International Journal of Quantum Information
Vol. 2, No. 4 (2004) 453–459

World Scientific
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EXAMPLES OF NON-UNIFORM LIMITING DISTRIBUTIONS FOR THE QUANTUM WALK ON EVEN CYCLES

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Received 27 March 2004

We present three different types of the evolution of a time-averaged probability distribution of the quantum walk on even cycles. We concentrate on initial states which lead to non-uniform limiting distributions.

Keywords: Quantum random walk; limiting distribution.

PACS Number(s): 03.67.Lx

1. Introduction

Recently, there has been a growing interest (for a review see Ref. 1) in the field of discrete quantum random walk initiated by the work of Aharonov *et al.*[2] In particular it is very interesting to establish the relation between quantum and classical walks. The differences between them are substantial and can be seen in, for example, spreading, mixing and hitting times[2–4] or limiting distributions.[5] Some of the features of quantum random walk has already been employed in the construction of quantum algorithms.[6–10] One of the differences between quantum and classical walks is that the former offers the possibility of starting the walk not from a single occupied node, but from the superposition of many nodes. The controlling of a quantum walk with the use of different initial states was investigated by Tragenna *et al.*[11] In a recent paper[5] we mentioned the possibility of generating highly non-uniform limiting distributions in a quantum walk on even cycles starting from superposition states. In the present paper we show how substantially the time evolution of the total variation distance of the time-averaged probability distribution depends on the initial conditions.

In particular we study a quantum random walk on an even cycle with d nodes. In the model of such a walk proposed in Ref. 2 nodes are represented by vectors $|v\rangle$, $v = 0, 1, \ldots, d-1$, which form an orthonormal basis of the Hilbert space H_V. An auxiliary two-dimensional Hilbert space H_A (coin space) is spanned by vectors $|s\rangle$, $s = 0, 1$. The initial state of the walk is a normalized vector:

$$|\Psi_0\rangle = \sum_{s,v} \gamma_{sv} |s, v\rangle = \sum_{s,v} \gamma_{sv} |s\rangle |v\rangle \tag{1}$$

from the tensor product $H = H_A \otimes H_V$. In a single step of the walk the state changes according to the equation:

$$|\Psi_{t+1}\rangle = U|\Psi_t\rangle, \tag{2}$$

where the operation $U = S(H \otimes I)$ first applies the Hadamard gate operator $H = \frac{1}{\sqrt{2}} \sum_{s,s'} (-1)^{ss'} |s\rangle\langle s'|$ to the vector from H_A, and then shifts the state by the operator

$$S = \sum_{s,v} |s\rangle\langle s| \otimes |v + 2s - 1 (\mathrm{mod}\ d)\rangle\langle v| \,. \tag{3}$$

Searching for eigenvectors and eigenvalues[5] we get

$$U|\phi_{jk}\rangle = c_{jk}|\phi_{jk}\rangle, \tag{4}$$

with eigenvalues

$$c_{jk} = \frac{1}{\sqrt{2}} \left((-1)^k \sqrt{1 + \cos^2(2\pi j/d)} - i \sin(2\pi j/d) \right), \tag{5}$$

for $k = 0, 1$, and $j = 0, 1, \ldots, d-1$, while the corresponding eigenvectors are

$$|\phi_{jk}\rangle = (a_{jk}|0\rangle + a_{jk} b_{jk}|1\rangle) \otimes \sum_v \omega_d^{jv} |v\rangle, \tag{6}$$

where $\omega_d = e^{2\pi i/d}$,

$$a_{jk} = 1 \Big/ \sqrt{d(1 + |b_{jk}|^2)}, \tag{7}$$

$$b_{jk} = \omega_d^j \left((-1)^k \sqrt{1 + \cos^2(2\pi j/d)} - \cos(2\pi j/d) \right). \tag{8}$$

The probability distribution on the nodes of the cycle after the first t steps of the walk is given by

$$p_t(v) = \sum_s |\langle s, v|\Psi_t\rangle|^2. \tag{9}$$

As was observed by Aharonov *et al.*,[2] for a fixed v, the probability $p_t(v)$ is "quasi-periodic" as a function of t and thus, typically, does not converge to a limit.

