Phase Spaces for Involutive Nonassociative Lambek Calculus

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Published in: Proc. Workshop on Logic and Algorithms in Computational Linguistics (LACompLing-2017), Stockholm University, 2017, pp. 21–45.

Abstract

Involutive Nonassociative Lambek Calculus is a nonassociative version of Noncommutative MLL [1], but the multiplicative constants are not admitted. It extends Nonassociative Lambek Calculus (NL) by two linear negations. InNL is a strongly conservative extension of NL. We define and study phase spaces for InNL. Using them, we prove the cut-elimination theorem for a one-sided sequent system for InNL; in fact, we prove the completeness theorem for a cut-free system. In a similar way, we prove completeness and cut elimination for an auxiliary system InNL(k), where $k \geq 2$ is even, an analogue of CNL from [22, 12]; CNL amounts to InNL(2). These proofs also work for richer systems with additives. Using InNL(k), we show that InNL is P-TIME (another proof was given in [13]) and the type grammars based on InNL generate the ($\epsilon-$free) context-free languages. These results remain true for the version of InNL with multiplicative constants.

1 Introduction and preliminaries

Type grammars (categorial grammars) are formal grammars which describe a language by assigning types to lexical atoms (words). Parsing is based on a fixed type logic, independent of the particular language. This embodies the paradigm of *parsing as deduction*. Types are formulas of the given type logic. Natural deduction systems for type logics strictly correspond to lambda terms in different versions of the lambda calculus (via the Curry-Howard isomorphism), which naturally connects proof-theoretic semantics with referential semantics; see van Benthem [7], Moortgat [28], Morrill [30].

The so-called Basic Categorial Grammars (BCGs), originated by Ajdukiewicz [3] and Bar-Hillel [4, 5], employ a simple redution procedure, based on the application laws $A, A\backslash B \Rightarrow B$ and $A/B, B \Rightarrow A$. This purely applicative logic was extended by Lambek [24] to Syntactic Calculus, admitting the connectives $\otimes$ (product), $\backslash$ (right division), and $/$ (left division), which satisfy the residuation rules: $A \otimes B \Rightarrow C$ iff $B \Rightarrow A\backslash C$ iff $A \Rightarrow C/B$, and the associative law for $\otimes$. The application laws take the forms $A \otimes (A\backslash B) \Rightarrow B$ and $(A/B) \otimes B \Rightarrow A$. One obtains new laws, e.g. $A \Rightarrow (B/A)\backslash B$, $A \Rightarrow B/(A\backslash B)$ (type raising), $A \Rightarrow B\backslash(B \otimes A)$, $A \Rightarrow (A \otimes B)/B$ (co-application), and $(A\backslash B) \otimes (B\backslash C) \Rightarrow A\backslash C$, $(A/B) \otimes (B/C) \Rightarrow A/C$ (composition).

Standard models for Syntactic Calculus are algebras of $\epsilon-$free languages on an alphabet (lexicon) Σ; $\Rightarrow$ is interpreted as set inclusion and $\otimes, \backslash, /$ as follows.

$$L_1 \cdot L_2 = \{uv : u \in L_1, v \in L_2\}$$

$$L_1\backslash L_2 = \{v \in \Sigma^+ : L_1 \cdot \{v\} \subseteq L_2\},\ L_1/L_2 = \{u \in \Sigma^+ : \{u\} \cdot L_2 \subseteq L_1\}$$

For example, if s, pn, and n are the categories of sentence, proper noun, and common noun, respectively (understood as some sets of strings), then $pn\backslash s$ can be interpreted as the category of verb phrase, $(pn\backslash s)/pn$ - transitive verb phrase, $np = s/(pn\backslash s)$ - noun phrase, and np/n - determiners. So the expression 'John admires Jane' can be assigned s, since the sequent

$$pn, (pn\backslash s)/pn, pn \Rightarrow s$$

is provable in L (in fact, in the purely applicative logic), whereas 'he admires her' leads to the sequent

$$s/(pn\backslash s), (pn\backslash s)/pn, (s/pn)\backslash s \Rightarrow s$$

provable in L with the aid of the composition laws.

These plane examples ignore fine aspects of grammar (tense, number, agreement etc.). We refer the reader to [28, 30, 31] for a deeper linguistic material and other interpretations, e.g. type-theoretic semantics.

Nowadays Syntactic Calculus is called Lambek Calculus (L) and its nonassociative version, due to Lambek [25], Nonassociative Lambek Calculus (NL). (The latter does not support the composition laws.) Both L and NL are treated as basic type logics for type grammars; L parses expressions represented as strings, whereas NL corresponds to phrase structures. One also considers different extensions of these logics, e.g. with multiplicative constants 1, 0, additives $\wedge, \vee, \top, \bot$, unary modal operators $\Diamond$, $\Box$, their residuals $\Box^{\downarrow}$, $\Diamond^{\downarrow}$, and others.

Product (also called multiplicative conjunction) need not satisfy $A \Rightarrow A \otimes A$ (contraction), $A \otimes B \Rightarrow A$, $A \otimes B \Rightarrow B$ (weakening), nor $A \otimes B \Rightarrow B \otimes A$ (exchange). These laws correspond to three structural rules, admitted in standard sequent systems for classical and intuitionistic logics. The logics that omit at least some of these rules are called *substructural logics* [19]. Some authors define substructural logics as axiomatic and rule extensions of L with 1, $\wedge$, $\vee$ (Full Lambek Calculus), but weaker systems may also be counted to this family. It contains many well-known non-classical logics, e.g. many-valued logics, fuzzy logics, relevance logics, with intuitionistic and classical logics in the limit.

With exchange, one proves $A\backslash B \Leftrightarrow B/A$ (i.e. both $\Rightarrow$ and $\Leftarrow$) and writes $A \to B$ for $A\backslash B$. Thus, implication $\to$ is a residual operation for the commutative product. If $\otimes$ is not commutative, then $\to$ splits in two implications $\backslash$ and $/$, also written as $\to$ and $\leftarrow$; hence we call them the right and the left implication, or division, respectively.

With constant 0, one defines negation $\neg A = A \to 0$ or, for the non-commutative case, the right negation $\sim A = A\backslash 0$ and the left negation $-A = 0/A$. Assuming the double negation law $\neg\neg A \Rightarrow A$ or $\sim -A \Rightarrow A$, $- \sim A \Rightarrow A$ (the converse laws are provable), one obtains the so-called 'classical' versions of these logics. This term is misleading, since they remain non-classical in general, i.e. essentially weaker than classical logic. Therefore in this paper we use the term 'involutive', after [19, 18].

The term 'substructural logic' appeared in the literature in the early 1990-ties. Earlier, Girard [20] introduced linear logics: Propositional Linear Logic is equivalent to Full Lambek Calculus with exchange, 'classical' negation and exponentials !, ? (unary modalities allowing the execution of structural rules for modal formulas). The fragment without exponentials is called Multiplicative-Additive Linear Logic (MALL) and MALL without $\wedge, \vee$ Multiplicative Linear Logic (MLL). Noncommutative MALL with one negation $\neg A = A\backslash 0 = 0/A$ was studied by Yetter [35] and with two negations by Abrusci [1]. The latter was called (Classical) Bilinear Logic by J. Lambek; see [26]. The former is also referred to as Cyclic Noncommutative MALL [2].

The present paper does not use Girard's original notation, which is non-orthodox (though widely adopted in the linear logic community). For example, Girard writes & for our $\wedge$ but $\oplus$ for our $\vee$, $\top$ for the top element but 0 for the bottom element (our $\perp$), 1 for the multiplicative unit (the unit element for $\otimes$) but $\perp$ for its dual (our 0, i.e. the unit element for the dual product). The notation used here is standard in the literature on substructural logics except that in what follows we write $A^{\sim}$ for $\sim A$ and A^{-} for $-A$, just to shorten formulas (linear logicians write $A^{\perp}$ and ${}^{\perp}A$). In particular, $\oplus$ denotes here dual product (par), defined by: $A \oplus B = (B^{\sim} \otimes A^{\sim})^{-}$.

From the very beginning, linear logics were regarded as logics closely related to type logics of type grammars. MALL is a conservative extension of (Full) Lambek Calculus with exchange, and both noncommutative versions of MALL are conservative extensions of (Full) Lambek Calculus [2]. Often Lambek calculi are characterized as the intuitionistic fragments of the corresponding linear logics.

Due to conservativity, every type grammar based on L with 1 can be translated into a grammar based on a linear logic, and the latter has precisely the same expressive and generative power as the former. Some authors directly apply noncommutative linear logics in type grammars; see e.g. [14, 15]. Pregroup grammars, due to Lambek [27], are based on Compact Bilinear Logic, i.e. an extension of Bilinear Logic, where $\otimes$ and $\oplus$ (hence also 1 and 0) collapse. Proof nets (a graph-theoretic representation of proofs in MLL) were adapted for noncommutative linear logics and Lambek calculi and used as logical forms of expressions [31]. The role of exponentials in PLL stimulated the study of multi-modal extensions of L and NL, where different modal operators (modes) are introduced for a controlled usage of structural rules (associativity, exchange); see [28, 31].

All linear logics, mentioned above, admit the associative law for $\otimes$. Nonassociative linear logics were not studied so extensively. de Groote and Lamarche [22] introduced Classical Nonassociative Lambek Calculus (CNL), being a nonassociative version of Cyclic Noncommutative MLL. CNL does not admit multiplicative constants. Their work is purely proof-theoretic, focusing on proof nets for CNL. They, however, present a one-sided sequent system for CNL, prove the cut-elimination theorem (using proof nets) and the P-TIME complexity of CNL; they also show that CNL is a conservative extension of NL.

