

# On Action Logic

Wojciech Buszkowski  
Adam Mickiewicz University, Poznań, Poland  
University of Warmia and Mazury, Olsztyn, Poland

## Abstract

Pratt [22] defines action algebras as Kleene algebras with residuals and action logic as the equational theory of action algebras. In opposition to Kleene algebras, action algebras form a (finitely based) variety. Jipsen [9] proposes a Gentzen-style sequent system for action logic but leaves it as an open question if this system admits cut-elimination and if action logic is decidable. We show that Jipsen's system does not admit cut-elimination. We prove that the equational theory of \*-continuous action algebras and the simple Horn theory of \*-continuous Kleene algebras are not recursively enumerable and they possess FMP, but action logic does not possess FMP.

*Keywords:* Kleene algebra, action algebra, action lattice, undecidability, finite model property

## 1 Introduction

Kleene algebras have been introduced by Kozen [11, 12] in order to axiomatize the equalities true for regular expressions. The Kozen completeness theorem states that  $\alpha = \beta$  is true for regular expressions iff this equality is true in all Kleene algebras. Kleene algebras form a quasi-variety but not a variety. It follows from the Kozen completeness theorem that the equational theory of Kleene algebras is decidable (PSPACE-complete).

Pratt [22] introduces action algebras as Kleene algebras with residuals (with respect to product) and shows that action algebras form a finitely based variety. In the language of Kleene algebras, the equalities true in all action algebras are the same as for Kleene algebras. Consequently, in the language of Kleene algebras enriched with residuals one succeeds in an elegant, finite and purely equational axiomatization of the algebra of regular expressions. On the other hand, many basic problems concerning action algebras and their logic are still open. For instance, it is not known if the equational theory of action algebras (action logic, ACT) is decidable.

Jipsen [9] proposes a Gentzen-style sequent system for action logic which arises from Full Lambek Calculus (FL) in the sense of Ono [19] by adding certain rules for \*. Actually, these rules closely follow Kozen's algebraic axioms

for  $*$ . The cut-elimination theorem and the decidability of this system are left open.

We show that the cut-elimination theorem does not hold for Jipsen's system. We consider logic  $\text{ACT}\omega$ , i.e. the equational theory  $*$ -continuous action algebras (roughly: action algebras with a standard  $*$ -operation). While in the case of Kleene algebras the (equational) logics of all algebras and of all  $*$ -continuous algebras are equal, we show that in the case of action algebras the situation is different.  $\text{ACT}\omega$  is not recursively enumerable (it is  $\Pi_1^0$ -complete), while ACT is  $\Sigma_1^0$ . Consequently, ACT is not complete with respect to  $*$ -continuous action algebras. In particular, it is not complete with respect to finite action algebras, which means that it does not possess Finite Model Property (FMP). Palka [20] shows that the equational theory of Kleene algebras possesses FMP, and this fact is equivalent to the Kozen completeness theorem.

It can be shown that the Horn theory of Kleene algebras is undecidable. By a simple atomic formula we mean a formula  $x_1 \cdot \dots \cdot x_n \leq x$  such that  $x_i, x$  are individual variables. By a simple Horn formula we mean a Horn formula  $\varphi_1 \& \dots \& \varphi_n \Rightarrow \varphi$  such that  $\varphi_i$  are simple atomic formulas and  $\varphi$  is an arbitrary atomic formula. As a consequence of the above results, we show that the simple Horn theory (i.e. the logic of simple Horn formulas) of  $*$ -continuous Kleene algebras is undecidable ( $\Pi_1^0$ -complete); for  $*$ -free fragments of action logic, such theories are decidable. This result is related to (but it is not a consequence of) those of Kozen [14] who proves the  $\Pi_1^0$ -completeness of a restricted Horn theory of  $*$ -continuous Kleene algebras with Horn formulas whose assumptions  $\varphi_i$  are equalities  $x \cdot y = y \cdot x$ , the  $\Pi_2^0$ -completeness of an analogous theory in which the assumptions are arbitrary monoid equations, and the  $\Pi_1^1$ -completeness of the general Horn theory of  $*$ -continuous Kleene algebras.

At the end, we show that  $\text{ACT}\omega$  possesses FMP. The proof is based on methods elaborated by Okada [17], Okada and Terui [18], and Belardinelli, Jipsen and Ono [2], providing an algebraic analysis of cut-elimination and FMP for substructural logics, and a recent result of Palka [21] on the elimination of negative occurrences of  $*$  in  $\text{ACT}\omega$ .

## 2 Preliminaries

A *Kleene algebra* is an algebra  $\mathcal{A} = (A, \vee, \cdot, *, 0, 1)$  such that  $(A, \vee, 0)$  is a join semilattice with the least element 0,  $(A, \cdot, 1)$  is a monoid with the identity element 1, the product  $\cdot$  distributes over  $\vee$ , 0 is an annihilator for product, and  $*$  is a unary operation, satisfying the conditions:

$$(K1) \quad 1 \vee aa^* \leq a^*, \quad 1 \vee a^*a \leq a^*,$$

$$(K2) \quad ab \leq b \Rightarrow a^*b \leq b, \quad ba \leq b \Rightarrow ba^* \leq b,$$

for all  $a, b \in A$ . Here  $\leq$  is a standard semilattice ordering:  $a \leq b$  iff  $a \vee b = b$ .

The above notion is due to Kozen [11, 12]. Kleene algebras can be axiomatized by the following axioms: semilattice axioms  $(a \vee b) \vee c = a \vee (b \vee c)$ ,

$a \vee b = b \vee a$ ,  $a \vee a = a$ ,  $a \vee 0 = a$ , monoid axioms  $(ab)c = a(bc)$ ,  $a1 = a = 1a$ , distribution axioms  $a(b \vee c) = ab \vee ac$ ,  $(a \vee b)c = ac \vee bc$ , annihilator axioms  $a0 = 0 = 0a$ , and (K1), (K2). (K2) are quasi-equalities, and the remaining axioms are equalities. A well-known construction of Conway's Leap (see [22]) shows that Kleene algebras are not closed under homomorphic images, and consequently, they do not form a variety.

Let  $\mathcal{A}$  be a Kleene algebra. An element  $a \in A$  is said to be *reflexive*, if  $1 \leq a$ , and *transitive*, if  $aa \leq a$ . In any Kleene algebra,  $a^*$  is the least reflexive and transitive element  $b$  such that  $a \leq b$ , which is equivalent to the following conditions:

$$(C1) \quad 1 \vee a \vee a^*a^* \leq a^*,$$

$$(C2) \quad 1 \vee a \vee bb \leq b \Rightarrow a^* \leq b.$$

A standard Kleene algebra is *the algebra of languages* on a finite alphabet  $\Sigma$ . A *language* on  $\Sigma$  is a set of finite strings on  $\Sigma$ . The largest language on  $\Sigma$  is denoted  $\Sigma^*$ . The empty string is denoted  $\lambda$ .  $\Sigma^+ = \Sigma^* - \{\lambda\}$ . For  $L, L_1, L_2 \subseteq \Sigma^*$ , one sets:  $L_1 \vee L_2 = L_1 \cup L_2$ ,  $L_1 \cdot L_2 = \{xy : x \in L_1 \& y \in L_2\}$ ,  $L^* = \bigcup_{n \in \omega} L^n$ ,  $0 = \emptyset$ ,  $1 = \{\lambda\}$ , where  $L^0 = 1$ ,  $L^{n+1} = L^n L$ . Another example is the algebra of all binary relations on a set  $U$ ; now, product is relational product,  $1$  is the identity relation, and the remaining notions are defined as above.

A Kleene algebra  $\mathcal{A}$  is said to be *\*-continuous*, if  $xa^*y = \text{l.u.b.}\{xa^ny : n \in \omega\}$ , for all  $x, y, a \in A$ . Clearly, the algebra of languages and the algebra of relations are *\*-continuous*. They are also *complete*, which means that every set of elements has l.u.b.

*Regular expressions* on  $\Sigma$  are variable-free terms of the (first-order) language of Kleene algebras enriched with all symbols from  $\Sigma$  as new individual constants. For  $a \in \Sigma$ , one sets  $L(a) = \{a\}$  and extends the mapping  $L$  to a (unique) homomorphism from the (variable-free) term algebra to the algebra of languages on  $\Sigma$ . For a regular expression  $\alpha$ , the language  $L(\alpha)$  is called *the language denoted by  $\alpha$* . The equality  $\alpha = \beta$  is said to be *true for regular expressions*, if  $L(\alpha) = L(\beta)$ . The Kozen completeness theorem states:  $\alpha = \beta$  is true for regular expressions iff it is true in all Kleene algebras. The 'if'-direction is obvious, while the 'only if'-direction is a deep result [11, 12], proved by means of some matrix representation of finite-state automata. It would be interesting to find a model-theoretic proof of this theorem.

A *residuated (join) semilattice* is an algebra  $\mathcal{A} = (A, \vee, \cdot, \rightarrow, \leftarrow, 1)$  such that  $(A, \cdot, 1)$  is a monoid,  $(A, \vee)$  is a join semilattice, and  $\rightarrow, \leftarrow$  are binary operations on  $A$ , satisfying the equivalences:

$$(RES) \quad ab \leq c \Leftrightarrow b \leq a \rightarrow c \Leftrightarrow a \leq c \leftarrow b,$$

for all  $a, b, c \in A$ . A more general notion is *a residuated monoid*:  $\vee$  is replaced by a partial ordering  $\leq$ . The operations  $\rightarrow$  and  $\leftarrow$  are called *the right* and *left*, respectively, *residuals* for product. An *action algebra* is an algebra which is both a Kleene algebra and a residuated semilattice. It is easy to show that, for action algebras, the distribution axioms and the annihilator axioms can be

omitted. Also, (K1), (K2) can be replaced with (C1), (C2) [22], which is not true for Kleene algebras ((C1), (C2) hold in right-handed Kleene algebras, and not every right-handed Kleene algebra is a Kleene algebra [10]). An action algebra is \*-continuous (as a Kleene algebra) iff  $a^* = \text{l.u.b.}\{a^n : n \in \omega\}$  (use the fact that, in action algebras, product distributes over infinite joins). Pratt [22] proves that action algebras form a finitely based variety.

An *action lattice* (resp. a *Kleene lattice*) is an action algebra (resp. a Kleene algebra) which is a lattice, this means, it admits a *meet* operation  $\wedge$ , satisfying the semilattice axioms except for the last one (with  $\vee$  replaced with  $\wedge$ ) and the absorption axioms  $a \vee (a \wedge b) = a$ ,  $a \wedge (a \vee b) = a$ . Similarly, residuated lattices are defined as residuated semilattices which are lattices with the meet operation  $\wedge$ . Some results on action lattices can be found in Kozen [13].

