A Tippe Top (TT) is a spinning toy, built as a truncated sphere with a small peg as a handle. The TT is started with the handle pointing upward. If spun fast enough, it will turn and spin on its handle. This is called inversion.
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We are interested in finding a rigorous way to describe the dynamics of inversion of a rolling and gliding TT.
Figure: Diagram of the TT. Note that $\mathbf{a} = R\hat{\mathbf{3}} - R\hat{\mathbf{z}}$. 
The TT is modelled as a sphere of radius $R$, axially symmetric distribution of mass $m$ and moments of inertia $I_1 = I_2, I_3$.

The center of mass ($CM$) is shifted along the symmetry axis by $\alpha R$, with $0 < \alpha < 1$. 
A model for the Tippe Top

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The center of mass ($CM$) is shifted along the symmetry axis by $\alpha R$, with $0 < \alpha < 1$.

We are interested in gliding and rolling TT, so it will be in contact with the plane at $A$.

We let $\mathbf{a} = R(\alpha \hat{\mathbf{3}} - \hat{2})$ be the vector from $CM$ to the point of contact $A$. 
Newton’s equations of motion:
\[
\begin{cases}
    m\ddot{s} = F - mg\hat{z}, \\
    \dot{L} = a \times F, \\
    \dot{\hat{3}} = \omega \times \hat{3} = \frac{1}{I_1} L \times \hat{3},
\end{cases}
\]
where \( F \) is the external force acting at the contact point \( A \).
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\]

where \( F \) is the external force acting at the contact point \( A \).

We write \( \mathbf{v}_A = \mathbf{s} + \omega \times \mathbf{a} \), where \( \mathbf{v}_A \) is the speed of the point \( A \).

Since the TT is always in contact with the plane, we have also the constraint \( \mathbf{v}_A \cdot \hat{z} = 0 \).
A model for the Tippe Top

We shall assume that the external force $\mathbf{F}$ has the form

$$\mathbf{F} = g_n \hat{z} - \mu g_n \mathbf{v}_A,$$

where $\mu$ and $g_n$ are positive. The vertical component is dynamically determined but the planar components has to be specified independently.

One can show that the gliding friction is the main mechanism which makes the inversion possible (Del Campo).
A model for the Tippe Top

For this model of the TT, the energy function

\[ E = \frac{1}{2} m \dot{s}^2 + \frac{1}{2} \omega \cdot \mathbf{L} + m g s \cdot \hat{z}, \]

becomes decreasing for \( \mathbf{F} = g_n \hat{z} - \mu g_n v_A \):

\[ \frac{d}{dt} E = \mathbf{v}_A \cdot \mathbf{F} = -\mu g_n v_A^2 < 0. \]

The equations of motion also admit Jellett’s integral of motion \( \lambda = -\mathbf{L} \cdot \mathbf{a} \).

\[ \frac{d}{dt} \lambda = 0. \]
The role of rolling solutions

A special case of the TT equations is found when imposing the constraint $v_A = 0$ (no gliding). Given the assumptions for the external force the constraint implies $F = gn\hat{z}$.
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Then the equations of motion are

\[
\begin{align*}
    m\ddot{s}_{\hat{x},\hat{y}} &= 0, \\
    \dot{L} &= R_\alpha g_n \hat{3} \times \hat{z}, \\
    \dot{\hat{3}} &= \frac{1}{I_1} L \times \hat{3}.
\end{align*}
\]

The solutions to this system can be interpreted as the asymptotic states of the TT system.
The role of rolling solutions

Asymptotic solutions for TT are either spinning in the upright or inverted position ($\theta = 0$ or $\theta = \pi$), or rolling around with a fixed center of mass with inclination angle $\theta \in (0, \pi)$. These are called tumbling solutions.
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When $\gamma = l_1 / l_3$ satisfies $1 - \alpha < \gamma < 1 + \alpha$, all values of $\theta \in (0, \pi)$ are admissible as tumbling solutions.