[12] continues this research. It defines phase spaces for CNL and employs them to prove cut elimination for a sequent system (dual to that of [22]) and the strong finite model property for CNL. It strengthens some results of [22]: (1) CNL is a strongly conservative extension of NL, (2) the finitary consequence relation for CNL is P-TIME. [12] also shows (3): the type grammars based on CNL generate the $\epsilon-$free context-free languages. The proof of (1) employs a simple construction of a phase space and can easily be adapted for richer logics (with additives, multiplicative constants etc.). In particular, one can prove in this way that both versions of noncommutative MALL are strongly conservative extensions of Full Lambek Calculus; [2] shows the weak conservativity by more involved proof-theoretic tools.

Phase spaces are certain frames (like Kripke frames for modal logics), used as basic models for linear logics. For nonassociative logics without multiplicative constants they can be defined more generally, which makes their relation to modal frames more explicit; see Section 2.

Recall that a logic possesses the finite model property, if it is complete with respect to its finite models, and the strong finite model property, if it is strongly complete with respect to its finite models. Models can be understood as abstract algebras (see below). The (resp. strong) completeness means that the formulas (or sequents) provable (resp. from a fixed set of assumptions) in this logic are precisely those which are valid in all models of this logic (resp. true in all models for any valuation which satisfies the assumptions). Let $\mathcal{L}$ and $\mathcal{L}'$ be logics such that the language of $\mathcal{L}'$ be richer than that of $\mathcal{L}$. $\mathcal{L}'$ is called a conservative extension of $\mathcal{L}$, if any formula (or sequent) in the language of $\mathcal{L}$ is provable in $\mathcal{L}$ if and only if it is provable in $\mathcal{L}'$, and a strongly conservative extension of $\mathcal{L}$, if this also holds for the provability from assumptions (in the language of $\mathcal{L}$). The finitary consequence relation means the provability from a finite set of assumptions.

The completeness shows that our logic adequately describes the intended reality and, for logics considered here, the (resp. strong) finite model property implies the decidability of (resp. the finitary consequence relation for) the given logic. If we know that the logic is decidable, then we may estimate its complexity, and we go this way in [12] and here.

In the present paper we study a weaker logic InNL, i.e. NL augmented with two negations $^{\sim}, ^{-}$, satisfying the following laws.

$$A^{\sim -} \Leftrightarrow A\,,\ A^{-\sim} \Leftrightarrow A \text{ (the double negation laws)} \tag{1}$$

$$A^{\sim}/B \Leftrightarrow A\backslash B^{-} \text{ (the contraposition law)} \tag{2}$$

CNL amounts to InNL with $A^{\sim} = A^{-}$. InNL is a nonassociative version of Noncommutative MLL [1]. CNL and InNL do not admit the multiplicative constants; the corresponding algebras need not have unit elements for product and dual product. Their extensions with these constants are denoted here by CNL1 and InNL1. Following the terminology of [19], CNL may also be called Cyclic InNL (CyInNL).

Although CNL and InNL are similar in many aspects, there is an essential difference: CNL possesses the strong finite model property, but InNL does not [12]. For $A^{\sim} \Rightarrow A^{-}$ entails $A^{-} \Rightarrow A^{\sim}$ in finite models (for any fixed A) but not in arbitrary models. Accordingly, not all methods applicable to CNL can be adapted for InNL. In particular, one can form finite sets of formulas closed under the CNL-negation (up to logical equivalence), which is impossible for the InNL-negations. We prove here that the pure InNL is P-TIME, but the complexity of the (finitary) consequence relation for InNL is left as an open problem.

InNL1 with additive (extensional) connectives $\wedge, \vee$ amounts to InGL (involutive groupoid logic) from [18]; it is a conservative extension of InNL1. [18] provides a two-sided sequent system for InGL, admitting cut elimination. By the subformula property, this system restricted to multiplicative connectives and constants is good for InNL1. We, however, present a simpler, one-sided sequent system, analogous to the system in [1] for Noncommutative MLL. This system seems more convenient for proof search and for proving metalogical theorems (decidability, finite model property).

Like in [12] for CNL, we define phase spaces for InNL and apply them in the proofs of our results, e.g. the cut-elimination theorem. They are also used to prove that the provability in InNL can be reduced to that in an auxiliary system InNL(k), whose (finitary) consequence relation is P-TIME; the type grammars based on InNL(k) generate the context-free languages. This yields the P-TIME complexity of InNL and the equivalence with context-free grammars.

The forthcoming paper [13] studies InNL by purely proof-theoretic tools. It contains a syntactic proof of the cut-elimination theorem and another proof of P-TIME comnplexity, not involving InNL(k). Phase spaces are not considered at all. The present paper fills up this gap.

Analogous results were earlier obtained for NL and its extensions (with additives, classical connectives, intuitionistic connectives, multiplicative constants, modal operators), using similar methods; see [9, 10]. Logics with linear negation(s), however, require some refinements, elaborated in [12, 13] and here. Let us note that NL with $\wedge, \vee$ is coNP-hard [11], and the finitary consequence relation for this logic is undecidable [16]; it remains decidable, if the distributive laws for $\wedge, \vee$ are assumed [10].

Now, we briefly describe the algebraic models of our logics.

The algebraic models of NL are *residuated groupoids*, i.e. (ordered) algebras $\mathbf{M} = (M, \otimes, \backslash, /, \leq)$ such that $(M, \leq)$ is a poset, and $\otimes, \backslash, /$ are binary operations on M, satisfying *the residuation laws*:

$$a \otimes b \leq c \text{ iff } b \leq a\backslash c \text{ iff } a \leq c/b$$

for all $a, b, c \in M$. A residuated groupoid $\mathbf{M}$ is *unital*, if it contains an element 1 such that $1 \otimes a = a = a \otimes 1$, for any $a \in M$. For any residuated groupoid $\mathbf{M}$, the product $\otimes$ is isotone in both arguments, hence $(M, \otimes, \leq)$ is a p.o. groupoid. We refer to $\backslash, /$ as the residual operations for product.

The algebraic models of InNL are *involutive residuated groupoids*, i.e. algebras $\mathbf{M} = (M, \otimes, \backslash, /, ^{\sim}, ^{-}, \leq)$ such that $(M, \otimes, \backslash, /, \leq)$ is a residuated groupoid, and $^{\sim}, ^{-}$ are antitone (i.e. order-reversing) unary operations on M, satisfying the double negation laws and the contraposition law:

$$\text{(DN) } a^{\sim -} = a = a^{- \sim}\,,$$

$$\text{(CON) } a^{\sim}/b = a\backslash b^{-}$$

for all $a, b \in M$. One easily derives other contraposition laws: $a\backslash b = a^{\sim}/b^{\sim}$, $a/b = a^{-}\backslash b^{-}$. In a unital involutive residuated groupoid, there holds $1^{\sim} = 1^{-}$, since $1^{\sim} = 1^{\sim}/1 = 1\backslash 1^{-} = 1^{-}$ (in any unital residuated groupoid, $a/1 = a = 1\backslash a$). One defines $0 = 1^{\sim}$.

One shows: $(b^{-} \otimes a^{-})^{\sim} = (b^{\sim} \otimes a^{\sim})^{-}$, for all a, b, and defines the dual product: $a \oplus b = (b^{-} \otimes a^{-})^{\sim}$ (also called *par* in the literature on linear logics). If $1 \in M$, then $0 \oplus a = a = a \oplus 0$. One obtains: $a\backslash b = a^{\sim} \oplus b$, $a/b = a \oplus b^{-}$. Hence $a\backslash b = (b^{-} \otimes a)^{\sim}$, $a/b = (b \otimes a^{\sim})^{-}$. In unital involutive residuated groupoids, there hold: $a^{\sim} = a\backslash 0$, $a^{-} = 0/a$.

An involutive residuated groupoid is said to be *cyclic*, if $a^{\sim} = a^{-}$, for any element a. Cyclic involutive residuated groupoids are algebraic models of CNL; see [12].

A residuated semigroup is an associative residuated groupoid (i.e. $\otimes$ is associative), and *a residuated monoid* is a unital residuated semigroup. One also considers algebras with operations $\wedge, \vee$ (meet and join), satisfying the lattice laws. *A residuated lattice* is a lattice-ordered residuated monoid. Residuated lattices are the algebraic models of Full Lambek Calculus.

Now, we present NL and InNL as intuitionistic sequent systems. The systems with negation(s) do not admit cut elimination.