The algebra of languages can be expanded to an action lattice by setting  $L_1 \wedge L_2 = L_1 \cap L_2$  and:

$$L_1 \rightarrow L_2 = \{x \in \Sigma^* : L_1\{x\} \subseteq L_2\}, \quad L_1 \leftarrow L_2 = \{x \in \Sigma^* : \{x\}L_2 \subseteq L_1\}.$$

Regular languages on  $\Sigma$ , i.e. languages denoted by regular expressions on  $\Sigma$ , form a subalgebra of this action lattice. The algebra of relations on  $U$  can also be expanded to an action lattice with the meet defined as the set-theoretic intersection and residuals defined as follows:

$$R_1 \rightarrow R_2 = \{(x, y) \in U^2 : R_1 \circ \{(x, y)\} \subseteq R_2\},$$

$$R_1 \leftarrow R_2 = \{(x, y) \in U^2 : \{(x, y)\} \circ R_2 \subseteq R_1\}.$$

Residuated lattices have extensively been studied as models of non-classical logics, especially substructural logics, i.e. logics whose sequent systems avoid some structural rules like Contraction, Weakening, Exchange; see e.g. the books [24, 23]. Ono [19] studies a Gentzen-style sequent system FL (Full Lambek Calculus) which is complete with respect to residuated lattices. Atomic formulas of FL are variables and constants 0 and 1. Formulas of FL are formed out of atomic formulas by means of the connectives  $\cdot, \rightarrow, \leftarrow, \vee, \wedge$ . We use characters  $p, q, r$  for atomic formulas and  $\varphi, \psi, \chi$  for formulas. Greek capitals represent finite strings of formulas. Sequents are expressions of the form  $\Gamma \vdash \varphi$ . The axioms of FL are:

$$(\text{Id}) \varphi \vdash \varphi, \quad (0\text{L}) \Gamma, 0, \Delta \vdash \varphi, \quad (1\text{R}) \vdash 1,$$

and the inference rules are the following:

$$(\cdot\text{L}) \frac{\Gamma, \varphi, \psi, \Delta \vdash \chi}{\Gamma, \varphi \cdot \psi, \Delta \vdash \chi}, \quad (\cdot\text{R}) \frac{\Gamma \vdash \varphi; \Delta \vdash \psi}{\Gamma, \Delta \vdash \varphi \cdot \psi},$$

$$(\rightarrow\text{L}) \frac{\Gamma, \psi, \Delta \vdash \chi; \Phi \vdash \varphi}{\Gamma, \Phi, \varphi \rightarrow \psi, \Delta \vdash \chi}, \quad (\rightarrow\text{R}) \frac{\varphi, \Gamma \vdash \psi}{\Gamma \vdash \varphi \rightarrow \psi},$$

$$(\leftarrow\text{L}) \frac{\Gamma, \psi, \Delta \vdash \chi; \Phi \vdash \varphi}{\Gamma, \psi \leftarrow \varphi, \Phi, \Delta \vdash \chi}, \quad (\leftarrow\text{R}) \frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \psi \leftarrow \varphi},$$

$$\begin{aligned}
& (\vee L) \frac{\Gamma, \varphi, \Delta \vdash \chi; \Gamma, \psi, \Delta \vdash \chi}{\Gamma, \varphi \vee \psi, \Delta \vdash \chi}, \quad (1L) \frac{\Gamma, \Delta \vdash \varphi}{\Gamma, 1, \Delta \vdash \varphi}, \\
& (\vee R_1) \frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi}, \quad (\vee R_2) \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi}, \\
& (\wedge L_1) \frac{\Gamma, \varphi, \Delta \vdash \chi}{\Gamma, \varphi \wedge \psi, \Delta \vdash \chi}, \quad (\wedge L_2) \frac{\Gamma, \psi, \Delta \vdash \chi}{\Gamma, \varphi \wedge \psi, \Delta \vdash \chi}, \\
& (\wedge R) \frac{\Gamma \vdash \varphi; \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi}.
\end{aligned}$$

FL admits cut-elimination, this means, the set of provable sequents is closed under the rule:

$$(\text{CUT}) \frac{\Gamma, \varphi, \Delta \vdash \psi; \Phi \vdash \varphi}{\Gamma, \Phi, \Delta \vdash \psi}.$$

This fact is well-known. The first cut-elimination theorem for systems of that kind was proved by Lambek [15] for the fragment without  $\wedge, \vee, 0, 1$ , not admitting empty antecedents of sequents; this system is usually referred to as the Lambek calculus (L). FL is not conservative over L, but FL is conservative over L' which is the  $(\cdot, \rightarrow, \leftarrow)$ -fragment of FL. Different variants of the Lambek calculus, their models and applications in formal grammars have been studied in e.g. [3, 4, 5, 7, 6, 25, 26, 16].

To formalize the logic of action lattices we add to this language the new connective  $*$ . Let ACT denote the system which admits all axioms and rules of FL (in the extended language) with (CUT) and the following ones:

$$\begin{aligned}
& (*1) \vdash \varphi^*, \quad (*2) \varphi, \varphi^* \vdash \varphi^*, \quad (*3) \varphi^*, \varphi \vdash \varphi^*, \\
& (\text{K2l}) \frac{\varphi, \psi \vdash \psi}{\varphi^*, \psi \vdash \psi}, \quad (\text{K2r}) \frac{\varphi, \psi \vdash \varphi}{\varphi, \psi^* \vdash \varphi}.
\end{aligned}$$

Let  $\mathcal{A}$  be an action lattice. Homomorphisms from the free algebra of formulas to  $\mathcal{A}$  are called *assignments* in  $\mathcal{A}$ . Assignments are extended to strings of formulas by setting:

$$f(\lambda) = 1, \quad f(\varphi_1, \dots, \varphi_n) = f(\varphi_1) \cdot \dots \cdot f(\varphi_n).$$

A sequent  $\Gamma \vdash \varphi$  is said to be *true* in a model  $(\mathcal{A}, f)$  if  $f(\Gamma) \leq f(\varphi)$ .

It is easy to show that ACT is complete with respect to the class of action lattices, this means, the sequents provable in ACT are precisely those sequents which are true in all models  $(\mathcal{A}, f)$  such that  $\mathcal{A}$  is an action lattice. Soundness is straightforward, since all axioms of ACT are true and all rules preserve the truth. Completeness can be shown by a standard construction of a Lindenbaum algebra. In a similar way, one shows that ACT without  $\wedge$  is complete with respect to the class of action algebras, FL is complete with respect to the class of residuated lattices (with 0), and so on.

By  $\text{ACT}\omega$  we denote the set of all sequents true in all models  $(\mathcal{A}, f)$  such that  $\mathcal{A}$  is a  $*$ -continuous action lattice. Clearly,  $\text{ACT} \subseteq \text{ACT}\omega$  (we identify the

logic ACT with its set of provable sequents).  $\text{ACT}\omega$  is closed under the following rules:

$$\begin{aligned}
(*\text{R}) \quad & \frac{\Gamma_1 \vdash \varphi; \dots; \Gamma_n \vdash \varphi}{\Gamma_1, \dots, \Gamma_n \vdash \varphi^*}, \text{ for any } n \geq 1, \\
(*\text{L}) \quad & \frac{(\Gamma, \varphi^n, \Delta \vdash \psi)_{n \in \omega}}{\Gamma, \varphi^*, \Delta \vdash \psi}.
\end{aligned}$$

$(*\text{R})$  is an infinite family of finitary rules.  $(*\text{L})$  is an infinitary rule (an  $\omega$ -rule); here  $\varphi^n$  stands for the string of  $n$  copies of  $\varphi$ ,  $\varphi^0$  is the empty string.  $(*1)$  can be treated as an additional instance of  $(*\text{R})$  for  $n = 0$ .

$(*\text{R})$  is derivable in ACT.  $(*\text{L})$  (together with  $(*\text{R})$ ) expresses \*-continuity. Palka [21] shows that  $\text{ACT}\omega$  can be axiomatized as the system FL enriched with  $(*1)$ ,  $(*\text{R})$  and  $(*\text{L})$  (without (CUT)). With (CUT), the completeness theorem can be proved in a straightforward way (see above). The cut-elimination theorem can be proved by a standard, syntactic argument. The set of provable sequents is the least fixpoint of a monotone operator  $C$  on the powerset of the set of sequents, defined as follows:  $C(X)$  is the set of all sequents which are conclusions of inference rules of the system with all premises belonging to  $X$  (axioms are treated as rules with the empty set of premises). One defines the hierarchy  $C^\alpha$ , for ordinals  $\alpha$ :  $C^0 = \emptyset$ ,  $C^{\alpha+1} = C(C^\alpha)$ ,  $C^\lambda = \bigcup_{\alpha < \lambda} C^\alpha$ , for limit ordinals  $\lambda$ . Then, the rank of a provable sequent equals the least  $\alpha$  such that this sequent belongs to  $C^\alpha$ . Palka proves that the rule (CUT) is admissible in the system by a triple induction: (1) on the complexity of  $\varphi$ , (2) on the rank of  $\Gamma, \varphi, \Delta \vdash \psi$ , (3) on the rank of  $\Phi \vdash \varphi$ . (The complexity of a formula is the total number of occurrences of logical connectives in this formula.) Then, we can identify  $\text{ACT}\omega$  with the system FL plus  $(*\text{L})$ ,  $(*\text{R})$ .

As a consequence,  $\text{ACT}\omega$  is a conservative supersystem of all its fragments, defined by a restriction of the language; in particular, it is a conservative supersystem of FL, and the same holds for systems without  $\wedge$ . Actually, these facts can also be shown by an algebraic argument, using the construction of a MacNeille completion [2]. We recall this notion in section 4.

$\text{ACT}\omega$  without  $\wedge$  and  $\rightarrow, \leftarrow$  is the logic of \*-continuous Kleene algebras which equals the logic of all Kleene algebras, and consequently, it is decidable. In the next section, we prove that  $\text{ACT}\omega$  is undecidable ( $\Pi_1^0$ -complete). Actually, this already holds for its fragments, restricted to  $(\rightarrow, \vee, *)$  and  $(\rightarrow, \wedge, *)$  ( $\rightarrow$  can be replaced with  $\leftarrow$ ).