Thus, instead of $p_t(v)$ the authors of Ref. 2 considered the time-averaged probability distribution:

$$\bar{p}_t(v) = \frac{1}{t+1}\sum_{i=0}^{t} p_t(v), \tag{10}$$

and its limiting distribution:

$$\pi(v) = \lim_{t\to\infty} \bar{p}_t(v). \tag{11}$$

In order to present the global properties of the walk let us also define the total variation distance:

$$\Delta_t = \frac{1}{2}\sum_{v=0}^{d-1}\left|\bar{p}_t(v) - \frac{1}{d}\right|, \tag{12}$$

which measures how far the time-averaged probability distribution is from the uniform distribution. Δ_t tends to a limit which will be denoted by $\Delta_\infty = \lim_{t\to\infty}\Delta_t$.

In Figs. 1–3, the evolution of the total variation distance Δ_t is presented for the case of three different initial states — $|\Psi_0^{(2d)}\rangle$, $|\Psi_0^{(2)}\rangle$ and $|\Psi_0^{(4)}\rangle$. Obviously, these states can be written as a superposition of the eigenvectors $|\phi_{jk}\rangle$ and are labeled by the number of superposed eigenvectors. Namely, $|\Psi_0^{(2d)}\rangle = \sum_{jk} g_{jk}\omega_d^{-v_0 j}|\phi_{jk}\rangle$ is the state with single occupied node v_0, where $g_{jk} = a_{jk}(1 + ib^*_{jk})/\sqrt{2}$, $|\Psi_0^{(2)}\rangle = \frac{1}{\sqrt{2}}\left(|\phi_{3,0}\rangle + |\phi_{9,0}\rangle\right)$ is a superposition of two degenerate eigenvectors, and $|\Psi_0^{(4)}\rangle = \frac{1}{\sqrt{2}}\left(|\phi_{3,0}\rangle + |\phi_{9,0}\rangle - |\phi_{15,0}\rangle - |\phi_{21,0}\rangle\right)$. In the case of the $|\Psi_0^{(2d)}\rangle$ initial state (Fig. 1), one observes the decay of the total variation distance to the nonzero value $\Delta_\infty^{(2d)}$.

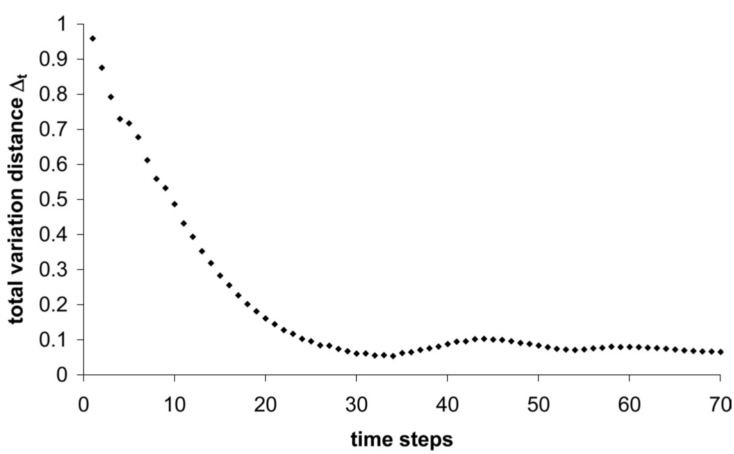

Fig. 1. Time evolution of the total variation distance Δ_t for the initial state $|\Psi_0^{(2d)}\rangle$.