The formulas of NL are built from variables $p, q, r, \ldots$ by means of connectives $\otimes, \backslash, /$. One defines *bunches*: (i) every formula is a bunch, (ii) if Γ and Δ are bunches, then (Γ, Δ) is a bunch. Bunches can be treated as the elements of the free groupoid generated by the set of formulas. *NL-sequents* are of the form $\Gamma \Rightarrow A$, where Γ is a bunch and A is a formula. *A context* is a bunch containing a special atom x (a place for substitution). We denote formulas by $A, B, C, \ldots$, bunches by Γ, Δ, Θ, and contexts by $\Gamma[\,], \Delta[\,]$ etc. $\Gamma[\Delta]$ denotes the result of substituting Δ for x in $\Gamma[\,]$. The axioms and the inference rules of NL are as follows.

$$\text{(NL-id) } A \Rightarrow A \quad \text{(NL-cut) } \frac{\Gamma[A] \Rightarrow B \quad \Delta \Rightarrow A}{\Gamma[\Delta] \Rightarrow B}$$

$$(\otimes \Rightarrow) \ \frac{\Gamma[(A, B)] \Rightarrow C}{\Gamma[A \otimes B] \Rightarrow C} \quad (\Rightarrow \otimes) \ \frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{(\Gamma, \Delta) \Rightarrow A \otimes B}$$

$$(\backslash \Rightarrow) \ \frac{\Gamma[B] \Rightarrow C \quad \Delta \Rightarrow A}{\Gamma[(\Delta, A \backslash B)] \Rightarrow C} \quad (\Rightarrow \backslash) \ \frac{(A, \Gamma) \Rightarrow B}{\Gamma \Rightarrow A \backslash B}$$

$$(/ \Rightarrow) \ \frac{\Gamma[A] \Rightarrow C \quad \Delta \Rightarrow B}{\Gamma[(A/B, \Delta)] \Rightarrow C} \quad (\Rightarrow /) \ \frac{(\Gamma, B) \Rightarrow A}{\Gamma \Rightarrow A/B}$$

This sequent system is due to Lambek [25]. NL1 is obtained by admitting the empty bunch ϵ, satisfying $(\epsilon, \Gamma) = \Gamma = (\Gamma, \epsilon)$, and the constant 1 (an atom) with two new rules and one new axiom.

$$(1 \Rightarrow) \ \frac{\Gamma[\Delta] \Rightarrow A}{\Gamma[(1, \Delta)] \Rightarrow A} \quad \frac{\Gamma[\Delta] \Rightarrow A}{\Gamma[(\Delta, 1)] \Rightarrow A} \quad (\Rightarrow 1) \ \epsilon \Rightarrow 1$$

We write $\Rightarrow A$ for $\epsilon \Rightarrow A$. NL1 is not a conservative extension of NL; $p/(q/q) \Rightarrow p$ is provable in NL1 but not in NL. Lambek [25] proved the cut-elimination theorem: every sequent provable in NL is provable without (NL-cut). His standard, syntactic proof can easily be adapted for NL1.

InNL (resp. InNL1) can be presented as an extension of NL (resp. NL1) with new unary connectives $^{\sim}, ^{-}$, new axioms (1), (2), and new inference rules:

$$\text{(r-CON) } \frac{A \Rightarrow B}{B^{\sim} \Rightarrow A^{\sim}} \quad \frac{A \Rightarrow B}{B^{-} \Rightarrow A^{-}}.$$

Notice that in InNL1 (r-CON) can be omitted; they are derivable in the resulting system. CNL (resp. CNL1) can be obtained from InNL (resp. InNL1) by adding the axioms $A^{\sim} \Leftrightarrow A^{-}$.

One easily shows that NL is strongly complete with respect to residuated groupoids, NL1 with respect to unital residuated groupoids, and InNL (resp. InNL1) with respect to (resp. unital) involutive residuated groupoids. Recall that *a valuation* in an algebra $\mathbf{M}$ is a homomorphism from the formula algebra to $\mathbf{M}$. It is extended for bunches by interpreting each comma as product. If the empty bunch is admitted, it is interpreted as 1. $\Gamma \Rightarrow A$ is *true* for a valuation μ in $\mathbf{M}$ if $\mu(\Gamma) \leq \mu(A)$.

[12] shows that CNL is *a strongly conservative extension* of NL. Also CNL1 is a strongly conservative extension of NL1. Since InNL (resp. InNL1) is intermediate between NL (resp. NL1) and CNL (resp. CNL1), then evidently InNL (resp. InNL1) is a strongly conservative extension of NL (resp. NL1).

NL is a basic logic of type grammars: formulas are interpreted as syntactic types of expressions, represented as phrase structures [28].

Now, we can precisely define type grammars. Let $\mathcal{L}$ be a logic, which admits sequents $\Gamma \Rightarrow A$. *A type grammar* (based on $\mathcal{L}$) is defined as a triple $G = (\Sigma, I, A_0)$ such that Σ is a finite alphabet (lexicon), I is a map which assigns a finite set of types (i.e. formulas of $\mathcal{L}$) to each $v \in \Sigma$, and A_0 is a designated type. One says that G *assigns* type A to the string $v_1 \ldots v_n$, where $v_i \in \Sigma$, if there exist types $A_1, \ldots, A_n$ such that $A_i \in I(v_i)$ for $i = 1, \ldots, n$ and $A_1, \ldots, A_n \Rightarrow A$ (for nonassociative logics, one adds: under some bracketing of $A_1, \ldots, A_n$). *The language of* G, denoted by $L(G)$, consists of all $a \in \Sigma^+$ such that G assigns A_0 to a. Two classes of grammars are said to be (weakly) *equivalent*, if they yield the same classes of languages.

Several basic classes of type grammars are equivalent to the class of $\epsilon-$free context-free grammars, e.g. Basic Categorial Grammars, the type grammars based on NL, NL1, L, L1, NL with operations of a distributive lattice, CNL, pregroup grammars, and others (also InNL, InNL1, as it will be shown here). This, however, does not mean that type grammars can be identified with context-free grammars. Usually, the former are not strongly equivalent to the latter, e.g. they do not yield the same phrase structure languages. Furthermore, type grammars (except pregroup grammars) are closely related to type-theoretic semantics. Nonetheless, due to this equivalence, different efficient parsing methods for context-free grammars can be applied to type grammars.

Moortgat [29] considers an extension of NL with $\oplus$, its (dual) residuals and some mixed associativity and commutativity axioms, due to Grishin [21], under the name Lambek-Grishin Calculus (LG). The corresponding type grammars are called symmetric type grammars. [6] contains a detailed study of these grammars; one section discusses InNL, called there Nonassociative Bilinear Logic. In fact, all new axioms of LG are provable in the associative and commutative version of InNL. Let us note that in InNL with Grishin axioms:

$$A \otimes (B \oplus C) \Rightarrow (A \otimes B) \oplus C \text{ and } (A \oplus B) \otimes C \Rightarrow A \oplus (B \otimes C)$$

one proves the associative laws for $\otimes$ and $\oplus$, hence the resulting logic amounts to Noncommutative MLL without constants.

2 Phase spaces

By *a phase space* we mean a frame $(M, \cdot, R)$ such that $(M, \cdot)$ is a groupoid and $R \subseteq M^2$. For $X \subseteq M$ we define:

$$X^{\sim} = \{b \in M : \forall_{a \in X} R(a, b)\},\ X^{-} = \{a \in M : \forall_{b \in X} R(a, b)\}.$$

This yields a Galois connection: $X \subseteq Y^{\sim}$ iff $Y \subseteq X^{-}$, for all $X, Y \subseteq M$. Consequently, ${}^{\sim}, {}^{-}$ reverse $\subseteq$ and $X^{\sim-\sim} = X^{\sim}$, $X^{-\sim-} = X^{-}$, for any $X \subseteq M$. The operations $\phi_R(X) = X^{-\sim}$ and $\psi_R(X) = X^{\sim-}$ are closure operations on $\mathcal{P}(M)$. Recall that an operation C on $\mathcal{P}(M)$ is called *a closure operation*, if it satisfies: (C1) $X \subseteq C(X)$, (C2) if $X \subseteq Y$ then $C(X) \subseteq C(Y)$, (C3) $C(C(X)) = C(X)$, for all $X, Y \subseteq M$. A set $X \subseteq M$ is said to be $C-$*closed*, if $X = C(X)$. A set X is ϕ_R-closed (resp. ψ_R-closed) if and only if $X = Y^{\sim}$ (resp. $X = Y^{-}$) for some $Y \subseteq M$.

Given a groupoid $(M, \cdot)$ and $X, Y \subseteq M$, one defines:

$$X \cdot Y = \{a \cdot b : a \in X, b \in Y\},$$

$$X \backslash Y = \{b \in M : X \cdot \{b\} \subseteq Y\},\ X/Y = \{a \in M : \{a\} \cdot Y \subseteq X\}.$$

$\mathcal{P}(M)$ with these operations and $\subseteq$ is a residuated groupoid.

The symbols ϕ_R, ψ_R appeared in Abrusci [1] who studied phase spaces for Noncommutative MALL. Like for other linear logics, his phase space is of the form $(M, \cdot, O)$, where $O \subseteq M$ (linear logicians write $\perp^{\bullet}$ for our O). One can define $R_O = \{(a, b) \in M^2 :\ a \cdot b \in O\}$. Then $X^{\sim} = X \backslash O$, $X^{-} = O/X$. Our notion is more general and natural for logics without multiplicative constants. [12] provides an example of a commutative, associative phase space $(M, \cdot, R)$ for CNL (see below) such that $R \neq R_O$, for any $O \subseteq M$. However, if $(M, \cdot)$ is a free groupoid, then always $R = R_O$ for $O = \{a \cdot b : R(a, b)\}$.

A closure operation C on $\mathcal{P}(M)$ is called *a nucleus*, if it satisfies: (C4) $C(X) \cdot C(Y) \subseteq C(X \cdot Y)$, for all $X, Y \subseteq M$. Assuming (C1)-(C3), (C4) is equivalent to: for any $C-$closed set X and any $Y \subseteq M$, the sets $Y \backslash X$ and X/Y are $C-$closed [10].

Let $(M, \cdot)$ be a groupoid and C be a closure operation on $\mathcal{P}(M)$. By M_C we denote the family of $C-$closed subsets of M. M_C is closed under infinite meets (intersections), hence it is a complete lattice. For $X, Y \in M_C$ one defines: $X \otimes_C Y = C(X \cdot Y)$ and $X/Y, X \backslash Y$ as above. If C is a nucleus, then M_C with these operations and $\subseteq$ is a residuated groupoid [19].