### 3 $\text{ACT}\omega$ is not recursively enumerable

We show that the following undecidable problem:

$$(\text{TOTAL}) \quad L(G) = (\Sigma_G)^*, \text{ for context-free grammars } G,$$

is reducible to  $\text{ACT}\omega$ .

A *context-free grammar* (CF-grammar) is a quadruple  $G = (\Sigma, N, S, P)$  such that  $\Sigma, N$  are finite alphabets,  $\Sigma \neq \emptyset$ ,  $\Sigma \cap N = \emptyset$ ,  $S \in N$ , and  $P \subseteq \Sigma \times (\Sigma \cup N)^*$  is

a finite relation. Elements of  $\Sigma$  (resp.  $N$ ) are called *terminal* (resp. *nonterminal*) symbols of  $G$ .  $S$  is the *initial* symbol of  $G$ .  $P$  is the set of *production rules* of  $G$ . The terminal alphabet of  $G$  will also be denoted by  $\Sigma_G$ .

Let  $G$  be a CF-grammar, and  $x, y \in (\Sigma \cup N)^*$ . We say that  $y$  is *directly derivable* from  $x$  in  $G$  if there exist  $(A, z) \in P$  and  $u, v \in (\Sigma \cup N)^*$  such that  $x = uAv$  and  $y = uzv$ . We say that  $y$  is *derivable* from  $x$  in  $G$  and write  $x \Rightarrow_G^* y$  if there exist  $x_0, \dots, x_n$ , ( $n \geq 0$ ), such that  $x = x_0$ ,  $y = x_n$ , and  $x_i$  is directly derivable from  $x_{i-1}$ , for all  $i = 1, \dots, n$ . The language generated by  $G$  is defined as the set:

$$L(G) = \{x \in \Sigma^+ : S \Rightarrow_G^* x\}.$$

The undecidability of (TOTAL) is a well-known undecidability result for CF-grammars (see [8]). Actually, this problem is  $\Pi_1^0$ -complete. We reduce this problem to an analogous problem for categorial grammars, i.e. formal grammars based on some logics of types. Since categorial grammars do not regard the empty string, then we need a  $\lambda$ -free version of (TOTAL).

A CF-grammar is said to be  $\lambda$ -free if it does not contain production rules of the form  $(A, \lambda)$  (nullary rules). A well-known algorithm of elimination of nullary rules transforms any CF-grammar  $G$  into a  $\lambda$ -free CF-grammar  $H$  such that  $\Sigma_H = \Sigma_G$  and  $L(H) = L(G) - \{\lambda\}$ . Another algorithm decides if  $\lambda \in L(G)$ , for CF-grammars  $G$ . We formulate a variant of (TOTAL):

$$(\text{TOTAL}^+) L(G) = (\Sigma_G)^+, \text{ for } \lambda\text{-free CF-grammars } G.$$

It is easy to show that (TOTAL<sup>+</sup>) is undecidable. Suppose that an algorithm for (TOTAL<sup>+</sup>) exists. We can use it to provide an algorithm for (TOTAL). Let  $G$  be a CF-grammar. We ask if  $L(G) = (\Sigma_G)^*$ . First, we check if  $\lambda \in L(G)$ . If not, then the answer is NO. If so, then we construct a grammar  $H$ , as above, and check if  $L(H) = (\Sigma_H)^+$ . If so, then the answer is YES; if not, the answer is NO. It is also easy to show that (TOTAL<sup>+</sup>) is  $\Pi_1^0$ -complete. Obviously, it is  $\Pi_1^0$ . It is  $\Pi_1^0$ -hard, by the following reduction. Fix a  $\lambda$ -free CF-grammar  $K$  such that  $L(K) = \emptyset$ , for example,  $K$  with the empty set of production rules. We define a computable mapping  $F$  from the set of CF-grammars to the set of  $\lambda$ -free CF-grammars. If  $\lambda \in L(G)$ , then  $F(G)$  is the grammar  $H$  for  $G$ ; otherwise,  $F(G) = K$ . Clearly,  $G$  satisfies (TOTAL) iff  $F(G)$  satisfies (TOTAL<sup>+</sup>).

Let  $\mathcal{L}$  be a logic in the language of ACT or its fragment. An  $\mathcal{L}$ -grammar on  $\Sigma$  (an alphabet) is defined as a finite relation  $G \subset \Sigma \times \text{FOR}_{\mathcal{L}}$ , where  $\text{FOR}_{\mathcal{L}}$  stands for the set of formulas of  $\mathcal{L}$ . The domain of  $G$  is called *the alphabet* of  $G$  and denoted  $\Sigma_G$ . Formulas are also called *types*. The codomain of  $G$  is called *the type lexicon* of  $G$ . Let  $v_1, \dots, v_n \in \Sigma_G$ . We say that  $G$  *assigns* a type  $\varphi$  to the string  $v_1 \dots v_n$  if there exist types  $\varphi_1, \dots, \varphi_n$  such that  $(v_i, \varphi_i) \in G$ , for all  $i = 1, \dots, n$ , and the sequent  $\varphi_1, \dots, \varphi_n \vdash \varphi$  is provable in  $\mathcal{L}$ . We fix an atomic formula  $s$  (a designated variable or a new constant). For any type  $\varphi$ , one defines the set:

$$L(G, \varphi) = \{x \in (\Sigma_G)^+ : G \text{ assigns type } \varphi \text{ to } x\}.$$

The language  $L(G, s)$  is called *the language* of  $G$ .

*Categorial grammars* are  $\mathcal{L}$ -grammars, for different logics  $\mathcal{L}$ . The simplest ones are AB-grammars (from the names of Ajdukiewicz and Bar-Hillel who formulated basic ideas). The logic AB is restricted to types formed out of variables by means of  $\rightarrow$  and  $\leftarrow$  only. In the linguistic literature, one often writes  $\varphi \backslash \psi$  for  $\varphi \rightarrow \psi$  and  $\psi / \varphi$  for  $\psi \leftarrow \varphi$ , but we will not use the latter notation. AB is usually presented as a rewriting system, based on Ajdukiewicz reduction rules:

$$(AJD) \varphi, (\varphi \rightarrow \psi) \Rightarrow \psi; (\psi \leftarrow \varphi), \varphi \Rightarrow \psi.$$

A sequent  $\Gamma \vdash \varphi$  is provable in AB iff  $\Gamma$  is reducible to  $\varphi$  by a finite number of applications of rules (AJD). It is well-known (see e.g. [3]) that AB is equivalent to the subsystem of FL, admitting  $(\rightarrow, \leftarrow)$ -formulas, axioms (Id) and rules  $(\rightarrow L)$ ,  $(\leftarrow L)$  only (no sequent with the empty antecedent can be proved in this system).

It is also well-known that languages of AB-grammars are precisely the languages generated by  $\lambda$ -free CF-grammars, i.e. the  $\lambda$ -free context-free languages. We will need the exact formulation of the nontrivial part of this theorem [1].

The Gaifman theorem: For any  $\lambda$ -free CF-grammar  $G$ , one constructs an AB-grammar  $H$  such that  $L(G) = L(H, s)$ , and all types in the type lexicon of  $H$  are of the form  $p, p \leftarrow q, (p \leftarrow q) \leftarrow r$ .

The Gaifman theorem easily follows for the Greibach Normal Form theorem for CF-grammars. By the latter theorem, for any  $\lambda$ -free CF-grammar  $G$ , one can construct a CF-grammar  $G'$  in the Greibach Normal Form with the same terminal alphabet and such that  $L(G) = L(G')$ . All production rules of  $G'$  are of the form  $(A, v)$ ,  $(A, vB)$  or  $(A, vBC)$ , where  $v$  is a terminal symbol, and  $A, B, C$  are nonterminal symbols. One constructs an AB-grammar  $H$  in the following way. Nonterminal symbols of  $G'$  are identified with different variables. For any  $v \in \Sigma_G$ , one puts: (1)  $(v, A) \in H$  iff  $(A, v)$  is a production of  $G'$ , (2)  $(v, A \leftarrow B) \in H$  iff  $(A, vB)$  is a production of  $G'$ , (3)  $(v, (A \leftarrow C) \leftarrow B) \in H$  iff  $(A, vBC)$  is a production in  $G'$ . By induction on  $n \geq 1$ , one proves:

$$A \Rightarrow_{G'}^* v_1 \dots v_n \text{ iff } v_1 \dots v_n \in L(H, A).$$

One identifies  $s$  with the initial symbol of  $G'$ . Consequently,  $L(H, s) = L(G') = L(G)$ .

The Greibach Normal Form theorem immediately follows from the Gaifman theorem, and the latter has been proved earlier (in 1960), by a direct transformation of CF-grammars in the Chomsky Normal Form to equivalent AB-grammars. In the Gaifman theorem, the types in the type lexicon of  $H$  can also be taken in the form  $p, p \rightarrow q, p \rightarrow (q \rightarrow r)$  (use a dual form of CF-grammars in the Greibach Normal Form).

We need some syntactic properties of FL; some of them are concerned with the  $(\rightarrow, \leftarrow)$ -fragment, so they are, in fact, properties of the Lambek calculus  $L'$ .



The order of a  $(\rightarrow, \leftarrow)$ -formula  $\varphi$  is recursively defined, as follows:

$$o(p) = 0, \quad o(\varphi \rightarrow \psi) = o(\psi \leftarrow \varphi) = \max(o(\psi), o(\varphi) + 1).$$

Then, all types mentioned in the Gaifman theorem are of order not greater than 1.

We write  $\Gamma \vdash_{\mathcal{L}} \varphi$  if  $\Gamma \vdash \varphi$  is provable in the logic  $\mathcal{L}$ . The following lemma was proved in [3]. We repeat the proof for the sake of completeness.

**Lemma 1** *Let  $\Gamma$  be a string of  $(\rightarrow, \leftarrow)$ -formulas of order not greater than 1 and  $p$  be a variable. Then,  $\Gamma \vdash_{FL} p$  iff  $\Gamma \vdash_{AB} p$ .*

Proof. The ‘if’-part holds, since AB is a subsystem of FL. We prove the ‘only if’-part, by induction on cut-free proofs in FL. If  $\Gamma \vdash p$  is an axiom (Id), then  $\Gamma \vdash_{AB} p$ . Otherwise,  $\Gamma \vdash p$  must be the conclusion of  $(\leftarrow L)$  or  $(\rightarrow L)$ . Consider the first case. The premises are:  $\Gamma_1, \varphi, \Gamma_2 \vdash p$  and  $\Phi \vdash q$ , where  $\Gamma = \Gamma_1, (\varphi \leftarrow q), \Phi, \Gamma_2$ . The antecedents of these premises are strings of formulas of order not greater than 1, so they are provable in AB, by the induction hypothesis. The AB-proof of  $\Gamma \vdash p$  looks, as follows. First, reduce  $\Phi$  to  $q$ . Second, apply (AJD):  $(\varphi \leftarrow q), q \Rightarrow \varphi$ . Third, reduce  $\Gamma_1, \varphi, \Gamma_2$  to  $p$ . The second case is dual. Q.E.D.