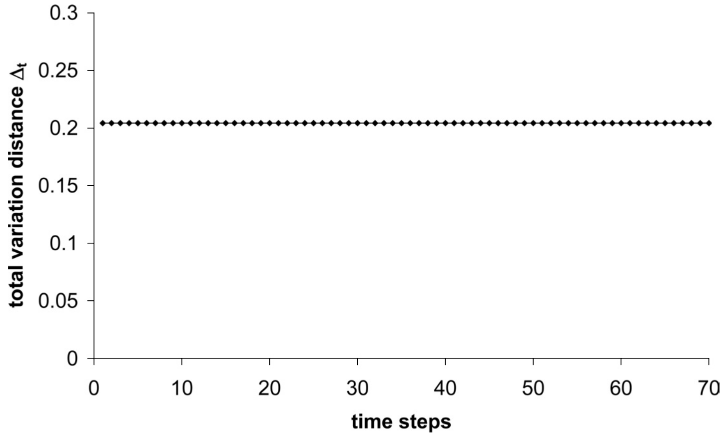

Fig. 2. Time evolution of the total variation distance Δ_t for the initial state $|\Psi_0^{(2)}\rangle$. Diamonds — numerical simulations, line — analytical value of Eq. (23).

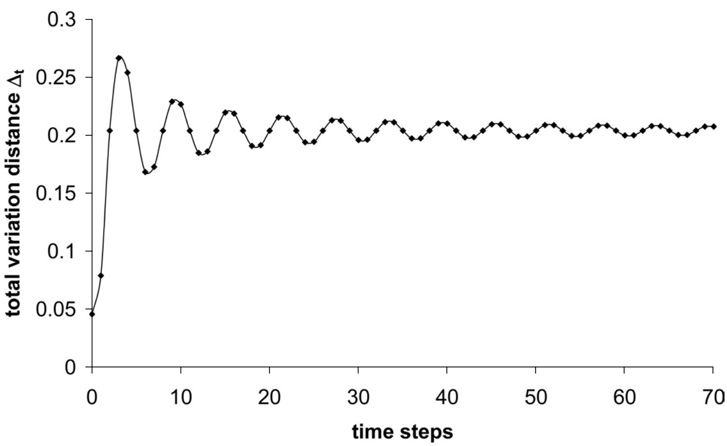

Fig. 3. Time evolution of the total variation distance Δ_t for the initial state $|\Psi_0^{(4)}\rangle$. Diamonds — numerical simulations, line — analytical value of Eqs. (12) and (26).

$\Delta_\infty^{(2d)}$ can be calculated with the use of our analytical form of limiting distribution $\pi(v)$ which, with some corrections to Eq. (22) in Ref. 5 can be written in the form:

$$\pi(v) = \frac{1 + f(s) - (-1)^{\xi} f(s')}{d}, \tag{13}$$

where

$$f(x) = \frac{\sqrt{2}}{1 - (-z)^{d/2}} z^x - \delta_{x0} - \frac{1}{d}, \tag{14}$$

$$\xi = \frac{(1 + (-1)^{d/2})}{2}, \tag{15}$$

i.e. $\xi = 1$ when $d/2$ is even, and $\xi = 0$ if $d/2$ is odd,

$$s = s(v) = \min(|v - v_0|, d - |v - v_0|), \tag{16}$$

denotes the distance between nodes v_0 and v, and $s' = d/2 - s$. If $d \gg 1$, we can write Δ_∞^{2d} in a simple form. When $\xi = 0$,

$$\Delta_\infty^{(2d)} = 1/d, \tag{17}$$

while in the case of $\xi = 1$:

$$\Delta_\infty^{(2d)} = \frac{2}{d} - \frac{4}{d^2}\left(1 - 2\frac{\log_2 d - 1/2}{\log_2 z}\right), \tag{18}$$

where $z = 3 - 2\sqrt{2}$. For the example given in Fig. 1, (18) gives the value $\Delta_\infty^{(2d)} = 0.054$. As is seen for the state with a single occupied node, the total variation distance decreases with time and its limiting value tends to zero as the graph size d grows. Let us now present two examples of a walk for which the dynamics of the total variation distance is dramatically different. Figure 2 presents the walk with constant $\Delta_t^{(2)} \neq 0$. This walk starts in a state of the form