A phase space $(M, \cdot, R)$ is called *a phase space for* InNL, if $\phi_R = \psi_R$ and the following condition holds:

(Shift) for all $a, b, c \in M$, $R(a \cdot b, c)$ iff $R(a, b \cdot c)$.

It is called a phase space for CNL, if (Shift) holds and R is symmetric.

For any phase space $(M, \cdot, R)$, R is symmetric if and only if $X^{\sim} = X^{-}$ for any $X \subseteq M$. Consequently, if R is symmetric, then $\phi_R = \psi_R$. So every phase space for CNL is a phase space for InNL.

A frame $(M, \cdot, 1, M)$ such that $(M, \cdot, 1)$ is a unital groupoid and $(M, \cdot, R)$ is a phase space is said to be *unital*. If $(M, \cdot, 1, R)$ is a unital phase space, satisfying (Shift), then $R = R_O$ for $O = \{a \in M : R(a, 1)\}$. Also $\{1\}^{\sim} = \{1\}\backslash O = O$, $\{1\}^{-} = O/\{1\} = O$, hence O is ϕ_R-closed and ψ_R-closed. This shows that our notion of a phase space is really more general just for non-unital phase spaces (not considered in the literature on linear logics).

Lemma 1. *Let $(M, \cdot, R)$ be a phase space. (Shift) is equivalent to each of the following conditions:*

(i) $X^{\sim}/Y = X\backslash Y^{-}$ for all $X, Y \subseteq M$,

(ii) $X\backslash Y^{\sim} = (Y \cdot X)^{\sim}$ for all $X, Y \subseteq M$,

(iii) $X^{-}/Y = (Y \cdot X)^{-}$ for all $X, Y \subseteq M$.

Proof. First, (Shift) is equivalent to (i) restricted to one-element sets: for all $a, b \in M$, $\{a\}^{\sim}/\{b\} = \{a\}\backslash\{b\}^{-}$. We have: $x \in \{a\}^{\sim}/b$ iff $x \cdot b \in \{a\}^{\sim}$ iff $R(a, x \cdot b)$. Also: $x \in \{a\}\backslash\{b\}^{-}$ iff $a \cdot x \in \{b\}^{-}$ iff $R(a \cdot x, b)$. So (Shift) is equivalent to: for all $a, b, x \in M$, $x \in \{a\}^{\sim}/\{b\}$ iff $x \in \{a\}\backslash\{b\}^{-}$.

The restricted (i) follows from (i). Also (i) follows from the restricted (i), since for $X = \{a_i\}_{i \in I}$, $Y = \{b_j\}_{j \in J}$, we obtain:

$$X^{\sim}/Y = (\bigcap_{i \in I}\{a_i\}^{\sim})/Y = \bigcap_{i \in I}\bigcap_{j \in J}\{a_i\}^{\sim}/\{b_j\} = \bigcap_{i \in I}\bigcap_{j \in J}\{a_i\}\backslash\{b_j\}^{-} = X\backslash Y^{-}.$$

Here we use the distributive laws for $\cdot, \backslash, /$ and:

$$(\bigcup_{i \in I} Z_i)^{\sim} = \bigcap_{i \in I} Z_i^{\sim},\ (\bigcup_{j \in J} Z_j)^{-} = \bigcap_{j \in J} Z_j^{-}.$$

So (Shift) is equivalent to (i). The remaining equivalences can be proved in a similar way. $\Box$

If $\phi_R = \psi_R$, then $M_{\phi_R} = M_{\psi_R}$; we denote this family by M_R. Clearly M_R is closed under $^{\sim}, ^{-}$. We write $\otimes_R$ for $\otimes_{\phi_R}$.

Theorem 1. *Let $(M, \cdot, R)$ be a phase space for InNL. Then, ϕ_R is a nucleus and $(M_R, \otimes_R, \backslash, /, ^{\sim}, ^{-}, \subseteq)$, where the operations are restricted to M_R, is an involutive residuated groupoid.*

Proof. Assume $X \in M_R$, $Y \subseteq M$. There exist $Z, U \subseteq M$ such that $X = Z^{\sim} = U^{-}$. By Lemma 1, $X/Y = U^{-}/Y = (Y \cdot U)^{-}$, hence $X/Y \in M_R$. Similarly, $Y\backslash X = Y\backslash Z^{\sim} = (Z \cdot Y)^{\sim}$, hence $Y\backslash X \in M_R$. Consequently, ϕ_R is a nucleus.

For $X \in M_R$, we have: $X^{-\sim} = \phi_R(X) = X$ and $X^{\sim-} = \psi_R(X) = X$, which yields (DN). (CON) follows from Lemma 1. $\Box$

We refer to the algebra M_R, defined above, as *the complex algebra* of the phase space $(M, \cdot, R)$. Let $\mathbf{M} = (M, \cdot, \backslash, /, ^{\sim}, ^{-}, \leq)$ be an involutive residuated groupoid. One defines the (canonical) phase space $(M, \cdot, R)$, by setting: $R(a, b)$ iff $a \leq b^{-}$ (equivalently: $b \leq a^{\sim}$). The mapping $h(a) = \{b \in M : b \leq a\}$ is an embedding of $\mathbf{M}$ into the complex algebra of this canonical phase space. In general, the canonical relation R cannot be represented as R_O for $O \subseteq M$.

An analogous theorem holds for phase spaces for CNL [12]. For a unital phase space $(M, \cdot, 1, R)$, one defines $1_R = \phi_R(\{1\})$. If ϕ_R is a nucleus, then 1_R is a unit for $\otimes_R$. By *a phase space for* InNL1 we mean a unital phase space, satisfying (Shift) and $\phi_R = \psi_R$. It follows from Theorem 1 that for any phase space $(M, \cdot, 1, R)$ for InNL1 the algebra M_R with 1_R and the remaining components as above is a unital involutive residuated groupoid.

Usually, the symmetry of R is easy to verify. It is more difficult to verify $\phi_R = \psi_R$. The following lemma will be used in Section 3; it refines Proposition 1.11 in [1].

Lemma 2. *Let $(M, \cdot, R)$ be a phase space, satisfying (Shift). Let G be a set of generators for M. Then, $\phi_R = \psi_R$ if and only if, for any $x \in G$, the set $\{x\}^{\sim}$ is ψ_R-closed and the set $\{x\}^{-}$ is ϕ_R-closed.*

Proof. The 'only if' part is easy. Assume $\phi_R = \psi_R$. Then, for $X \subseteq M$, $X^{\sim}$ is ϕ_R-closed, hence ψ_R-closed, and X^{-} is ψ_R-closed, hence ϕ_R-closed.

We prove the 'if' part. We show: if X is ϕ_R-closed (resp. ψ_R-closed) and $Y \subseteq M$, then $Y\backslash X$ (resp. X/Y) is ϕ_R-closed (resp. ψ_R-closed). Take $X = Z^{\sim}$. Then $Y\backslash X = (Z \cdot Y)^{\sim}$, by Lemma 1, hence $Y\backslash X$ is ϕ_R =closed. Take $X = Z^{-}$. Then $X/Y = (Y \cdot Z)^{-}$, by Lemma 1, hence X/Y is ψ_R-closed.

Assume the right-hand side of the equivalence. We prove that for any $x \in M$, $\{x\}^{\sim}$ is ψ_R-closed and $\{x\}^{-}$ is ϕ_R-closed. This holds for $x \in G$. Assume that this holds for y, z. We show that this folds for $y \cdot z$. We have: $\{y\}^{\sim} = \psi_R(\{y\}^{\sim})$, hence, using Lemma 1, $\{y \cdot z\}^{\sim} = \{z\}\backslash\{y\}^{\sim} = \{z\}\backslash\{y\}^{\sim\sim-} = \{z\}^{\sim}/\{y\}^{\sim\sim}$. The last set is ψ_R-closed, since $\{z\}^{\sim}$ is so. A similar argument shows that $\{y \cdot z\}^{-}$ is ϕ_R-closed.

We show that every ϕ_R-closed set is ψ_R-closed, and conversely. Let X be ϕ_R-closed. Then $X = Y^{\sim}$ for some Y. We have: $Y^{\sim} = \bigcap_{y \in Y}\{y\}^{\sim}$. Since every set $\{y\}^{\sim}$ is ψ_R-closed, then X is so. The converse can be proved in a similar way.

We have shown $M_{\phi_R} = M_{\psi_R}$. Since $\phi_R(X)$ equals the least ϕ_R-closed set containing X, and similarly for ψ_R, we obtain $\phi_R = \psi_R$. $\square$

This lemma remains true for unital phase spaces. Since the algebra M_R is a complete lattice, we obtain some models for NL, NL1 with additives. If $\cdot$ is associative (commutative), then $\otimes_R$ is so. If $\cdot$ is associative, then (Shift) obviously holds for R_O. If $\cdot$ is commutative and admits 1, then

R_O is symmetrical. Therefore, phase spaces for PLL can be (and have been) defined as quadruples $(M, \cdot, 1, O)$ such that $(M, \cdot, 1)$ is a commutative monoid and $O \subseteq M$.

3 Sequent systems

We present our one-sided sequent system for InNL. Propositional variables are denoted by $p, q, r, \ldots$. Atomic formulas (atoms) are of the form $p^{(n)}$, where p is a variable and $n \in \mathbb{Z}$. One interprets $p^{(n)}$, for $n \geq 0$, as $p^{\sim \cdots \sim}$ with $^\sim$ occurring n times, and for $n < 0$, as $p^{- \cdots -}$ with $^-$ occurring $|n|$ times. The connectives are $\otimes, \oplus$.