The notion of a subformula of a formula is defined in the natural way. We distinguish *positive* and *negative* occurrences of subformulas in a formula. Let us define these notions for all formulas of ACT. If  $\varphi$  is an atomic formula, then  $\varphi$  is the only positive subformula of  $\varphi$ , and  $\varphi$  has no negative subformulas. If  $\varphi \equiv \psi \circ \chi$ , where  $\circ \in \{\cdot, \vee, \wedge\}$ , then the positive subformulas of  $\varphi$  are  $\varphi$  and all positive subformulas of  $\psi$  or  $\chi$ , and the negative subformulas of  $\varphi$  are all negative subformulas of  $\psi$  or  $\chi$ . If  $\varphi \equiv \psi \rightarrow \chi$  or  $\varphi \equiv \chi \leftarrow \psi$ , then the positive subformulas of  $\varphi$  are  $\varphi$ , all positive subformulas of  $\chi$ , and all negative subformulas of  $\psi$ , and the negative subformulas of  $\varphi$  are all negative subformulas of  $\chi$  and all positive subformulas of  $\psi$ . If  $\varphi \equiv \psi^*$ , then the positive subformulas of  $\varphi$  are  $\varphi$  and all positive subformulas of  $\psi$ , and the negative subformulas of  $\varphi$  are all negative subformulas of  $\psi$ .

The positive (resp. negative) subformulas of a sequent  $\Gamma \vdash \varphi$  are all positive (resp. negative) subformulas of  $\varphi$  and all negative (resp. positive) subformulas of formulas appearing in  $\Gamma$ . This definition is justified by the fact that  $\Gamma \vdash \varphi$  is provable iff  $\vdash \varphi \leftarrow \Gamma$  is provable, for any logic closed under (CUT),  $(\leftarrow L)$  and  $(\leftarrow R)$ . The formula  $\varphi \leftarrow \Gamma$  is defined, by induction on the length of  $\Gamma$ :

$$\varphi \leftarrow \lambda \equiv \varphi, \quad \varphi \leftarrow (\Gamma, \psi) \equiv (\varphi \leftarrow \psi) \leftarrow \Gamma.$$

**Lemma 2** *Let  $\varphi \wedge \psi$  be a negative subformula of a sequent  $\Gamma \vdash \chi$ , provable in the  $(\wedge, \rightarrow, \leftarrow)$ -fragment of FL and containing no positive subformulas of the form  $\varphi_1 \wedge \varphi_2$ . Let  $\Gamma_1 \vdash \chi_1$  (resp.  $\Gamma_2 \vdash \chi_2$ ) be the sequent resulting from  $\Gamma \vdash \chi$  after one has replaced the subformula  $\varphi \wedge \psi$  by  $\varphi$  (resp.  $\psi$ ). Then,  $\Gamma_1 \vdash_{FL} \chi_1$  or  $\Gamma_2 \vdash_{FL} \chi_2$ .*

Proof. Induction on cut-free derivations in the  $(\rightarrow, \leftarrow, \wedge)$ -fragment of FL.  $\Gamma \vdash \chi$  cannot be an axiom (Id) or the conclusion of  $(\wedge R)$ , since it contains no positive occurrences of  $\wedge$ . If it is the conclusion of  $(\wedge L)$ , not introducing the designated formula  $\varphi \wedge \psi$ ,  $(\rightarrow L)$ ,  $(\rightarrow R)$ ,  $(\leftarrow L)$  or  $(\leftarrow R)$ , then we apply the induction hypothesis directly. If it is the conclusion of  $(\wedge L)$ , introducing the designated formula  $\varphi \wedge \psi$ , then our thesis is obvious. Q.E.D.

**Corollary 1** *For each  $i = 1, \dots, n$ , let  $\varphi_i \equiv \varphi_{i,1} \wedge \dots \wedge \varphi_{i,k_i}$ ,  $k_i \geq 1$  and all formulas  $\varphi_{i,j}$  and  $\varphi$  be  $(\rightarrow, \leftarrow)$ -formulas. Then,  $\varphi_1, \dots, \varphi_n \vdash_{FL} \varphi$  iff, for each  $i = 1, \dots, n$ , there exists  $j_i \in \{1, \dots, k_i\}$  such that  $\varphi_{1,j_1}, \dots, \varphi_{n,j_n} \vdash_{FL} \varphi$ .*

Proof. The ‘if’-part holds, by  $(\wedge L)$ . The ‘only if’-part is proved by induction on the number of occurrences of  $\wedge$  in  $\varphi_1, \dots, \varphi_n \vdash \varphi$ , using lemma 2. Q.E.D.

We define a natural equivalence relation on formulas:  $\varphi \sim_{FL} \psi$  iff  $\varphi \vdash_{FL} \psi$  and  $\psi \vdash_{FL} \varphi$ , and similarly for any logic  $\mathcal{L}$ . Let  $\psi \equiv \psi_1 \vee \dots \vee \psi_k$ ,  $k \geq 1$ . By  $\psi^n$ , for  $n \geq 1$ , we denote now the formula  $\psi \cdots \psi$  with  $n$  copies of  $\psi$ . We reserve the variable  $\mathbf{m}$  for finite sequences of integers. If  $\mathbf{m} = (i_1, \dots, i_n)$ , then we denote  $\psi(\mathbf{m}) \equiv \psi_{i_1} \cdots \psi_{i_n}$ ; we assume  $\mathbf{m} \in \{1, \dots, k\}^n$ ,  $n \geq 1$ . We set  $[k] = \{1, \dots, k\}$ .  $\bigvee$  stands for an iterated  $\vee$ .

**Lemma 3** *Let  $\psi \equiv \bigvee_{i \in [k]} \psi_i$ . Then,  $\psi^n \sim_{FL} \bigvee_{\mathbf{m} \in [k]^n} \psi(\mathbf{m})$ , for any  $n \geq 1$ .*

Proof. An easy induction on  $n$ , using the distribution of product over finite joins. Q.E.D.

**Lemma 4** *Let  $\chi \equiv \bigvee_{i \in [m]} \chi_i$ . Then,  $\chi \vdash_{FL} \varphi$  iff  $\chi_i \vdash_{FL} \varphi$ , for all  $i \in [m]$ .*

Proof. The ‘if’-part holds, by  $(\vee L)$ . The ‘only if’-part holds, since  $\chi_i \vdash_{FL} \chi$ , by  $(\vee R_1)$ ,  $(\vee R_2)$ , and FL is closed under (CUT). Q.E.D.

We are ready to prove the main lemma. Let  $H$  be an AB-grammar such that all types in its type lexicon are  $(\leftarrow)$ -formulas of order not greater than 1. Let  $\Sigma_H = \{v_1, \dots, v_k\}$ . Let  $\varphi_{i,1}, \dots, \varphi_{i,k_i}$  be all types  $\varphi$  such that  $(v_i, \varphi) \in H$ . Define  $\psi_i \equiv \varphi_{i,1} \wedge \dots \wedge \varphi_{i,k_i}$  and  $\psi \equiv \bigvee_{i \in [k]} \psi_i$ .

**Lemma 5**  *$L(H, s) = (\Sigma_H)^+$  iff  $\psi^*, \psi \vdash s$  belongs to  $ACT\omega$ .*

Proof. Clearly,  $\psi^n \vdash \psi^*$  belongs to  $ACT\omega$ , for all  $n \in \omega$ . Since  $ACT\omega$  is closed under  $(*L)$ , then we obtain:

- (1)  $\psi^*, \psi \vdash s$  is in  $ACT\omega$  iff, for all  $n \in \omega$ ,  $\psi^n, \psi \vdash s$  is in  $ACT\omega$ ,
- (2)  $\psi^*, \psi \vdash s$  is in  $ACT\omega$  iff, for all  $n \geq 1$ ,  $\psi^n \vdash s$  is in  $ACT\omega$ .

By  $(\cdot L)$ ,  $(\cdot R)$  and (CUT), it does not matter how do we understand  $\psi^n$ : as a string or as a product. Formulas  $\psi^n$  are  $*$ -free, and  $ACT\omega$  is conservative over FL, whence  $\psi^n \vdash s$  is in  $ACT\omega$  iff  $\psi^n \vdash_{FL} s$ . By lemmas 3 and 4, we get:

- (3) for  $n \geq 1$ ,  $\psi^n \vdash_{FL} s$  iff, for all  $\mathbf{m} \in [k]^n$ ,  $\psi(\mathbf{m}) \vdash_{FL} s$ .

By the definition of  $L(H, s)$ , lemma 1 and corollary 1, we have:

(4) for  $\mathbf{m} = (i_1, \dots, i_n)$ ,  $\psi(\mathbf{m}) \vdash_{FL} s$  iff  $v_{i_1} \dots v_{i_n} \in L(H, s)$ .

We finish the proof.  $L(H, s) = (\Sigma_H)^+$  iff, for all  $n \geq 1$  and all  $(i_1, \dots, i_n) \in [k]^n$ , there holds  $v_{i_1} \dots v_{i_n} \in L(H, s)$  iff (by (4)), for all  $n \geq 1$  and all  $\mathbf{m} \in [k]^n$ , there holds  $\psi(\mathbf{m}) \vdash_{FL} s$  iff (by (3)), for all  $n \geq 1$ ,  $\psi^n \vdash_{FL} s$  iff (by (2))  $\psi^*, \psi \vdash s$  is in  $ACT\omega$ . Q.E.D.

We can prove the main result of this section.

**Theorem 1** *ACT $\omega$  is not recursively enumerable.*

Proof. By the Gaifman theorem and lemma 5,  $(TOTAL^+)$  is reducible to  $ACT\omega$ . Consequently,  $ACT\omega$  is undecidable. Since FL is decidable, then, by (1),  $ACT\omega$ , restricted to sequents of the form  $\psi^*, \psi \vdash s$ , where  $\psi$  is \*-free, is  $\Pi_1^0$ . So,  $ACT\omega$  cannot be  $\Sigma_1^0$ . Q.E.D.