Examining the relative stability of the solutions gives that when $\lambda > \sqrt{mgR^3 l_3 \alpha (1+\alpha)^2} / \sqrt{1+\alpha-\gamma}$, then only the inverted spinning position $\theta = \pi$ is stable.
We can consider the pure rolling condition, $v_A = \dot{s} + \omega \times a = 0$, for the TT system without specifying the external force.
Dynamics of the TT

We can consider the pure rolling condition, $\mathbf{v}_A = \dot{\mathbf{s}} + \omega \times \mathbf{a} = 0$, for the TT system without specifying the external force.

This constraint reduces the equations of motion to a closed system (the rTT system) in $\omega$ and $\hat{3}$:

\[
\begin{align*}
(\mathbb{I}\omega)' &= \mathbf{a} \times \left( mg\hat{z} - m(\omega \times \mathbf{a}), \right) \\
\hat{3} &= \omega \times \hat{3},
\end{align*}
\]

and the external force is dynamically determined.
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This constraint reduces the equations of motion to a closed system (the rTT system) in $\omega$ and $\hat{3}$:

\[
\begin{aligned}
(\mathbb{I}\omega)' &= \mathbf{a} \times \left( mg\hat{2} - m(\omega \times \mathbf{a}) \right), \\
\hat{3}' &= \omega \times \hat{3},
\end{aligned}
\]

and the external force is dynamically determined. This system is integrable and solvable through separation of variables.
The rTT equations admits three integrals of motion, Jellett’s:

\[ \lambda = -\mathbf{L} \cdot \mathbf{a} = Rl_1 \dot{\phi} \sin^2 \theta - R(\alpha - \cos \theta) l_3 \omega_3, \]

the energy integral

\[ E = \frac{1}{2} m \dot{s}^2 + \frac{1}{2} \omega \cdot \mathbf{L} + mgs \cdot \hat{z}, \]

and Routh’s integral

\[
D = \omega_3 \sqrt{l_1 l_3 + mR^2 l_3 (\alpha - \cos \theta)^2 + mR^2 l_1 (1 - \cos^2 \theta)} \\
= l_3 \omega_3 \sqrt{d(\cos \theta)}.
\]

We use these to get a separable differential equation.
Dynamics of the TT

\[
E = g(\cos \theta)\dot{\theta}^2 + V(\cos \theta, D, \lambda).
\]

Where

\[
g(\cos \theta) = \frac{1}{2} I_3 (\sigma((\alpha - \cos \theta)^2 + 1 - \cos^2 \theta) + \gamma) > 0
\]

\[
V(\cos \theta, D, \lambda) = mgR(1 - \alpha \cos \theta) + \frac{I_3 D^2 - m\lambda^2}{2l_1l_3}
\]

\[
+ \frac{(\lambda \sqrt{d(\cos \theta)} + RD(\alpha - \cos \theta))^2}{2l_3 R^2 \gamma^2 \sin^2 \theta}
\]

and \(d(\cos \theta) = \gamma + \sigma(\alpha - \cos \theta)^2 + \sigma \gamma \sin^2 \theta > 0\) with parameters \(\sigma = \frac{mR^2}{I_3}\) and \(\gamma = \frac{l_1}{I_3}\).
Dynamics of the TT

When we consider the rolling and gliding TT, $D$ and $E$ are no longer integrals of motion, but we can still treat $D(t)$ and $\tilde{E}(t)$ as functions of time so the same elimination as above gives us the Main Equation of the TT (METT):

$$\tilde{E}(t) = g(\cos \theta)\dot{\theta}^2 + V(\cos \theta, D(t), \lambda).$$
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We cannot solve this equation, but we might study types of solutions through the functions $D(t)$ and $\tilde{E}(t)$.