The elements of the free groupoid generated by all formulas are called *bunches*. The meta-logical notation is like for NL. *Sequents* are the bunches containing at least two formulas. Often we omit the outer parentheses in sequents.

The axioms of our cut-free system are:

$$\text{(id) } p^{(n)}, p^{(n+1)} \text{ for all variables } p \text{ and } n \in \mathbb{Z}.$$

The inference rules are as follows.

$$\text{(r-}\otimes\text{)} \ \frac{\Gamma[(A, B)]}{\Gamma[A \otimes B]}$$

$$\text{(r-}\oplus 1\text{)} \ \frac{\Gamma[B] \quad \Delta, A}{\Gamma[(\Delta, A \oplus B)]} \quad \text{(r-}\oplus 2\text{)} \ \frac{\Gamma[A] \quad B, \Delta}{\Gamma[(A \oplus B, \Delta)]}$$

$$\text{(r-shift)} \ \frac{(\Gamma, \Delta), \Theta}{\Gamma, (\Delta, \Theta)}$$

The algebraic models are involutive residuated groupoids. A valuation μ must satisfy: $\mu(p^{(n+1)}) = \mu(p^{(n)})^\sim$, for any atom $p^{(n)}$. μ is extended for sequents by: $\mu((\Gamma, \Delta)) = \mu(\Gamma) \otimes \mu(\Delta)$. We define: $\mathbf{M}, \mu \models \Gamma, \Delta$ iff $\mu(\Gamma) \leq \mu(\Delta)^-$ (equivalently: $\mu(\Delta) \leq \mu(\Gamma)^\sim$).

This system is a dual Schütte style system: the sequent Γ can be written as $\Gamma \Rightarrow$, while in Schütte style systems as $\Rightarrow \Gamma$. We prefer the dual style, since it is more compatible with the syntax of intuitionistic systems (see Section 1). A similar system for Bilinear Logic, i.e. an associative version of InNL1 with additives, was presented by Lambek [26].

We denote this system by S-InNL. Negations $^\sim, ^-$ are defined in metalanguage.

$$(p^{(n)})^\sim = p^{(n+1)} \quad (p^{(n)})^- = p^{(n-1)}$$

$$(A \otimes B)^\sim = B^\sim \oplus A^\sim \quad (A \oplus B)^\sim = B^\sim \otimes A^\sim$$

$$(A \otimes B)^- = B^- \oplus A^- \quad (A \oplus B)^- = B^- \otimes A^-$$

One easily proves $A^{\sim -} = A$, $A^{- \sim} = A$, by induction on A. If μ is a valuation in an involutive residuated groupoid, then:

$$\mu(A^{\sim}) = \mu(A)^{\sim}, \;\; \mu(A^{-}) = \mu(A)^{-}, \text{ for any formula } A. \tag{3}$$

(3) can be proved by induction on A. We write $\vdash \Gamma$, if Γ is provable in S-InNL. By induction on derivations in S-InNL one easily proves that (r-$\otimes$) is reversible: if $\vdash \Gamma[A \otimes B]$ then $\vdash \Gamma[(A, B)]$. Also $\vdash A^{-}, A$ and $\vdash A, A^{\sim}$ for any formula A (use induction on A).

With the cut-rules

$$(\text{cut}^{\sim}) \; \frac{\Gamma[A] \;\; \Delta, A^{\sim}}{\Gamma[\Delta]} \quad (\text{cut}^{-}) \; \frac{\Gamma[A] \;\; A^{-}, \Delta}{\Gamma[\Delta]}$$

S-InNL is strongly complete with respect to involutive residuated groupoids (see Theorem 3).

We want to prove that the cut-rules are admissible in S-InNL by a model-theoretic argument. We need an auxiliary system S_0, which arises from S-InNL, after one has replaced (r-shift) with the following rules:

$$(\text{r-}\oplus 3) \; \frac{A, \Gamma \;\; B, \Delta}{A \oplus B, (\Delta, \Gamma)} \quad (\text{r-}\oplus 4) \; \frac{\Gamma, A \;\; \Delta, B}{(\Delta, \Gamma), A \oplus B}$$

(r-$\oplus$3) is derivable in S-InNL, by (r-$\oplus$2) and (r-shift), and (r-$\oplus$4) is derivable, by (r-$\oplus$1) and (r-shift). By $\vdash_0$ we denote the provability in S_0.

In opposition to an analogous result for CNL in [12], for InNL we need two proof-theoretic lemmas. The complete proofs of these lemmas can be found in [13].

Lemma 3. *The rule (r-shift) is admissible in S_0.*

Proof. We prove: $\vdash_0 (\Gamma_1, \Gamma_2), \Gamma_3$ iff $\vdash_0 \Gamma_1, (\Gamma_2, \Gamma_3)$. The only-if part is proved by induction on the proof of $(\Gamma_1, \Gamma_2), \Gamma_3$ in S_0, and the converse implication in a similar way. We only consider the case for (r-$\oplus$2).

We consider two subcases. 1°. The active bunch $(A \oplus B, \Delta)$ occurs in Γ_i for some $1 \leq i \leq 3$. We apply the induction hypothesis. 2°. $(A \oplus B, \Delta) = (\Gamma_1, \Gamma_2)$. Then $\Gamma_1 = A \oplus B$, $\Gamma_2 = \Delta$, and the premises are A, Γ_3 and B, Δ. One derives $A \oplus B, (\Delta, \Gamma_3)$, by (r-$\oplus$3). $\square$

Consequently, $\vdash \Gamma$ if and only if $\vdash_0 \Gamma$, for any sequent Γ. We need two new rules.

$$(\text{r-}^{\sim\sim}) \; \frac{A, \Gamma}{\Gamma, A^{\sim\sim}} \quad (\text{r-}^{--}) \; \frac{\Gamma, A}{A^{--}, \Gamma}$$

Lemma 4. *The rules (r-$^{\sim\sim}$) and (r-$^{--}$) are admissible in S-InNL.*

Proof. It suffices to prove that these rules are admissible in S_0. For (r-$^{\sim\sim}$), we prove: if $\vdash_0 D, \Theta$ then $\vdash_0 \Theta, D^{\sim\sim}$, by the outer induction on the number of connectives in D and the inner induction on the proof of D, Θ in S_0. For (r-$^{--}$) the argument is similar.

Assuming our claim for all D' having less connectives than D, we run the inner induction. If D, Θ is an axiom (id), where $D = p^{(n)}$, $\Theta = p^{(n+1)}$, then $\Theta, D^{\sim\sim}$ equals $p^{(n+1)}.p^{(n+2)}$, which is an axiom, too.

Assume that D, Θ is the conclusion of a rule. If D is not the active formula of the rule (this may only happen for (r-$\otimes$), (r-$\oplus$1), (r-$\oplus$2)), then one applies the (inner) induction hypothesis to the premise which contains D (this must be $\Gamma[B]$ in (r-$\oplus$1) and $\Gamma[A]$ in (r-$\oplus$2)), next applies the same rule.

Assume that D is the active formula of the rule. This may happen for (r-$\otimes$) and (r-$\oplus$3). We only consider (r-$\otimes$). So $D = A \otimes B$ and the premise is $(A, B), \Theta$. By Lemma 3 and the outer induction hypothesis, we obtain $\vdash_0 A, (B, \Theta)$, $\vdash_0 (B, \Theta), A^{\sim\sim}$, $\vdash_0 B, (\Theta, A^{\sim\sim})$, $\vdash_0 (\Theta, A^{\sim\sim}), B^{\sim\sim}$, $\vdash_0 \Theta, (A^{\sim\sim}, B^{\sim\sim})$, hence $\vdash_0 \Theta, D^{\sim\sim}$, by (r-$\otimes$). $\Box$

As a consequence, $\vdash A^{-}, \Gamma$ if and only if $\vdash \Gamma, A^{\sim}$; we write $\Gamma \Rightarrow A$ for A^{-}, Γ. We define:

$$[A] = \{\Gamma : \vdash \Gamma \Rightarrow A\}.$$

A sequent Γ is said to be *valid* in InNL, if $\mathbf{M}, \mu \models \Gamma$ for all involutive residuated groupoids $\mathbf{M}$ and all valuations μ in $\mathbf{M}$. We prove that S-InNL is weakly complete.

Theorem 2. *For any sequent* Γ, $\vdash \Gamma$ *if and only if* Γ *is valid in InNL.*

Proof. It is easy to verify that the axioms (id) are valid and all rules preserve validity (in fact, they preserve the truth for μ in $\mathbf{M}$). We only consider (r-shift) in the top-down direction. Assume $\mathbf{M}, \mu \models (\Gamma, \Delta), \Theta$. Then $\mu(\Gamma) \otimes \mu(\Delta) \leq \mu(\Theta)^{-}$. So $\mu(\Delta) \leq \mu(\Gamma) \backslash \mu(\Theta)^{-} = \mu(\Gamma)^{\sim} / \mu(\Theta)$. We obtain $\mu(\Delta) \otimes \mu(\Theta) \leq \mu(\Gamma)^{\sim}$, which yields $\mathbf{M}, \mu \models \Gamma, (\Delta, \Theta)$. Hence the only-if part holds.

For the 'if' part, we construct a counter-model for any unprovable sequent. We consider a phase space $(M, \cdot, R)$ such that $(M, \cdot)$ is the free groupoid generated by the set of formulas (so $\Gamma \cdot \Delta = (\Gamma, \Delta)$) and: $R(\Gamma, \Delta)$ iff $\vdash \Gamma, \Delta$. Due to (r-shift), this phase space satisfies (Shift).