Actually, the proof yields the  $\Pi_1^0$ -hardness of  $ACT\omega$ . My student Palka [21] proves that the whole  $ACT\omega$  is  $\Pi_1^0$  in the following way. Formulas of the form  $\chi^*$  are called \*-formulas. We define  $\varphi^{\leq n} \equiv \varphi^0 \vee \dots \vee \varphi^n$ ,  $n \in \omega$  (we assume  $\varphi^0 \equiv 1$ ). Let  $P_n(\psi)$  (resp.  $N_n(\psi)$ ) be the formula arising from  $\psi$  by a (successive) replacement of any positive (resp. negative) \*-subformula  $\chi^*$  by  $\chi^{\leq n}$ . This rough formulation can be replaced by a strict, recursive definition whose most characteristic cases are:  $P_n(\psi^*) \equiv (P_n(\psi))^{\leq n}$ ,  $N_n(\psi^*) \equiv (N_n(\psi))^*$ . For a string  $\Gamma = \psi_1, \dots, \psi_m$ , we set  $P_n(\Gamma) = P_n(\psi_1), \dots, P_n(\psi_m)$ . The following theorem is proved in [21]:

(\*E) for any sequent  $\Gamma \vdash \varphi$ , this sequent is in  $ACT\omega$  iff, for all  $n \in \omega$ , the sequent  $P_n(\Gamma) \vdash N_n(\varphi)$  is in  $ACT\omega$ .

The ‘only if’ part is easy, since  $P_n(\psi) \vdash \psi$  and  $\psi \vdash N_n(\psi)$  are in  $ACT\omega$ , and the ‘if’-part is proved by a transfinite induction, using cut-elimination. Since the right-hand sequent in (\*E) contains no negative \*-subformula, it is provable in  $ACT\omega$  iff it is provable in  $ACT\omega$  without the rule (\*L); let us denote the latter system by  $ACT\omega^-$ .  $ACT\omega^-$  is a finitary system with an effective proof-search procedure, and consequently, the right-hand condition of (\*E) is  $\Pi_1^0$ . Therefore,  $ACT\omega$  is  $\Pi_1^0$ -complete.

From the proof of theorem 1, it follows that the  $(\vee, \wedge, \leftarrow, *)$ -fragment of  $ACT\omega$  is  $\Pi_1^0$ -hard, and  $\leftarrow$  can be replaced with  $\rightarrow$ . We show that  $\wedge$  and  $\vee$  can be eliminated (not both together). To eliminate  $\wedge$  we need two other lemmas from [3] (lemma 3 and a direct consequence of lemmas 4 and 5).

**Lemma 6** *Let  $\Gamma = \varphi_1, \dots, \varphi_n$ . In the  $(\leftarrow)$ -fragment of FL,  $(s \leftarrow \Gamma), \Delta \vdash s$  is provable iff there exist  $\Delta_1, \dots, \Delta_n$  such that  $\Delta = \Delta_1, \dots, \Delta_n$  and  $\Delta_i \vdash \varphi_i$  is provable, for all  $i = 1, \dots, n$ .*

Proof. The ‘if’-part holds, by (Id),  $(\leftarrow L)$ , and the ‘only if’-part can be proved by induction on cut-free proofs. Q.E.D.

**Lemma 7** *In the same system,  $s \leftarrow (s \leftarrow \varphi_1), \dots, s \leftarrow (s \leftarrow \varphi_n) \vdash s$  is provable iff  $\varphi_1, \dots, \varphi_n \vdash s$  is provable.*

Proof. We prove the following equivalence:

$$s \leftarrow (s \leftarrow \varphi_{i+1}), \dots, s \leftarrow (s \leftarrow \varphi_n), \varphi_1, \dots, \varphi_i \vdash_{FL} s, \text{ iff } \varphi_1, \dots, \varphi_n \vdash_{FL} s,$$

where all formulas are  $(\leftarrow)$ -formulas. This is proved by induction on  $0 \leq n-i \leq n$ . For  $i = n$ , there is nothing to prove. Take  $i < n$ . Assume the left-hand side. By lemma 6:

$$s \leftarrow (s \leftarrow \varphi_{i+2}), \dots, s \leftarrow (s \leftarrow \varphi_n), \varphi_1, \dots, \varphi_i \vdash_{FL} s \leftarrow \varphi_{i+1},$$

which yields:

$$s \leftarrow (s \leftarrow \varphi_{i+2}), \dots, s \leftarrow (s \leftarrow \varphi_n), \varphi_1, \dots, \varphi_{i+1} \vdash_{FL} s,$$

whence the right-hand side holds, by the induction hypothesis. Assume the right-hand side. By the induction hypothesis, we get:

$$s \leftarrow (s \leftarrow \varphi_{i+2}), \dots, s \leftarrow (s \leftarrow \varphi_n), \varphi_1, \dots, \varphi_{i+1} \vdash_{FL} s,$$

whence, by  $(\leftarrow R)$ ,  $(\leftarrow L)$  and (Id), the left-hand side holds. Q.E.D.

Now, replace the AB-grammar  $H$  with all types being  $(\leftarrow)$ -formulas of order not greater than 1 with an FL-grammar  $H'$ , defined as follows:  $(v, \chi) \in H$  iff  $(v, s \leftarrow (s \leftarrow \chi)) \in H'$ , and construct the formula  $\psi'$  for  $H'$  in the same way as the formula  $\psi$  for  $H$ . By lemmas 1 and 7,  $L(H, s) = L(H', s)$ , so  $L(H, s) = (\Sigma_H)^+$  iff  $(\psi')^*, \psi' \vdash s$  is in  $ACT\omega$ . But,  $\wedge$  in  $\psi'$  only appears in contexts:

$$(s \leftarrow (s \leftarrow \varphi_{i,1})) \wedge \dots \wedge (s \leftarrow (s \leftarrow \varphi_{i,k_i})),$$

which can be replaced by equivalent formulas:

$$s \leftarrow [(s \leftarrow \varphi_{i,1}) \vee \dots \vee (s \leftarrow \varphi_{i,k_i})],$$

because  $a \leftarrow (b \vee c) = (a \leftarrow b) \wedge (a \leftarrow c)$  is true in action lattices.

The elimination of  $\vee$  (keeping  $\wedge$ ) is simpler. According to the Kleene algebra law:

$$(a \vee b)^* = (a^*b)^*a^*,$$

for the formula  $\psi \equiv \psi_1 \vee \dots \vee \psi_k$ , constructed before lemma 5,  $\psi^*$  is equivalent to a  $(\wedge, \leftarrow, *)$ -formula  $\chi$ . We have:  $\psi^*, \psi \vdash s$  is in  $ACT\omega$  iff  $\chi, \psi \vdash s$  is in  $ACT\omega$  iff, for all  $i = 1, \dots, k$ ,  $\chi, \psi_i \vdash s$  is in  $ACT\omega$  iff, for all  $i = 1, \dots, k$ ,  $\chi \vdash s \leftarrow \psi_i$  is in  $ACT\omega$  iff  $\chi \vdash \bigwedge_{i \in [k]} (s \leftarrow \psi_i)$  is in  $ACT\omega$ .

**Theorem 2** *The  $(\vee, \leftarrow, *)$ -fragment and the  $(\wedge, \leftarrow, *)$ -fragment of  $ACT\omega$  are  $\Pi_1^0$ -complete, and  $\leftarrow$  can be replaced with  $\rightarrow$ .*

I do not know if the multiplicative fragment of  $\text{ACT}\omega$  (i.e. the  $(\vee, \wedge)$ -free fragment) is undecidable.

The  $(\wedge, \rightarrow, \leftarrow)$ -free fragment of  $\text{ACT}\omega$  is the logic of  $*$ -continuous Kleene algebras, which is decidable, by the Kozen completeness theorem. Below we show that the simple Horn theory of  $*$ -continuous Kleene algebras is  $\Pi_1^0$ -complete (see the introduction for references to analogous results of Kozen).

Simple atomic formulas (see the introduction) are expressible in the language of FL by sequents of the form  $\Gamma \vdash p$  such that  $p$  is a variable and  $\Gamma$  is a string of variables; we call them *simple sequents*. Let  $X$  be a set of simple sequents. By  $\text{ACT}\omega(X)$  we denote the system  $\text{ACT}\omega$  enriched with all sequents from  $X$  as new axioms (not closed under substitution). If (CUT) is admitted, then the sequents provable in  $\text{ACT}\omega(X)$  are precisely those sequents which are true in all models  $(\mathcal{A}, f)$  such that  $\mathcal{A}$  is a  $*$ -continuous action lattice, and all sequents from  $X$  are true in  $(\mathcal{A}, f)$ . If  $X$  is closed under (CUT), then the cut-elimination theorem holds for  $\text{ACT}\omega(X)$ . The proof is the same as for  $\text{ACT}\omega$  with one new case:  $C \equiv p$  and both premises of (CUT) are new axioms. Then, the conclusion is a new axiom, too. Further, theorem (\*E) can be generalized to  $\text{ACT}\omega(X)$ ,  $X$  closed under (CUT). Consequently, if  $X$  is a recursive set of simple sequents, then  $\text{ACT}\omega(X)$  is  $\Pi_1^0$  [21].

Let us return to the construction of formula  $\psi$  before lemma 5. For each  $i = 1, \dots, k$ , introduce a new variable  $p_i$  and consider all sequents  $p_i \vdash \varphi_{i,j}$ , for  $j = 1, \dots, k_i$ . By the form of the grammar  $H$ ,  $\varphi_{i,j} \equiv q_{i,j} \leftarrow \Phi_{i,j}$ , where  $q_{i,j}$  is a variable and  $\Phi_{i,j}$  is a string of at most two variables. So, sequent  $p_i \vdash \varphi_{i,j}$  is deductively equivalent (in FL or  $\text{ACT}\omega$ ) to the simple sequent  $p_i, \Phi_{i,j} \vdash q_{i,j}$ . Let  $X(H)$  denote the smallest set of simple sequents which contains all sequents constructed in this way and is closed under (CUT). Then,  $X(H)$  is a recursive set, since it can be generated by a CF-grammar. We write simply  $X$  for  $X(H)$ .