We find that the equations of motion for TT are equivalent to the system

$$\frac{d}{dt} \lambda(\theta, \dot{\theta}, \phi, \omega_3) = 0,$$
$$\frac{d}{dt} D(\theta, \omega_3) = \frac{\gamma m}{\alpha \sqrt{d(\hat{z} \times \mathbf{a}) \cdot \dot{\mathbf{v}}_A}},$$
$$\frac{d}{dt} \tilde{E}(\theta, \dot{\theta}, \phi, \omega_3) = m(\omega \times \mathbf{a}) \cdot \dot{\mathbf{v}}_A,$$
$$\frac{d}{dt} m\dot{s}_{\hat{x},\hat{y}} = -\mu g_n \mathbf{v}_A.$$
Dynamics of the TT

The Main Equation for the TT describes nutational motion of the symmetry axis \( \hat{3} \) on the unit sphere between angles \( \theta_1, \theta_2 \) satisfying \( \tilde{E}(t) = V(\cos \theta, D(t), \lambda) \). During inversion, this nutational band goes from the north to the south pole.
Dynamics of the TT

The Main Equation for the TT describes nutational motion of the symmetry axis $\hat{3}$ on the unit sphere between angles $\theta_1, \theta_2$ satisfying $\tilde{E}(t) = V(\cos \theta, D(t), \lambda)$. During inversion, this nutational band goes from the north to the south pole.

The idea is to study the METT for a class of curves $(D(t), \tilde{E}(t))$ satisfying boundary conditions describing inversion solutions, which we can estimate.
Dynamics of the TT

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symmetry axis $\hat{3}$ on the unit sphere between angles $\theta_1$, $\theta_2$ satisfying
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The idea is to study the METT for a class of curves $(D(t), \tilde{E}(t))$
satisfying boundary conditions describing inversion solutions, which
we can estimate.

As the TT is inverting $D(t)$ goes from the initial value $D_0 \approx \frac{\lambda \sqrt{d(1)}}{RL_3(1-\alpha)}$ to
the final value $D_1 \approx -\frac{\lambda \sqrt{d(-1)}}{RL_3(1+\alpha)}$ and the energy $\tilde{E}(t)$ is decreasing
from $\tilde{E}_0 \approx \frac{\lambda^2}{2R^2l_3(1-\alpha)^2} + mgR(1 - \alpha)$ to
$\tilde{E}_1 \approx \frac{\lambda^2}{2R^2l_3(1+\alpha)^2} + mgR(1 + \alpha) < \tilde{E}_0$. 
Dynamics of the TT

Figure: Evolution of $V(\cos \theta, D(t), \lambda)$, $\theta \in (0, \pi)$, for $D$ between $D_0$ and $D_1$. 
We find that when $1 - \alpha^2 < \gamma < 1$, the function $d(z = \cos \theta)$ can be written as an even square if $\sigma = \frac{1-\gamma}{\gamma+\alpha^2-1}$.
Nutational estimates for simplified METT

We find that when \( 1 - \alpha^2 < \gamma < 1 \), the function \( d(z = \cos \theta) \) can be written as an even square if \( \sigma = \frac{1-\gamma}{\gamma+\alpha^2-1} \).

This makes the potential function \( V(z, D, \lambda) \) rational in \( z \). Rearranging the separation equation we have

\[
\dot{z}^2 = \frac{(1 - z^2)}{g(z)} \left( E - V(z, D, \lambda) \right) := f(z).
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Rearranging the separation equation we have

$$\dot{z}^2 = \frac{(1 - z^2)}{g(z)} \left( E - V(z, D, \lambda) \right) := f(z).$$

Investigation on the solutions for this type of TT is then reduced to studying the roots of the function $f(z)$ for $z \in [-1, 1]$, $z_0$, and $z_1$.  