For any formula A, we have: $[A] = \{\Gamma : R(A^{-}, \Gamma)\} = \{\Gamma : R(\Gamma, A^{\sim})\}$, which yields:

$$[A] = \{A^{-}\}^{\sim} = \{A^{\sim}\}^{-},\ [A^{\sim}] = \{A\}^{\sim},\ [A^{-}] = \{A\}^{-}. \tag{4}$$

Consequently, $[A]$ is ϕ_R−closed and ψ_R−closed, for any formula A. We obtain $\phi_R = \psi_R$, by Lemma 2. So $(M, \cdot, R)$ is a phase space for InNL.

By Theorem 1, the algebra $(M_R, \otimes_R, \backslash, /, ^\sim, ^-, \subseteq)$ is an involutive residuated groupoid. We define a valuation μ.

$$\mu(p) = [p] \text{ where } p = p^{(0)}$$

$$\mu(p^{(n+1)}) = \mu(p^{(n)})^\sim \text{ for } n \geq 0, \quad \mu(p^{(n-1)}) = \mu(p^{(n)})^- \text{ for } n \leq 0$$

By induction on the number of connectives in A, we prove:

$$A \in \mu(A) \subseteq [A] \text{ for any formula } A\,. \tag{5}$$

$A = p^{(n)}$. We proceed by induction on $|n|$. For $n = 0$, we have $\mu(p) = [p]$ and $p \in [p]$, since $p, p^\sim$ is an axiom. Assume (5) for $n \geq 0$. From $\mu(p^{(n)}) \subseteq [p^{(n)}] = \{p^{(n+1)}\}^-$ (use (4)) we obtain $\{p^{(n+1)}\}^{-\sim} \subseteq \mu(p^{(n)})^\sim = \mu(p^{(n+1)})$ (use (3)), and $p^{(n+1)} \in \phi_R(\{p^{(n+1)}\}) = \{p^{(n+1)}\}^{-\sim}$, hence $p^{(n+1)} \in \mu(p^{(n+1)})$. From $\{p^{(n)}\} \subseteq \mu(p^{(n)})$ we obtain $\mu(p^{(n+1)}) \subseteq \{p^{(n)}\}^\sim = [p^{(n+1)}]$ (use (3), (4)). In a similar way, assuming (5) for $n \leq 0$, we obtain (5) for $n - 1$.

$A = B \otimes C$. We need the following property.

(P1) If $X \in M_R$ and $(B, C) \in X$, then $B \otimes C \in X$.

Assume that $X \in M_R$ and $(B, C) \in X$. Fix Y such that $X = Y^\sim$. For all $\Gamma \in Y$, $\vdash \Gamma, (B, C)$, hence $\vdash \Gamma, B \otimes C$, by (r-$\otimes$). So $B \otimes C \in X$.

By the induction hypothesis, (5) holds for B and C. From $B \in \mu(B)$ and $C \in \mu(C)$ we obtain $(B, C) \in \mu(B) \cdot \mu(C) \subseteq \mu(B) \otimes \mu(C) = \mu(B \otimes C)$, hence $B \otimes C \in \mu(B \otimes C)$, by (P1). Also $\mu(B) \subseteq [B] = \{B^-\}^\sim$ and $\mu(C) \subseteq [C] = \{C^-\}^\sim$. Hence, for all $\Gamma \in \mu(B)$, $\vdash B^-, \Gamma$, and for all $\Delta \in \mu(C)$, $\vdash C^-, \Delta$. By (r-$\oplus$3), $\vdash C^- \oplus B^-, (\Gamma, \Delta)$, for all $\Gamma \in \mu(B)$, $\Delta \in \mu(C)$. Consequently, $\mu(B) \cdot \mu(C) \subseteq \{(B \otimes C)^-\}^\sim = [B \otimes C]$, which yields $\mu(B \otimes C) = \phi_R(\mu(B) \cdot \mu(C)) \subseteq [B \otimes C]$, since $[B \otimes C]$ is ϕ_R-closed.

$A = B \oplus C$. Since B and B^- have the same number of connectives, and similarly for C and C^-, then (5) holds for B^- and C^- by the induction hypothesis. So $C^- \otimes B^- \in \mu(C^- \otimes B^-) \subseteq [C^- \otimes B^-]$. We obtain $[C^- \otimes B^-]^\sim \subseteq \mu(C^- \otimes B^-)^\sim \subseteq \{C^- \otimes B^-\}^\sim$, which yields (5) for $B \oplus C$, since $B \oplus C \in \{B \oplus C\}^{-\sim} = [C^- \otimes B^-]^\sim$, $\mu(C^- \otimes B^-)^\sim = \mu(B \oplus C)$ and $\{C^- \otimes B^-\}^\sim = [B \oplus C]$. This finishes the proof of (5).

Assume $\not\vdash \Gamma, \Delta$. From $A \in \mu(A)$, for any formula A, one easily proves $\Theta \in \mu(\Theta)$ for any bunch Θ, by induction on the number of commas in Θ. So $\Gamma \in \mu(\Gamma)$ and $\Delta \in \mu(\Delta)$. By the assumption, $\Gamma \notin \mu(\Delta)^-$, hence $\mu(\Gamma) \not\subseteq \mu(\Delta)^-$. Consequently, Γ, Δ is not valid. $\square$

Corollary 1. *The rules (cut$^\sim$), (cut$^-$) are admissible in S-InNL.*

Proof. Both rules preserve the truth for μ in $\mathbf{M}$, hence the validity. $\square$

This model-theoretic proof of the cut-elimination theorem, similar to the proofs for associative linear logics in [23, 32], is not constructive. A constructive, syntactic proof is given in [13].

Theorem 3. *S-InNL with (cut$^{\sim}$), (cut^{-}) is strongly complete with respect to involutive residuated groupoids.*

Proof. Let Φ be a set of sequents. With the cut-rules, every sequent Γ, Δ is deductively equivalent to a sequent A, B; A arises from Γ by replacing each comma by $\otimes$, and so for B and Δ. We assume that all sequents in Φ are of the latter form.

Now, $\vdash$ denotes the provability in S-InNL with the cut-rules. Notice that (r-$^{\sim\sim}$), (r-$^{--}$) are derivable. For (r-$^{\sim\sim}$), we derive $\Gamma, A^{\sim\sim}$ from $A^{\sim}, A^{\sim\sim}$ and A, Γ, by (cut^{-}).

We consider a phase space $(M, \cdot, R)$, defined as in the proof of Theorem 2 except that: $R(\Gamma, \Delta)$ iff $\Phi \vdash \Gamma, \Delta$. We define $[A] = \{\Gamma : \Phi \vdash \Gamma \Rightarrow A\}$, where $\Gamma \Rightarrow A$ is as above. The equations (4) hold. Accordingly, $(M, \cdot, R)$ is a phase space for InNL. We consider the involutive residuated groupoid M_R and the valuation μ, defined there. One proves (5) and a stronger claim:

$$\mu(A) = [A] \text{ for any formula } A. \tag{6}$$

We need the following property.

(P2) If $X \in M_R$ and $A \in X$, then $[A] \subseteq X$.

Assume that $X \in M_R$ and $A \in X$. Fix Y such that $X = Y^{\sim}$. Then, $\Phi \vdash \Gamma, A$ for all $\Gamma \in Y$, and consequently, $\Phi \vdash \Gamma, \Delta$ for all $\Gamma \in Y$, $\Delta \in [A]$ (use (cut^{-})). So $[A] \subseteq X$.

Accordingly, $A \in \mu(A)$ yields $[A] \subseteq \mu(A)$. Hence (5) entails (6).

If $(A, B) \in \Phi$, then $[A] \subseteq [B^{-}]$ (use (cut^{-}), (r-$^{--}$)), hence $\mu(A) \subseteq \mu(B)^{-}$, by (6), (3). If $\Phi \not\vdash \Gamma, \Delta$, then $\mu(\Gamma) \not\subseteq \mu(\Delta)^{-}$, as in the proof of Theorem 2. $\square$

The sequent system S-InNL1 is an extension of S-InNL. We add the empty bunch ϵ and constants 1 and 0. Sequents are all nonempty bunches. We add one new axiom and two new rules.

$$\text{(a.0) } 0 \quad \text{(r-1) } \frac{\Gamma[\Delta]}{\Gamma[(1, \Delta)]} \quad \frac{\Gamma[\Delta]}{\Gamma[(\Delta, 1)]}$$

We assume $(\epsilon, \Gamma) = \Gamma = (\Gamma, \epsilon)$. In rules (r-$\oplus$1), (r-$\oplus$2) we admit $\Delta = \epsilon$, and similarly for (cut$^{\sim}$), (cut^{-}). In the derivable rules (r-$\oplus$3), (r-$\oplus$4) both Γ and Δ may be empty; in (r-$^{\sim\sim}$), (r-$^{--}$) Γ may be empty.

The algebraic models are unital involutive residuated monoids. We define: $\mathbf{M}, \mu \models \Gamma$ iff $\mu(\Gamma) \leq 0$. In metalanguage $1^{\sim} = 1^{-} = 0$, $0^{\sim} = 0^{-} = 1$.

All results, proved above and appropriately modified, are true for S-InNL1. The proofs are quite similar, but they need certain modifications (more cases).