Let us consider the  $(\wedge, \rightarrow, \leftarrow)$ -free fragment of  $\text{ACT}\omega(X)$ . It is a conservative subsystem of  $\text{ACT}\omega(X)$ , and consequently, it is  $\Pi_1^0$ . By  $\chi$  we denote the formula  $p_1 \vee \dots \vee p_k$ . For the formula  $\psi$  constructed for  $H$ , we prove:

$$\chi^*, \chi \vdash s \text{ is in } \text{ACT}\omega(X) \text{ iff } \psi^*, \psi \vdash s \text{ is in } \text{ACT}\omega.$$

We prove the ‘only if’-part. Assume that  $\chi^*, \chi \vdash s$  is provable in  $\text{ACT}\omega(X)$ . By completeness, this sequent is true in all models  $(\mathcal{A}, f)$  such that  $\mathcal{A}$  is a  $*$ -continuous action lattice and all sequents from  $X$  are true in  $(\mathcal{A}, f)$ . Since no variable  $p_i$  occurs in formulas  $\varphi_{i,j}$ , then we can stipulate:

$$f(p_i) = f(\varphi_{i,1} \wedge \dots \wedge \varphi_{i,k_i}),$$

for  $i = 1, \dots, k$ . All sequents from  $X$  are true in models, satisfying the latter postulate, so  $\chi^*, \chi \vdash s$  must be true in these models, but this means that  $\psi^*, \psi \vdash s$  is in  $\text{ACT}\omega$ . The ‘if’-part holds, by (CUT) and the monotonicity of  $*$ , since  $\chi \vdash \psi$  is provable in  $\text{ACT}\omega(X)$ .

Further, both  $\chi$  and the sequents from  $X$  are  $(\wedge, \rightarrow, \leftarrow)$ -free, whence  $\chi^*, \chi \vdash s$  is provable in  $\text{ACT}\omega(X)$  iff it is provable in the Kleene algebra fragment of  $\text{ACT}\omega(X)$  (use cut-elimination). This finishes the proof of the next theorem.

**Theorem 3** *The simple Horn theory of \*-continuous Kleene algebras is not recursively enumerable (it is  $\Pi_1^0$ -complete).*

We turn to ACT. It follows from theorem 1 that  $\text{ACT}\omega$  is essentially stronger than ACT, since ACT is  $\Sigma_1^0$ . The decidability of ACT remains an open problem. Recall that a logic possesses FMP iff it is complete with respect to its finite models. Finite action algebras (lattices) are complete as posets, and complete action algebras (lattices) are \*-continuous. This yields the following fact.

**Corollary 2** *ACT does not possess FMP.*

Proof. Assume that ACT possesses FMP. Then, the sequents provable in ACT are precisely those sequents which are true in all \*-continuous action lattices. So, ACT equals  $\text{ACT}\omega$ . By theorem 1, this is impossible. Q.E.D.

At the end of this section, we show that a sequent system for ACT, proposed by Jipsen [9], does not admit cut elimination. This system arises from FL (in the language of ACT) by adding the following inference rules for \*.

$$\begin{aligned} \text{(R1)} \quad & \frac{\Gamma \vdash 1}{\Gamma \vdash \varphi^*}, \quad \text{(R2)} \quad \frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi^*}, \\ \text{(R3)} \quad & \frac{\Gamma \vdash \psi; \varphi, \psi \vdash \psi}{\varphi^*, \Gamma \vdash \psi}, \quad \text{(R4)} \quad \frac{\Gamma \vdash \psi; \psi, \varphi \vdash \psi}{\Gamma, \varphi^* \vdash \psi}. \end{aligned}$$

Let us denote this system by ACT'. With (CUT), ACT' is equivalent to ACT. Let us consider the sequent:

$$(s \leftarrow (s \leftarrow (s \leftarrow s)))^*, s \leftarrow (s \leftarrow s) \vdash s.$$

We show that the above sequent is provable in ACT' with (CUT), but not without (CUT). First, the following sequent:

$$s \leftarrow (s \leftarrow (s \leftarrow s)), s \leftarrow (s \leftarrow s) \vdash s \leftarrow (s \leftarrow s),$$

is provable in FL, and consequently, in ACT' without (CUT). By (Id) and (R3), the sequent:

$$(s \leftarrow (s \leftarrow (s \leftarrow s)))^*, s \leftarrow (s \leftarrow s) \vdash s \leftarrow (s \leftarrow s),$$

is provable in ACT' without (CUT). Since  $s \leftarrow (s \leftarrow s) \vdash s$  is provable in FL, by (Id), ( $\leftarrow$ R) and ( $\leftarrow$ L), then the first sequent is provable in ACT' with (CUT). Suppose that it is provable in ACT' without (CUT). It is not an axiom. It can be the conclusion of (R3) or ( $\leftarrow$ L). In the first case, the right premise is:

$$s \leftarrow (s \leftarrow (s \leftarrow s)), s \vdash s.$$

This is impossible, since the latter sequent is not provable in FL. In the second case, the left premise is:

$$(s \leftarrow (s \leftarrow (s \leftarrow s)))^*, s \vdash s.$$

Consequently, this sequent is true in action lattices. Since  $\varphi \vdash \varphi^*$  is also true, then the unprovable sequent, considered above, must be true in action lattices, which is impossible, by the completeness of FL.

## 4 FMP of $\text{ACT}\omega$

In this section, we prove that  $\text{ACT}\omega$  possesses FMP. First, we prove FMP for a weaker logic, being an extension of  $\text{ACT}\omega^-$ .

By a *weak action lattice* we mean an algebra  $\mathcal{A} = (A, \vee, \wedge, *, \rightarrow, \leftarrow, 0, 1)$  such that the  $*$ -free reduct of  $\mathcal{A}$  is a residuated lattice with the least element 0, and  $*$  fulfills the following conditions:

(WA1) for all  $a \in A$  and  $n \in \omega$ ,  $a^n \leq a^*$ ,

(WA2) for all  $a, b \in A$ , if  $a \leq b$  then  $a^* \leq b^*$ .

Clearly, every action lattice is a weak action lattice. The converse does not hold, as we show for a moment. We consider a system WACT, which arises from  $\text{ACT}\omega^-$  (i.e.  $\text{ACT}\omega$  without the rule  $(*L)$ ) by adding the rule:

$$(\text{MON}) \frac{\varphi \vdash \psi}{\varphi^* \vdash \psi^*},$$

which is derivable in  $\text{ACT}\omega$ , using  $(*L)$ . So, WACT is an extension of  $\text{ACT}\omega^-$  and a subsystem of  $\text{ACT}\omega$ .

**Lemma 8** *If  $\Gamma \vdash \varphi$  contains no negative  $*$ -subformula, then  $\Gamma \vdash \varphi$  is provable in WACT iff it is provable in  $\text{ACT}\omega$ .*

*Proof.* The ‘only if’-part is obvious. The ‘if’-part follows from the fact that  $\text{ACT}\omega$  coincides with  $\text{ACT}\omega^-$  in the scope of sequents without negative occurrences of  $*$ . Q.E.D.

The cut-elimination theorem holds for WACT. It is easy to provide a syntactic proof, using three kinds of induction: (1) on the complexity of  $\varphi$ , (2) on the proof of the left premise of (CUT), (3) on the proof of the right premise of (CUT). In comparing with an analogous proof for FL, the new case of (1) is  $\varphi \equiv \chi^*$ . We switch on induction (2). There is one interesting case:  $\Gamma, \chi^*, \Delta \vdash \psi$  is the conclusion of (MON). Then,  $\Gamma = \lambda$ ,  $\Delta = \lambda$ ,  $\psi = \delta^*$ , and the premise is  $\chi \vdash \delta$ . Then, we switch on induction (3). There are three interesting cases: (A)  $\Phi \vdash \chi^*$  is an axiom  $(*1)$ , (B) it is the conclusion of  $(*R)$ , (C) it is the conclusion of (MON). For (A),  $\Phi = \lambda$ , whence the conclusion of (CUT) is again axiom  $(*1) \vdash \delta^*$ . For (B), the  $n$  premises of  $(*R)$  are  $\Phi_1 \vdash \chi, \dots, \Phi_n \vdash \chi$ , where  $\Phi = \Phi_1, \dots, \Phi_n$ . By the induction hypothesis of (1),  $\Phi_i \vdash \delta$  is provable in WACT, for all  $i = 1, \dots, n$ , and consequently,  $\Phi \vdash \psi$  is provable in WACT, by  $(*R)$ . For (C),  $\Phi = \gamma^*$ , and  $\gamma^* \vdash \chi^*$  is the conclusion of (MON) with the premise  $\gamma \vdash \chi$ . By the induction hypothesis of (1),  $\gamma \vdash \delta$  is provable in WACT, and consequently,  $\Phi \vdash \psi$  is provable in WACT, by (MON).

Accordingly, one can show the completeness of WACT with respect to weak action lattices. Let us notice that the cut-elimination theorem for WACT is not needed in our proof of FMP. Like in [2], cut-elimination follows from our further completeness results.

We adopt methods for proving cut-elimination and FMP for intuitionistic substructural logics, in particular for FL, elaborated in [17, 18, 2]; see also [5, 7] for different proofs.

Let  $\mathcal{A}$  be a residuated lattice. A *closure operator* on  $\mathcal{A}$  is a mapping  $C : A \mapsto A$ , satisfying the following conditions:

- (c1)  $x \leq C(x)$ ,
- (c2) if  $x \leq y$  then  $C(x) \leq C(y)$ ,
- (c3)  $C(C(x)) \leq x$ ,
- (c4)  $C(x) \cdot C(y) \leq C(x \cdot y)$ ,

for all  $x, y \in A$ . An element  $x \in A$  is said to be *closed*, if  $C(x) = x$ . Let  $C(\mathcal{A})$  denote the set of closed elements of  $\mathcal{A}$  with respect to  $C$ . One defines operations on  $C(\mathcal{A})$ :

$$a \cdot_C b = C(a \cdot b), \quad a \vee_C b = C(a \vee b).$$

The set  $C(\mathcal{A})$  is closed under meet and residuals. The algebra  $C(\mathcal{A})$  with operations  $\vee_C, \wedge, \cdot_C, \rightarrow, \leftarrow$  and designated elements  $C(0), C(1)$  is a residuated lattice, and it will be denoted by  $C_{\mathcal{A}}$ .

Let  $(M, \cdot, 1)$  be a monoid. For  $X, Y \subseteq M$ , one defines operations:  $X \vee Y = X \cup Y$ ,  $X \wedge Y = X \cap Y$ ,  $X \cdot Y = \{x \cdot y : x \in X, y \in Y\}$ ,  $X \rightarrow Y = \{z : X \cdot \{z\} \subseteq Y\}$ ,  $X \leftarrow Y = \{z : \{z\} \cdot Y \subseteq X\}$ . Then, the algebra  $(P(M), \cup, \cap, \cdot, \rightarrow, \leftarrow, \emptyset, \{1\})$  is a complete residuated lattice, and it will be referred to as *the residuated lattice*  $P(M)$ .