When these roots satisfy $-1 < z_0 < z_1 < 1$, they correspond to bounding circles on $S^2$, between which the symmetry axis performs nutational motion.
Nutational estimates for simplified METT

\[ z_0 \leq z \leq z_1 \]

\[ f(z) \]

**Figure:** Graph of the function \( f(z) \) with two roots in \([-1, 1]\).
If the TT is rolling fast (so $\omega_3$ is large), we can make precise estimates of such quantities as the width of the nutational band, the frequency of nutations and the average rate of precession. This is similar to estimates made for the fast-spinning Lagrange top (Arnold, Goldstein, etc.).
Nutational estimates for simplified METT

We assume first that the initial precession speed is zero, $\dot{\phi}(0) = 0$, which gives cusps on the lower bounding circle.
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Proposition

For a rolling sphere with such parameters that \( f(z) \) is rational in \( z \) and initial conditions \( \theta(0) = \theta_0, \dot{\theta}(0) = \dot{\phi}(0) = 0 \) with large value of spin \( \omega_3 \), we have the following estimates:

i) The width of the nutational band \( z_1 - z_0 = \frac{\kappa_1}{\omega_3^2} + O\left(\frac{1}{\omega_3^4}\right) \).

ii) The depth of the nutational motion \( E - \min_{\theta \in (0, \pi)} V(\cos \theta, D, \lambda) \leq mgR \alpha (z_1 - z_0) \), so it decreases with \( \omega_3 \) as \( \frac{1}{\omega_3^2} \).

iii) The frequency of nutation \( \frac{2\pi}{T} = \kappa_2 \omega_3 + O\left(\frac{1}{\omega_3}\right) \).

iv) The average rate of precession \( \bar{\dot{\phi}} = -\frac{\kappa_3}{\omega_3} + O\left(\frac{1}{\omega_3^2}\right) \).

Here \( \kappa_1, \kappa_2 \) and \( \kappa_3 \) are positive.
Interestingly enough, if we assume that the initial precession speed is non-zero but small, \( \dot{\phi}(0) = c \), we can make similar estimates:

**Proposition**

For a rolling sphere with such parameters that \( f(z) \) is rational in \( z \) and initial conditions \( \theta(0) = \theta_0 \), \( \dot{\theta}(0) = 0 \), \( \dot{\phi}(0) = c \) with large value of spin \( \omega_3 \), we have the following estimates:

1. The width of the nutational band \( z_1 - z_0 = \kappa_1 \omega_3 + O(1/\omega_3^2) \).
2. The depth of the nutational motion \( E - \min_{\theta \in (0, \pi)} V(\cos \theta, D, \lambda) \leq mgR \alpha(z_1 - z_0) \), so it decreases with \( \omega_3 \) as \( 1/\omega_3 \).
3. The frequency of nutation \( 2\pi T = \kappa_2 \omega_3 + O(1) \).

Here \( \kappa_1, \kappa_2 \) are positive.
Interestingly enough, if we assume that the initial precession speed is non-zero but small, $\dot{\phi}(0) = c$, we can make similar estimates:

**Proposition**

*For a rolling sphere with such parameters that $f(z)$ is rational in $z$ and initial conditions $\theta(0) = \theta_0$, $\dot{\theta}(0) = 0$, $\dot{\phi}(0) = c$ with large value of spin $\omega_3$, we have the following estimates:*

i) *The width of the nutational band* $z_1 - z_0 = \frac{\kappa_1}{\omega_3} + O\left(\frac{1}{\omega_3^2}\right)$.

ii) *The depth of the nutational motion*

$E - \min_{\theta \in (0,\pi)} V(\cos \theta, D, \lambda) \leq mgR\alpha(z_1 - z_0)$, *so it decreases with* $\omega_3$ *as* $\frac{1}{\omega_3}$.

iii) *The frequency of nutation* $\frac{2\pi}{T} = \kappa_2 \omega_3 + O(1)$.

*Here* $\kappa_1$, $\kappa_2$ *are positive.*
Nutational estimates for simplified METT

Figure: Diagram of high frequency behaviour for simplified potential $V(\cos \theta, D, \lambda)$ when $\dot{\phi}(0) = c$ is small