These systems can be augmented with additive connectives $\wedge, \vee$, interpreted as meet and join in lattice-ordered (l.o.) involutive residuated groupoids. All results of Sections 2 and 3 remain true for these extensions. In a similar way, one can prove these results for associative (commutative) logics of this kind.

4 InNL versus InNL(k)

Let a be an element of an involutive residuated groupoid. We use the notation $a^{(n)}$ in the same meaning as in Section 3, e.g. $a^{(2)} = a^{\sim\sim}$, $a^{(-2)} = a^{--}$. Let k be a positive integer. An involutive residuated groupoid is said to be $k-$*cyclic*, if $a^{(k)} = a$ for any $a \in M$. If k is odd, then $a \leq b$ entails $b^{(k)} \leq a^{(k)}$. So for an odd k, $\leq$ is the identity relation and $a \otimes b = b \oplus a$ in any $k-$cyclic involutive residuated groupoid. In what follows we assume that k is even. Observe that $a^{-} = a^{(k-1)}$ in $k-$cyclic algebras of this kind.

InNL(k) is InNL from Section 1 with the new axiom $A^{(k)} \Leftrightarrow A$. This logic is strongly complete with respect to $k-$cyclic involutive residuated groupoids. Clearly InNL(2) amounts to CNL.

We introduce a sequent system S-InNL(k). The atoms are of the form $p^{(n)}$ for $0 \leq n < k$. All axioms and rules are as for S-InNL, but n, $n+1$ are computed modulo k. So the axioms are: $p^{(n)}, p^{(n+1)}$, for $0 \leq n < k-1$ and $p^{(k-1)}, p^{(0)}$. We assume $p^{(0)} = p$.

The metalanguage negations $^{\sim}, ^{-}$ are defined as in Section 3 (we compute $n+1$ and $n-1$ modulo k). All syntactic equations from Section 3 remain true. Also $A^{(k)} = A$, since k is even. We also obtain (3) for any valuation in a $k-$cyclic involutive residuated groupoid. Lemmas 3 and 4 remain true; now S_0 arises from S-InNL(k) like the former S_0 from S-InNL.

Theorem 4. *S-InNL(k) is weakly complete with respect to $k-$cyclic involutive residuated groupoids.*

Proof. The whole proof of Theorem 2 can be repeated, except that we cannot prove (at this moment) that the algebra $(M_R, \otimes_R, \backslash, /, ^{\sim}, ^{-}, \subseteq)$ is $k-$cyclic. Let $\mathbf{M}_R^{\mu}$ be the subalgebra whose universe consists of $\mu(A)$ for all formulas A. Clearly $\mathbf{M}_R^{\mu}$ is an involutive residuated groupoid and μ is a valuation in $\mathbf{M}_R^{\mu}$. By (3), $\mu(A)^{(k)} = \mu(A^{(k)}) = \mu(A)$, hence $\mathbf{M}_R^{\mu}$ is $k-$cyclic. If $\not\vdash \Gamma, \Delta$, then $\mathbf{M}_R^{\mu}, \mu \not\models \Gamma, \Delta$, and consequently, Γ, Δ is not valid. $\square$

Accordingly, (cut$^{\sim}$), (cut^{-}) are admissible in S-InNL(k). With these rules S-InNL(k) is strongly complete with respect to $k-$cyclic involutive residuated groupoids. Again the proof of Theorem 3 can be repeated, and we consider $\mathbf{M}_R^\mu$ as above. Consequently, S-InNL(2) is a cut-free system for CNL, different from those in [22, 12]. It can be shown that M_R in the proof of Theorem 4 is $k-$cyclic (use the infinite De Morgan laws for $^\sim$ in M_R).

We return to S-InNL. For any bunch Γ and $m \in \mathbb{Z}$, by $I(\Gamma, m)$ we denote the bunch arising from Γ, after one has replaced each atom $p^{(n)}$ by $p^{(n+m)}$.

Lemma 5. *For any sequent Γ and $m \in \mathbb{Z}$, Γ is provable in S-InNL if and only if $I(\Gamma, m)$ is provable in S-InNL.*

Proof. Assume that Γ is provable in S-InNL. In a proof of Γ substitute $p^{(n+m)}$ for $p^{(n)}$, for any atom $p^{(n)}$. We obtain a proof of $I(\Gamma, m)$. This yields the only-if part and entails the 'if' part, since $\Gamma = I(I(\Gamma, m), -m)$. □

Let Γ be a sequent, and let m be the greatest positive integer n such that some atom $p^{(-n)}$ occurs in Γ; if there is none, we set $m = 0$. The provability of Γ is equivalent to the provability of $I(\Gamma, m)$, which contains no $p^{(n)}$ with $n < 0$; such a sequent is said to be $^{-}-$*free*.

Lemma 6. *Let Γ be a $^{-}-$free sequent. Let m be the greatest integer n such that some atom $p^{(n)}$ occurs in Γ. Let $k > m + 1$. Then, Γ is provable in S-InNL if and only if Γ is provable in S-InNL(k).*

Proof. Assume that Γ is provable in InNL. Then Γ is valid in InNL, and consequently, in $k-$cyclic involutive residuated groupoids. By Theorem 4, Γ is provable in S-InNL(k).

Assume that Γ is provable in S-InNL(k). There exists a (cut-free) proof of Γ such that every formula in this proof is a subformula of a formula in Γ. Consequently, no atom $p^{(k-1)}$ appears in this proof, hence no axiom $p^{(k-1)}, p$ is used. This proof is a proof in S-InNL. □

Therefore, if a sequent Γ is not valid in InNL, then it is not valid in some $k-$cyclic involutive residuated groupoid. This result for InGL was obtained in [18] with the aid of a two-sided sequent system for InGL. Our proof seems simpler and works for this richer logic as well.

InNL(k) has similar properties as CNL (which amounts to InNL(2)):

- (T1) an interpolation property: if $\Phi \vdash \Gamma[\Delta]$, where $\Gamma[\Delta] \neq \Delta$, then there exists $D \in T$ (an interpolant of Δ) such that $\Phi \vdash \Gamma[D]$ and $Phi \vdash \Delta \Rightarrow D$, where T is the closure of the set of formulas appearing in $\Phi \cup \{\Gamma[\Delta]\}$ under subformulas and $^\sim$ (T is finite, if one uses S-InNL(k)),

- (T2) the strong finite model property: the sequents provable in InNL(k) from a finite set Φ coincide with those which follow from Φ in finite models,

- (T3) the languages generated by the type grammars based on InNL(k) are precisely the ($\epsilon-$free) context-free languages, (T4) the (finitary) consequence relation for InNL(k) is P-TIME.

(T1)-(T4) can be proved quite similarly as their particular cases for CNL in [12]. We skip the details. Let us only note some essential points.

Let k be fixed. Given a finite set of formulas T, its closure under subformulas and $^{\sim}$ can be computed in polynomial time (in the size of T and k), since one applies $^{\sim}$ $k-1$ times to each formula; we denote this closure by $\bar{T}$.

(T1) can be proved by induction on proofs in S-InNL(k). If Δ is a formula, then $D = \Delta$. Let us only consider the rule (r-shift): top-down, with premise $(\Gamma, \Delta'), \Theta$ and conclusion $\Gamma, (\Delta', \Theta)$, where $\Delta = (\Delta', \Theta)$. Let D' be an interpolant of Γ in the premise. Then $D^{\sim}$ is an interpolant of (Δ', Θ) in the conclusion.

(T2) can be proved like Theorem 3 for S-InNL(k) except that M is the free groupoid generated by $\bar{T}$, where T is the set of formulas appearing in $\Phi \cup \{\Gamma\}$. We only consider $\bar{T}-$proofs, i.e. the proofs in S-InNL(k) which employ formulas from $\bar{T}$ only. We write $\Phi \vdash^{\bar{T}} \Gamma$, if there exists a $\bar{T}-$proof of Γ from Φ. We define $[A]_{\bar{T}} = \{\Gamma \in M : \Phi \vdash^{\bar{T}} \Gamma\}$, and replace $[A]$ by $[A]_{\bar{T}}$ in the whole reasoning. We obtain $\mu(A) = [A]_{\bar{T}}$ for any $A \in \bar{T}$. Thus, if $\Phi \vdash^{\bar{T}} \Gamma$ does not hold, then Γ is not true for μ. Although the resulting frame is infinite, its complex algebra is finite. By (T1), there exist only finitely many sets of the form $\{\Gamma\}^{\sim}$, since $\{\Gamma\}^{\sim}$ equals the union of sets $[D]_{\bar{T}}$ for some formulas $D \in \bar{T}$. Consequently, there are only finitely many closed sets, hence M_R is finite.

(T3) follows from (T1). Let $G = (\Sigma, I, A_0)$ be a type grammar based on InNL(k). Let T denote the set of all formulas involved in I plus A_0. By (T1), every $\bar{T}-$sequent $\Gamma \Rightarrow A$, provable in InNL(k) can be proved on the basis of provable $\bar{T}-$sequents of the form $(A_1, A_2) \Rightarrow B$ and $B \Rightarrow C$ (restricted sequents) with the aid of (NL-cut). This yields a context-free grammar equivalent to G. Its terminal symbols are those of G, the nonterminal symbols are the formulas from $\bar{T}$, the production rules are the reversed, restricted $\bar{T}-$sequents provable in InNL(k), and A_0 is the start symbol. It is known that every $\epsilon-$free context-free language is generated by a type grammar based on NL, and NL can be replaced by InNL(k), since InNL(k) is a strongly conservative extension of NL.

Finally, (T4) holds, since all restricted $\bar{T}-$sequents, provable in InNL(k), can be proved in S-InNL(k), limited to restricted $\bar{T}-$sequents; see the proof for CNL in [12].