Let  $\mathcal{A}$  be a residuated lattice. A closure operator on the residuated lattice  $P(\mathcal{A})$  (determined by the monoid  $(\mathcal{A}, \cdot, 1)$ ) can be defined as follows:

$$C(X) = \bigcap \{[a] : a \in A \& X \subseteq [a]\},$$

where  $[a] = \{x \in A : x \leq a\}$  is the (lower) principal cone generated by  $a$ . The mapping  $f(a) = [a]$  is an embedding of  $\mathcal{A}$  into the complete residuated lattice  $C_{P(\mathcal{A})}$ ; this embedding preserves all existing infinite joins and meets in  $\mathcal{A}$ . The latter lattice is called *the MacNeille - completion* of  $\mathcal{A}$ .

Let  $\mathcal{A}$  be an action lattice.  $C_{P(\mathcal{A})}$  is a complete residuated lattice, so it admits a unique operation  $*$ , fulfilling conditions (C1) and (C2) (see section 2), which means that  $C_{P(\mathcal{A})}$  has a unique expansion to an action lattice. The MacNeille embedding  $f(a) = [a]$  need not preserve  $*$ . If  $\mathcal{A}$  is  $*$ -continuous, then  $[a^*] = [a]^*$ , which means that  $f$  preserves  $*$ . Consequently,  $*$ -continuous action lattices are precisely the subalgebras of complete action lattices. The same holds for action algebras, but not for Kleene algebras.

Every  $*$ -continuous Kleene algebra (lattice)  $\mathcal{A}$  can be embedded into a complete action lattice  $C_{P(\mathcal{A})}$ , for a different closure operator  $C$  on  $P(\mathcal{A})$ . For  $x, y, a \in A$ , denote:

$$[x - y, a] = \{z \in A : xzy \leq a\},$$



and define:

$$C(X) = \bigcap \{[x - y, a] : x, y, a \in A \& X \subseteq [x - y, a]\}.$$

The so-defined operator  $C$  is a closure operator on  $P(A)$ , and the mapping  $f(a) = [a]$  is an embedding of  $\mathcal{A}$  into  $C_{P(A)}$ , which preserves the whole Kleene algebra structure.

An analogous construction will be used in our proof of FMP for WACT. We fix a sequent  $\Gamma_0 \vdash \varphi_0$ . Let  $S$  denote the set of all subformulas of formulas occurring in this sequent plus formulas 0, 1.  $S^*$  denotes the set of all finite strings of formulas from  $S$ . Let  $T$  be the set of all sequents which appear in the proof-search tree for  $\Gamma_0 \vdash \varphi_0$  in WACT. This means that  $T$  is the smallest set of sequents fulfilling the conditions:

$$(T1) (\Gamma_0 \vdash \varphi_0) \in T,$$

(T2) for any instance of an inference rule of WACT, if the conclusion of this rule belongs to  $T$ , then all premises of this rule belong to  $T$ .

Since all rules of WACT increase the complexity of sequents (i.e. the total number of occurrences of atomic formulas), and WACT has the subformula property, then  $T$  is a finite set, and all sequents in  $T$  are formed out of formulas from  $S$ . We consider the free monoid  $(S^*, \cdot, \lambda)$ , where  $\cdot$  stands for concatenation. Following [2], we define a relation  $\preceq \subseteq S^* \times S$ :  $\Gamma \preceq \varphi$  iff either  $\Gamma \vdash \varphi$  does not belong to  $T$ , or  $\Gamma \vdash \varphi$  is provable in WACT. Clearly,  $\Gamma \preceq \varphi$  holds for all but finitely many pairs  $(\Gamma, \varphi) \in S^* \times S$ .

A closure operator  $C$  on  $P(S^*)$  is defined as follows:

$$C(X) = \bigcap \{[\Gamma - \Delta, \varphi] : \Gamma, \Delta \in S^*, \varphi \in S, X \subseteq [\Gamma - \Delta, \varphi]\},$$

where  $[\Gamma - \Delta, \varphi] = \{\psi \in S : \Gamma, \psi, \Delta \preceq \varphi\}$ . It is easy to prove (c1)-(c4).

The relation  $\preceq$  satisfies all axioms of WACT and is closed under all inference rules of WACT such that all formulas appearing in these axioms and rules belong to  $S$ . We prove this fact for axioms and rules concerning  $*$ . For axiom  $\vdash \varphi^*$ , this sequent is provable in WACT, whence  $\lambda \preceq \varphi^*$ . Let us consider rule (\*R). Assume  $\Gamma_i \preceq \varphi$ , for all  $i = 1, \dots, n$ . If  $\Gamma_1, \dots, \Gamma_n \vdash \varphi^*$  does not belong to  $T$ , then  $\Gamma_1, \dots, \Gamma_n \preceq \varphi^*$ . Otherwise, all sequents  $\Gamma_i \vdash \varphi_i$ ,  $i = 1, \dots, n$ , belong to  $T$ , whence they are provable in WACT. Then,  $\Gamma_1, \dots, \Gamma_n \vdash \varphi^*$  is provable in WACT, and consequently,  $\Gamma_1, \dots, \Gamma_n \preceq \varphi^*$ . For (MON), the reasoning is similar.

We consider the complete residuated lattice  $C_{P(S^*)}$  and the assignment  $f(p) = [p]$ , for variables  $p \in S$ . Now,  $[\psi]$  denotes the set  $[\lambda - \lambda, \psi]$ . As usual,  $f$  is uniquely extended to a homomorphism from the algebra of  $*$ -free formulas of WACT to  $C_{P(S^*)}$ . As in [2], one proves:

$$(QE) \varphi \in f(\varphi) \subseteq [\varphi], \text{ for any } *\text{-free formula } \varphi \in S.$$

The label (QE) comes from ‘quasi-embedding’. The proof goes by induction on formula  $\varphi$ . We consider one case:  $\varphi \equiv \psi \rightarrow \chi$ . Let  $\Gamma \in f(\psi)$ . By the induction hypothesis,  $\Gamma \in [\psi]$ , and consequently,  $\Gamma \preceq \psi$ . Let  $[\Delta - \Delta', \delta]$  contain  $f(\chi)$ . By the induction hypothesis,  $\chi \in f(\chi)$ , and consequently,  $\Delta, \chi, \Delta' \preceq \delta$ . Since  $\preceq$  satisfies ( $\rightarrow$ L), then  $\Delta, \Gamma, \psi \rightarrow \chi, \Delta' \preceq \delta$ , which means that the string  $\Gamma, \psi \rightarrow \chi$  belongs to  $[\Delta - \Delta', \delta]$ . Therefore, this string belongs to  $f(\chi)$ . We have shown  $\varphi \in f(\varphi)$ . Let  $\Gamma \in f(\varphi)$ . By the induction hypothesis,  $\psi \in f(\psi)$ , and consequently, the string  $\psi, \Gamma$  belongs to  $f(\chi)$ . By the induction hypothesis,  $f(\chi) \subseteq [\chi]$ , which yields  $\psi, \Gamma \preceq \chi$ . Since  $\preceq$  satisfies ( $\rightarrow$ R), then  $\Gamma \preceq \varphi$ . We have shown  $f(\varphi) \subseteq [\varphi]$ .

We define an operation  $*$  on  $C_{P(S^*)}$ :

$$(W^*) \quad X^* = \bigcap \{[\varphi^*] : \varphi^* \in S \& X \subseteq [\varphi]\}.$$

We show that  $*$  fulfills (WA1). Let  $X \subseteq S^*$  be closed. Let  $n = 0$ . Let  $X \subseteq [\varphi]$ ,  $\varphi^* \in S$ . Since  $1 \vdash \varphi^*$  is provable in WACT, by ( $*1$ ) and (1L), then  $1 \preceq \varphi^*$ . Consequently,  $1 \in X^*$ . So,  $\{1\} \subseteq X^*$ , which yields  $C(\{1\}) \subseteq C(X^*) = X^*$ . Let  $n > 0$ . Let  $X \subseteq [\varphi]$ . We have  $X^n = X \cdot_C \dots \cdot_C X$ , where  $X$  appears  $n$  times. Let  $\Gamma_i \in X$ , for  $i = 1, \dots, n$ . Then,  $\Gamma_i \preceq \varphi$ , for  $i = 1, \dots, n$ . Since  $\preceq$  is closed under ( $*R$ ), we get  $\Gamma_1, \dots, \Gamma_n \preceq \varphi^*$ . Consequently, the string  $\Gamma_1, \dots, \Gamma_n$  belongs to  $X^*$ . This yields  $X \cdot \dots \cdot X \subseteq X^*$ . Then,  $C(X \cdot \dots \cdot X) \subseteq X^*$ . Using (c4), one easily shows  $X^n \subseteq C(X \cdot \dots \cdot X)$ , by induction on  $n \geq 1$ . It is obvious that  $*$  fulfills (WA2).

We have shown that  $C_{P(S^*)}$  with the operation  $*$  defined by (W $^*$ ) is a weak action lattice. Since there are only finitely many pairwise distinct sets  $[\Gamma - \Delta, \varphi]$ , then  $C_{P(S^*)}$  is a finite algebra. We extend the mapping  $f$  to all formulas of WACT, by setting  $f(\varphi^*) = (f(\varphi))^*$ , where the right-hand  $*$  is defined by (W $^*$ ). We show:

(QE $^*$ )  $\varphi \in f(\varphi) \subseteq [\varphi]$ , for any formula  $\varphi \in S$ .

The inductive proof of (QE) must be supplied with the new case:  $\varphi = \psi^*$ . We show  $\psi^* \in f(\psi^*)$ . Let  $f(\psi) \subseteq [\chi]$ ,  $\chi^* \in S$ . By the induction hypothesis,  $\psi \in f(\psi)$ , whence  $\psi \preceq \chi$ . Since  $\preceq$  is closed under (MON), then  $\psi^* \preceq \chi^*$ . Consequently,  $\psi^* \in (f(\psi))^* = f(\psi^*)$ . By the induction hypothesis,  $f(\psi) \subseteq [\psi]$ , and consequently,  $f(\psi)^* \subseteq [\psi^*]$ .

**Theorem 4** *WACT possesses FMP.*

Proof. Assume that  $\Gamma_0 \vdash \varphi_0$  is not provable in WACT. We define  $C_{P(S^*)}$  and  $f$  as above. Since  $\Gamma_0 \vdash \varphi_0$  belongs to  $T$ , then  $\Gamma_0 \not\preceq \varphi_0$ . By (QE $^*$ ),  $\Gamma_0 \in f(\Gamma_0)$  and  $\Gamma_0 \notin f(\varphi_0)$ . Then,  $f(\Gamma_0) \not\subseteq f(\varphi_0)$ . Q.E.D.