$$\frac{1}{\sqrt{2}}\left(|\phi_{m,k}\rangle + |\phi_{d/2-m,k}\rangle\right), \tag{19}$$

for $m = 3$ and $k = 0$. The state described by (19) for $m = 0, \ldots, m_{\max}$ as well as the state

$$\frac{1}{\sqrt{2}}\left(|\phi_{d/2+m,k}\rangle + |\phi_{d-m,k}\rangle\right), \tag{20}$$

for $m = 1, \ldots, m_{\max}$, ($m_{\max} = \lfloor (d-2)/4 \rfloor$) consists of two degenerated eigenvectors. As is obvious, the evolution of the superposition of any number of degenerated eigenvectors only leads to global phase changes so the dynamics of the probability distribution is frozen and $\pi(v) = \bar{p}_t(v) = p_0(v)$. For the states given by (19) and (20), the limiting distribution takes the form:

$$\pi(v) = \frac{1}{d} + \frac{(-1)^v \sin\alpha}{d\sqrt{1+\cos^2\alpha}}\sin(\alpha(2v+1)), \tag{21}$$

where $\alpha = 2\pi m/d$. Thus

$$\Delta_t^{(2)} = \frac{\sin\alpha}{d\sqrt{1+\cos^2\alpha}}\sum_v |\sin(\alpha(2v+1))|. \tag{22}$$

When m divides $d/2$, the summation can easily be performed, leading to

$$\Delta_t^{(2)} = \frac{m}{d}\frac{1}{\sqrt{1+\cos^2\alpha}}\left(1 - \cos(2\alpha(\eta+1))\right), \tag{23}$$

where

$$\eta = \left\lfloor \frac{d}{4m} - \frac{1}{2} \right\rfloor. \tag{24}$$

For $m = 3$ and $d = 24$, (23) gives 0.204.

The last example we want to present (see Fig. 3) is the most spectacular one. The dynamics of the total variation distance in this case resembles the motion of the damped harmonic oscillator with shifted equilibrium. Let us emphasize that the limiting value of the total variation distance $\Delta_\infty^{(4)}$ is much higher than the initial one $\Delta_0^{(4)}$. The initial state $|\phi_0^{(4)}\rangle$ is of the kind:

$$\frac{1}{2}\left(|\phi_{m,k}\rangle + |\phi_{d/2-m,k}\rangle - |\phi_{d/2+m,k,k}\rangle - |\phi_{d-m,k}\rangle\right). \tag{25}$$

In order to understand the dynamics of $\Delta_t^{(4)}$ we found the following formula for the time-averaged probability distribution for the initial states of the form given by (25):

$$\bar{p}_t(v) = A(v) + B(v)\frac{\sin(2\varphi_{mk}(t+1))}{t+1}, \tag{26}$$

where $A(v) = \pi(v)$, $B(v) = (p_0(v) - \pi(v))\,/\sin(2\varphi_{mk})$ and φ_{mk} is the phase of the eigenvalue c_{mk} $\left(c_{mk} = e^{i\varphi_{mk}}\right)$. In Fig. 3, $\Delta_t^{(4)}$ calculated with the use of (12) and (26) is presented together with the results of numerical calculation.

2. Conclusion

In conclusion, we have shown how the initial conditions determines the dynamics of the quantum walk on cycles. We have given specific examples for three different kinds of behavior of the total variation distance between the given distribution and a uniform distribution: decaying, constant and damped oscillation.

Acknowledgments

The authors wish to thank the State Committee for Scientific Research (KBN) for its support: M. Bednarska and T. Łuczak were supported by Grant No. 2 P03A 016 23; A. Grudka and A. Wójcik by Grant No. 0 T00A 003 23. P. Kurzyński would like to thank Adam Mickiewicz University and the National University of Singapore for support.

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