From (T2)-(T4) we infer the main results of this section. Analogous results can be obtained for InNL1.

(T5) InNL possesses the finite model property.

(T6) The type grammars based on InNL generate the $\epsilon-$free context-free languages.

(T7) InNL is P-TIME.

REMARK. This proof of P-TIME complexity assumes that the size of $p^{(n)}$ is defined as $|n|+1$, where n denotes the absolute value of n. This is reasonable, if $|n|$ is small (which is the case for types appearing in linguistic applications) or $p^{(n)}$ is understood as a metalanguage abbreviation of p with $|n|$ negations. Another proof, given in [13], yields a polynomial algorithm with $|n|$ defined as the length of the binary (or decimal) representation of n.

Accordingly, a context-free grammar equivalent to the given type grammar G based on InNL can be constructed in polynomial time in the size of G. On the contrary, an analogous construction for the type grammars based on L (due to Pentus [33]) is exponential; no polynomial construction exists, if the hypothesis P=NP is false. This follows from the fact that L is NP-complete [34].

5 Final comments

We briefly discuss the significance of our results for type grammars.

Since a leitmotive of type grammars is a reduction of parsing to formal proofs in appropriate logical calculi, independent of the particular language, this theory develops and studies different logics suitable for this purpose. This research is located on the borderline of logic and linguistics: logical ideas are applied in grammar and, conversely, linguistics stimulates new logical formalisms. Linear logics belong to this area. This paper contributes to nonassociative linear logics, which have not been much worked out earlier. Our interest in these logics is motivated by their close relations to Nonassociative Lambek Calculus and other systems of this kind.

InNL and CNL are strongly conservative extensions of NL. In Section 1, we have already explained that this allows us to use the former instead of the latter in type grammars. If one translates NL-types into InNL-types in the formalism of S-InNL, then formulas with negations appear. For example, $pn\backslash s$ is translated as $pn^{\sim}\oplus s$ and $(pn\backslash s)/pn$ as $(pn^{\sim}\oplus s)\oplus pn^{-}$. If $(n\backslash n)/(s/pn)$ is assigned to 'whom' in 'girl whom John admires', this type is translated as $(n^{\sim}\oplus n)\oplus(pn^{--}\otimes s^{-})$. Types of the latter form were used in e.g. [14, 15]. In pregroup grammars $\otimes$ and $\oplus$ collapse; one writes AB for both $A\otimes B$ and $A\oplus B$, A^r for $A^{\sim}$ and A^l for A^{-}. So the type of 'whom' is written as $n^r n[pn]^{ll}s^l$ (product is associative).

Due to conservativity, our results for InNL, CNL are stronger than their analogues for NL: the former entail the latter, but not conversely. For example, from the fact that InNL-grammars are equivalent to context-free grammars it follows that NL-grammars are so (for NL-grammars, this was known earlier). Interestingly, these conservativity results also hold for associative (commutative) linear logics versus their intuitionistic fragments, but it was not known until [2] has appeared. ([12] shows a simpler, general method for such conservativity proofs with the aid of phase spaces.) Certainly, the authors of [32] did not know it. They developed intuitionistic phase spaces in order to prove the finite model property for intuitionistic linear logics, which was earlier proved by Lafont [23] for MALL (with a claim that this proof also works for noncommutative logics). Intuitionistic phase spaces are interesting for themselves, but the main results of [32] (the finite model property) simply follow from Lafont's results, by conservativity.

Logics with negation(s) exhibit interesting symmetries (dualities), absent in their intuitionistic fragments. For instance, the application law $A \otimes (A\backslash B) \Rightarrow B$ and the co-application law $B \Rightarrow (B \otimes A)/A$ are related by the duality: $a \leq b$ iff $b^- \leq a^-$.

Why the consequence relation is relevant in type grammars? At this point, the present author differs in opinions from most linguists working in type grammar. They, as a rule, adhere to *radical lexicalism*: the whole language-dependent grammatical information has to be contained in the type lexicon (i.e. the map I), and parsing is to be done within a pure logic. In some situations, it seems reasonable to admit parsing within *a theory*, i.e. a logic augmented with nonlogical assumptions (axioms). Lambek [27] used such assumptions in pregroup grammars, e.g. $s_1 \Rightarrow s$, $s_2 \Rightarrow s$, where s is the type of statement, s_1 of statement in present tense, s_2 of statement in past tense, and many other arrows of this kind (poset arrows). One can consider L, NL etc. augmented with the assumptions corresponding to the production rules of a fixed context-free grammar. Then, the type logic can infer interesting consequences. For instance, L transforms the given context-free grammar into an equivalent basic categorial grammar [8]. One can approximate a stronger logic by a weaker, but more efficient, logic. In particular, L (which is NP-complete) can be approximated by NL augmented with finitely many sequents provable in L (but not closed under substitution). For instance, $(pn\backslash s)/pn \Leftrightarrow pn\backslash(s/pn)$ is provable in L, but not in NL. One can add this particular law to NL, without assuming the general pattern $(A\backslash B)/C \Leftrightarrow A\backslash(B/C)$ (which yields the associative law for $\otimes$), just for a more flexible treatment of transitive verb phrases in the nonassociative framework. Since the finitary consequence relation for NL is P-TIME [9], this approximation leads to an efficient parsing procedure. The same holds for Cyclic MLL versus CNL. For InNL, the complexity of the consequence relation is not known.

The operations $X^{\sim}$ and X^{-}, defined in Section 2, are a special case of polarities $X^{\rhd}$ and $Y^{\lhd}$, considered in lattice theory. For $R \subseteq U \times V$, $X \subseteq U$, $Y \subseteq V$, one defines:

$$X^{\rhd} = \{v \in V : \forall_{u \in X}\, R(u,v)\},\ \ Y^{\lhd} = \{u \in U : \forall_{v \in Y}\, R(u,v)\}.$$

This yields a Galois connection and two closure operations ${}^{\rhd\lhd}$ and ${}^{\lhd\rhd}$; let us denote them by ψ and ϕ. One obtains two dual (complete) lattices: the first consists of $\psi-$closed subsets of U and the second of $\phi-$closed subsets of V (so-called *concept lattices*).

Clark [17] considered *syntactic concept lattices*, where $U = \Sigma^*$, $V = \Sigma^* \times \Sigma^*$ (the set of contexts) and, for a fixed language $L \subseteq \Sigma^*$, one defines:

$$R_L(u, (v_1, v_2)) \text{ iff } v_1 u v_2 \in L\,.$$

The $\psi-$closed subsets of Σ^* are interpreted as the syntactic categories determined by the language L (the main category). In fact, the lattice of syntactic categories is a residuated lattice with $\backslash, /$ defined as in the algebra of languages. Syntactic concept lattices are models of Full Lambek Calculus. It is easy to adapt this construction for languages of phrase structures, as processed by nonassociative type grammars.

Accordingly, algebras of closed sets, defined similarly as in Section 2, possess sound linguistic interpretations. In our phase spaces $U = V$, and we impose additional constraints upon R: (Shift), $\phi = \psi$. The relation R_L, defined as above (for nonassociative contexts), need not satisfy them, if L is, say, the set of declarative sentences of a natural language, represented as phrase structures. Nonetheless, every nonassociative syntactic concept lattice can be isomorphically embedded into a lattice-ordered involutive residuated groupoid (this follows from the strong conservativity). A linguist, working with InNL instead of NL, may interpret the formulas of InNL in the extended model. This resembles the mathematical construction of complex numbers, extending real numbers. Although complex numbers do not possess such intuitive interpretations as reals (e.g. as lengths, weights etc.), they give rise to a more regular and deeper mathematical theory.

At the end, let us point out evident connections with classical modal logics. It is well-known that $\otimes, \backslash, /$ can be treated as binary modal operators. We focus on linear negations. Given a modal frame (M, R), where $R \subseteq M^2$, and $X \subseteq M$one defines $\Diamond(X) = \{u \in M : \exists_{v \in X} R(u, v)\}$, $\Box(X) = (\Diamond(X^c))^c$ (here X^c denotes the complement of X), and similarly $\Diamond^{\downarrow}, \Box^{\downarrow}$ with respect to the converse relation $R^{\cup}$. These operations satisfy the unary residuation laws: $\Diamond(X) \subseteq Y$ iff $X \subseteq \Box^{\downarrow}(Y)$, and $\Diamond^{\downarrow}(X) \subseteq Y$ iff $X \subseteq \Box(Y)$ (so $\Diamond, \Box^{\downarrow}$ and $\Diamond^{\downarrow}, \Box$ form residuation pairs).

For a phase space, $(M, \cdot, R)$, we have $X^{\sim} = \Box^{\downarrow}(X^c)$, $X^{-} = \Box(X^c)$, where the modalities are defined for (M, R^c). Consequently $\phi_R(X) = \Box^{\downarrow}\Diamond(X)$, $\psi_R(X) = \Box\Diamond^{\downarrow}(X)$. The condition $\phi_R = \psi_R$ can be expressed by the modal axiom $\Box^{\downarrow}\Diamond p \Leftrightarrow \Box\Diamond^{\downarrow} p$, where $\Leftrightarrow$ stands for biconditional of classical logic. Also $p^{\sim} \Leftrightarrow \Box^{\downarrow}\neg p$, $p^{-} \Leftrightarrow \Box\neg p$, where $\neg$ stands for classical negation. (Shift) can be expressed by $p^{\sim}/q \Leftrightarrow p\backslash q^{-}$.

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