Let  $\mathcal{A}$  be a complete weak action lattice. Then, its  $*$ -free reduct is a complete residuated lattice, so it admits a unique expansion to a complete action lattice  $\mathcal{A}'$  in which  $a^*$  equals the l.u.b. of all  $a^n$ ,  $n \in \omega$ . To avoid collision of symbols, we denote the latter operation by  $(^*)$ . By (WA1),  $a^{(*)} \leq a^*$ , for all  $a \in \mathcal{A}$ .

Let  $\mathcal{A}$  be a finite weak action lattice, and let  $f$  be the assignment in  $\mathcal{A}$ . The extension of  $f$  to all formulas fulfills  $f(\varphi^*) = (f(\varphi))^*$ , where  $*$  denotes the

weak  $*$ -operation in  $\mathcal{A}$ . Since  $\mathcal{A}$  is finite, then it is complete, so one can define a unique operation  $(*)$ , as in the preceding paragraph. Let  $g$  be the assignment fulfilling  $g(p) = f(p)$ , for variables  $p$ , which is extended to a homomorphism from the algebra of formulas to  $\mathcal{A}'$ , using the clause  $g(\varphi^*) = (g(\varphi))^*$ .

**Lemma 9** *If  $\varphi$  contains no negative occurrence of  $*$ , then  $g(\varphi) \subseteq f(\varphi)$ . If  $\varphi$  contains no positive occurrence of  $*$ , then  $f(\varphi) \subseteq g(\varphi)$ .*

*Proof.* Induction on formula  $\varphi$ . If  $\varphi$  is atomic, then  $f(\varphi) = g(\varphi)$ . For  $\varphi \equiv \psi \circ \chi$ , where  $\circ$  is a binary connective, one uses the induction hypothesis and monotonicity conditions for residuated lattices. Let  $\varphi \equiv \psi^*$ . Assume that  $\psi^*$  contains no negative occurrences of  $*$ . Then,  $\psi$  contains no negative occurrences of  $*$ . By the induction hypothesis,  $g(\psi) \subseteq f(\psi)$ . We get:

$$g(\psi^*) = (g(\psi))^* \subseteq (f(\psi))^* \subseteq (f(\psi))^*,$$

where the last inequality follows from (WA2). Since  $\psi^*$  contains at least one positive occurrence of  $*$ , we are done. Q.E.D.

**Theorem 5** *ACT $\omega$  possesses FMP.*

*Proof.* Assume that  $\Gamma \vdash \varphi$  is not provable in ACT $\omega$ . By (\*E), there exists  $n \in \omega$  such that  $P_n(\Gamma) \vdash N_n(\varphi)$  is not provable in ACT $\omega$ . By lemma 8,  $P_n(\Gamma) \vdash N_n(\varphi)$  is not provable in WACT. Then, by theorem 4, there exist a finite weak action lattice  $\mathcal{A}$  and an assignment  $f$  such that  $f(P_n(\Gamma)) \not\subseteq f(N_n(\varphi))$ . We consider the finite action lattice  $\mathcal{A}'$ , which is the only expansion of the  $*$ -free reduct of  $\mathcal{A}$ , and an assignment  $g$ , defined as before lemma 9. By lemma 9,  $f(\psi) \subseteq g(\psi)$ , for any formula  $\psi$  appearing in  $P_n(\Gamma)$ , and  $g(N_n(\varphi)) \subseteq f(N_n(\varphi))$ . Consequently,  $g(P_n(\Gamma)) \not\subseteq g(N_n(\varphi))$ . As we have observed in section 3 (under (\*E)),  $P_n(\psi) \vdash \psi$  and  $\psi \vdash N_n(\psi)$  are in ACT $\omega$ , for any formula  $\psi$ , which yields  $g(P_n(\Gamma)) \subseteq g(\Gamma)$  and  $g(\varphi) \subseteq g(N_n(\varphi))$ . Therefore,  $g(\Gamma) \not\subseteq g(\varphi)$ . Q.E.D.

It is obvious that theorems 4 and 5 remain true for algebras without  $\wedge$  and the corresponding systems; in particular, the  $\wedge$ -free fragment of ACT $\omega$  possesses FMP, this means, it is complete with respect to finite action algebras.

To provide sequent systems, we have considered sequents  $\Gamma \vdash \varphi$  which correspond to atomic formulas  $\alpha \leq \beta$ . All results remain true for equalities  $\alpha = \beta$ , since  $\alpha \leq \beta$  is an equality, and conversely,  $\alpha = \beta$  is equivalent to  $\alpha \leq \beta \& \beta \leq \alpha$ .

Let ACT $*$  denote the class of  $*$ -continuous action algebras (lattices) and ACT $^f$  the class of finite action algebras (lattices). It follows from theorem 5 that  $\text{HSP}(\text{ACT}^*) = \text{HSP}(\text{ACT}^f)$ , this means, the two classes generate the same variety.

At the end, we explain how to prove theorem 5 for simple Horn theories of Kleene algebras (lattices) or action algebras (lattices). Let  $Y$  be a finite set of new axioms of the form  $p_1, \dots, p_n \vdash p$  such that  $n \geq 1$  and  $p_i, p$  are variables. Let  $X$  be the smallest set of sequents of this form which contains  $Y$  and is closed under (CUT). We have noticed in section 3 that ACT $\omega(X)$  admits cut-elimination, (\*E) is true for systems of that kind. The constructions and proofs

of section 4 can be repeated with slight changes. The set  $S$  should additionally contain all variables appearing in sequents from  $Y$ . Since  $\text{ACT}\omega(X)$  is closed under (CUT), then it is closed under the following rule: from  $\Gamma_1 \vdash p_1, \dots, \Gamma_n \vdash p_n$  infer  $\Gamma_1, \dots, \Gamma_n \vdash p$ , for any axiom  $p_1, \dots, p_n \vdash p$  from  $Y$ . We add these rules to inference rules regarded in the construction of  $T$ . Then, the relation  $\preceq$ , defined as above, is closed under these new rules. Since  $f(p) = g(p) = [p]$ , then axioms from  $Y$  are true both in model  $(\mathcal{A}, f)$  and in model  $(\mathcal{A}', g)$ .

## References

- [1] Bar-Hillel, Y., Gaifman, H., Shamir, E, ‘On categorial and phrase structure grammars’, *Bulletin Res. Council Israel* F9 (1960), 155-166.
- [2] Belardinelli, F., Jipsen, P., Ono, H., ‘Algebraic aspects of cut elimination’, *Studia Logica* 77 (2004), 209-240.
- [3] Buszkowski, W., ‘The equivalence of unidirectional Lambek categorial grammars and context-free grammars’, *Zeitschrift f. math. Logik und Grundlagen der Mathematik* 31 (1985), 369-384.
- [4] Buszkowski, W., ‘Completeness results for Lambek Syntactic Calculus’, *Zeitschrift f. math. Logik und Grundlagen der Mathematik* 32 (1986), 13-28.
- [5] Buszkowski, W., ‘The Finite Model Property for BCI and Related Systems’, *Studia Logica* 57 (1996), 303-323.
- [6] Buszkowski, W., ‘Mathematical Linguistics and Proof Theory’, in: *Handbook of logic and language*, (van Benthem, J., ter Meulen, A., eds.), Elsevier, Amsterdam, 1997, 683-736.
- [7] Buszkowski, W., ‘Finite Models of Some Substructural Logics’, *Mathematical Logic Quarterly* 48 (2002), 63-72.
- [8] Hopcroft, J.E., Ullman, J.D., *Introduction to Automata Theory, Languages and Computation*, Addison-Wesley, Reading, 1979.
- [9] Jipsen, P., ‘From Semirings to Residuated Kleene Algebras’, *Studia Logica* 76 (2004), 291-303.
- [10] Kozen, D., ‘On Kleene algebras and closed semirings’, in: *Proc. MFCS 1990, Lecture Notes in Comp. Science* 452, 1990, 26-47.
- [11] Kozen, D., ‘A completeness theorem for Kleene algebras and the algebra of regular events’, in: *Proc. 6th Symp. Logic in Comp. Sci.*, IEEE, 1991, 214-225.
- [12] Kozen, D., ‘A Completeness Theorem for Kleene Algebras and the Algebra of Regular Events’, *Information and Computation* 110:2 (1994), 366-390.

- [13] Kozen, D., ‘On Action Algebras’, manuscript, presented at: *Logic and the Flow of Information*, Amsterdam, 1991.
- [14] Kozen, D. ‘On the complexity of reasoning in Kleene algebra’, *Information and Computation* 179 (2002), 152-162.
- [15] Lambek, J., ‘The mathematics of sentence structure’, *The American Mathematical Monthly* 65 (1958), 154-170.
- [16] Moortgat, M., ‘Categorical Type Logics’, in: *Handbook of logic and language*, (van Benthem, J., ter Meulen, A., eds.), Elsevier, Amsterdam, 1997, 93-177.
- [17] Okada, M., ‘Phase semantic cut-elimination and normalization proofs in first- and higher-order linear logic’, *Theoretical Computer Science* 227:1-2 (1999), 333-396.
- [18] Okada, M., Terui, K., ‘The finite model property for various fragments of intuitionistic linear logic’, *Journal of Symbolic Logic* 64:2 (1999), 780-802.
- [19] Ono, H., ‘Semantics for Substructural Logics’, in: [24], 259-291.
- [20] Palka, E., ‘On Finite Model Property of the Equational Theory of Kleene Algebras’, *Fundamenta Informaticae* 68.3 (2005), 221-230.
- [21] Palka, E., ‘An infinitary sequent system for the equational theory of \*-continuous action lattices’, manuscript.
- [22] Pratt, V., ‘Action Logic and Pure Induction’, in: *Logics in AI. Proc. of JELIA’90, Lecture Notes in Artif. Intelligence* 478, 1991, 97-120.
- [23] Restall, G., *An Introduction to Substructural Logics*, Routledge, London and New York, 2000.
- [24] *Substructural Logics*, (Došen, K., Schroeder-Heister, P., eds.), Clarendon Press, Oxford, 1993.
- [25] van Benthem, J., *Essays in Logical Semantics*, D. Reidel, Dordrecht, 1986.
- [26] van Benthem, J., *Language in Action. Categories, Lambdas and Dynamic Logic*, North-Holland, Amsterdam, 1991.