the sum in (4.39) does not exceed

$$\eta(\log n)^3 n \sum_{p_1\leq n/\lambda_1} \Psi_\eta(\lambda_1 p_1) \ll \eta(\log n)^3 n \sum_{m\leq N} \Psi_\eta(\lambda_1 m),$$

and Lemma 4.5 yields

$$\sum_{l_2=0}^{L-1} r(n; 0, l_2) \ll n^2\eta^3(\log n)^3.$$

The same argument also yields

$$\sum_{l_1=0}^{L-1} r(n; l_1, 0) \ll n^2\eta^3(\log n)^3,$$

and Lemma 4.2 follows.

References

[1] G. I. Arkhipov, K. Buriev and V. N. Chudarikov (1999). *On the exceptional set in the generalized binary Goldbach conjecture* (Russian). Dokl. Akad. Nauk 365, 151–153; English transl. Dokl. Math. 59 (1999) 193–195.

[2] J. Brüdern, R. J. Cook and A. Perelli (1997). *The values of binary linear forms at prime arguments.* In: *Sieve methods, exponential sums and their applications in number theory*, G. Greaves, G. Harman and M. N. Huxley eds. Cambridge, University Press.

[3] J. Brüdern and A. Perelli (1998). *Goldbach numbers in sparse sequences.* Ann. Inst. Fourier Grenoble, 48, 353–378.

[4] X. Cao and W. Zhai (1999). *On an equation with prime numbers.* To appear.

[5] N. G. Chudakov (1938). *On the density of the set of even numbers which are not representable as the sum of two odd primes.* Izv. Akad. Nauk. SSSR Ser. Mat. 2, 25–40.

[6] J. G. van der Corput (1938). *Sur l'hypothèse de Goldbach.* Proc. Akad. Wet. Amsterdam 41, 76–80.

[7] H. Davenport and H. Heilbronn (1937). *Note on a result in the additive theory of numbers.* Proc. London Math. Soc. (2) 43, 142–151.

[8] H. Davenport and H. Heilbronn (1946). *On indefinite quadratic forms in five variables.* J. London Math. Soc. 21, 185–193.

[9] T. Estermann (1938). *On Goldbach's problem: Proof that almost all even positive integers are sums of two primes.* Proc. London Math. Soc. (2) 44, 307–314.

[10] J. B. Friedlander and D. A. Goldston (1997). *Sums of three or more primes.* Trans. American Math. Soc. 349, 287–310.

[11] P. X. Gallagher (1975). *Primes and powers of 2.* Invent. Math. 29, 125–142.

[12] A. Kumchev and T. Nedeva (1998). *On an equation with prime numbers.* Acta Arith. 83, 117–126.

[13] H. Li (1999). *The exceptional set of Goldbach numbers.* Quart. J. Math. Oxford 50 (1999), 471–482.

[14] H. L. Montgomery and R. C. Vaughan (1973). *Error terms in additive prime number theory.* Quart. J. Math. Oxford (2) 24, 207–216.

[15] H. L. Montgomery and R. C. Vaughan (1975). *The exceptional set in Goldbach's problem.* Acta Arith. 27, 353–370.

[16] N. Romanov (1934). *Über einige Sätze der additiven Zahlentheorie.* Math. Ann. 109, 668–678.

[17] W. M. Schmidt (1980). *Diophantine approximation.* Springer, Berlin.

[18] R. C. Vaughan (1974). *Diophantine approximation by prime numbers*, I. Proc. London Math. Soc. (3) 28, 373–384.

[19] R. C. Vaughan (1997). *The Hardy-Littlewood method.* Cambridge, University Press (2nd edn).

[20] I. M. Vinogradov (1937). *Representation of an odd number as a sum of three primes.* C. R. Acad. Sci. URSS 15, 6–7.

[21] A. Zaccagnini (1992). *On the exceptional set for the sum of a prime and a kth power.* Mathematika 39, 400–421.

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