On the Folkman Number $f(2,3,4)$

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Abstract

Let $f(2,3,4)$ denote the smallest integer n such that there exists a K_4-free graph of order n for which any 2-coloring of its edges yields at least one monochromatic triangle. It is well-known that such a number must exist. For a long time the best known upper bound, provided by J. Spencer, said that $f(2,3,4) < 3 \cdot 10^9$. Recently, L. Lu announced that $f(2,3,4) < 10\,000$. In this note, we will give a computer assisted proof showing that $f(2,3,4) < 1000$. To prove it we will generalize the idea of Goodman giving a necessary and sufficient condition for a graph G to yield a monochromatic triangle for every edge coloring.

1 Introduction

Let $\mathcal{F}(r,k,l)$, $k < l$, be a family of K_l-free graphs with the property that if $G \in \mathcal{F}(r,k,l)$, then every r-coloring of the edges of G must yield at least one monochromatic copy of K_k. J. Folkman showed in [4] that $\mathcal{F}(2,k,l) \neq \emptyset$. The general case, i.e. $\mathcal{F}(r,k,l) \neq \emptyset$, $r \geq 2$, was settled by J. Nešetřil and the second author [13]. Let $f(r,k,l) = \min_{G \in \mathcal{F}(r,k,l)} |V(G)|$. The problem of determining the numbers $f(r,k,l)$ in general includes the classical Ramsey numbers and thus is not easy. In this note we focus on the case where $r = 2$

and $k = 3$. We will write $G \to \triangle$ and say that G arrows a triangle if every 2-coloring of G yields a monochromatic triangle. Since the Ramsey number $R(3,3) = 6$ clearly $f(2,3,l) = 6$, for $l > 6$. The value of $f(2,3,6) = 8$ was determined by R. Graham [7], and $f(2,3,5) = 15$ by K. Piwakowski, S. Radziszowski and S. Urbański [14]. In the remaining case, the upper bounds on $f(2,3,4)$ obtained in [4] and [13] are extremely large (iterated tower function). Consequently, in 1975, P. Erdős [3] offered \$100 for proving or disproving that $f(2,3,4) < 10^{10}$. Applying Goodman's idea [6] (of counting triangles in a graph and in its complement) for random graphs P. Frankl and the second author [5] came relatively close to the desired bound showing that $f(2,3,4) < 8 \times 10^{11}$. This result was improved by J. Spencer [18], who refined the argument and proved $f(2,3,4) < 3 \times 10^9$ giving a positive answer to the question of Erdős [3]. Subsequently, F. Chung and R. Graham [1] conjectured that $f(2,3,4) < 10^6$ and offered \$100 for a proof or disproof. Recently, L. Lu [11] showed that $f(2,3,4) < 10\,000$ (A weaker result, $f(2,3,4) < 1.3 \times 10^5$, also answering Chung and Graham's question was independently found in an earlier version of this paper, see, e.g., [2]). All these proofs [5, 11, 18] are based on the modification of Goodman's idea [6]. The idea explores the local property of every vertex neighborhood in a graph (see Corollary 2.2).

In this note, we will present a K_4-free graph G_{941} of order 941 and give a computer assisted proof that $G_{941} \in \mathcal{F}(2,3,4)$. This yields that $f(2,3,4) \leq 941$. To prove it we will develop a technique, which is a generalization of ideas from [6, 13, 18]. More precisely, for every graph G we will construct a graph H with the property that G arrows a triangle if and only if the maxcut of H is less than twice number of triangles in G.

2 Computer assisted proof of $f(2,3,4) < 1000$

2.1 Counting blue and red triangles

In order to find an upper bound on the number $f(2,3,4)$, we will use an idea of [6]. For any blue-red coloring of G let $T_{BR}(v)$, $T_{BB}(v)$ and $T_{RR}(v)$ count the number of triangles containing vertex v, for which two edges incident to v are colored blue-red, blue-blue and red-red, respectively. Also let T_{Blue} (T_{Red}) be the number of blue (red) monochromatic triangles. The sum $\sum_{v \in V(G)} T_{BR}(v)$ counts 2 times the number of nonmonochromatic triangles. This is because each such triangle is counted once for two different vertices. On the other hand, the sum $\sum_{v \in V(G)} \big(T_{BB}(v) + T_{RR}(v)\big)$ counts 3 times the number of monochromatic triangles and once the number of nonmonochromatic triangles. Hence,

$$\sum_{v\in V(G)} T_{BR}(v) = 2\sum_{v\in V(G)} \big(T_{BB}(v)+T_{RR}(v)\big) - 6\big(T_{Blue}+T_{Red}\big). \tag{1}$$

Consequently, $G \to \triangle$ if and only if for every edge coloring of G the following holds

$$\sum_{v\in V(G)} T_{BR}(v) < 2\sum_{v\in V(G)} \big(T_{BB}(v)+T_{RR}(v)\big). \tag{2}$$

Denote by $N(v)$ the set of neighbors of a vertex $v \in V$ and let $G[N(v)]$ be a subgraph of G induced on $N(v)$. Moreover, for a given cut $C \subset V(G)$ let

$$M_C(G) = \big\{\{x,y\} \in E(G) \mid x \in C \text{ and } y \in V \setminus C\big\},$$

and that

$$M(G) = \max_{C\subset V} M_C(G),$$

i.e. $M(G)$ is the value corresponding to the solution of the maxcut problem for G.

Proposition 2.1 (Frankl & Rödl 1986 [5]; Spencer 1988 [18]). *Let $G = (V,E)$ be a graph that satisfies*

$$\sum_{v\in V(G)} M(G[N(v)]) < \frac{2}{3}\sum_{v\in V(G)} \big|E(G[N(v)])\big|. \tag{3}$$

Then, $G \to \triangle$.

An easy consequence of Proposition 2.1 gives the following corollary.

Corollary 2.2. *Let $G = (V,E)$ be a graph which satisfies*

$$M(G[N(v)]) < \frac{2}{3}\big|E(G[N(v)])\big| \tag{4}$$

for every vertex $v \in V(G)$. Then, $G \to \triangle$.

Note that in particular Corollary 2.2 gives a sufficient condition for a K_4-free graph to be in $\mathcal{F}(2,3,4)$. We will extend this idea and give a necessary and sufficient condition for a graph G to yield a monochromatic triangle for every edge coloring. More precisely, for every graph $G = (V,E)$ with $t_\triangle = t_\triangle(G)$ triangles, we construct a graph H with $|E|$ vertices such that $G \to \triangle$ if and only if the maxcut of H is less than $2t_\triangle$.

Let G be a graph with the vertex set $V(G) = \{1, 2, \dots, n\}$. For every vertex $i \in V(G)$, let G_i be a graph with

$$V(G_i) = \big\{\{i, j\} \mid j \in N(i)\big\}$$

and

$$E(G_i) = \big\{\{\{i, j\}, \{i, k\}\} \mid \text{if } ijk \text{ is a triangle in } G\big\}.$$

Clearly G_i is isomorphic to the subgraph $G[N(i)]$ of G induced on the neighborhood $N(i)$. Now we define a graph H as follows. Let

$$V(H) = E(G)$$

and

$$E(H) = \bigcup_{i \in V(G)} E(G_i).$$

In other words, H is a graph with the set of vertices being the set of edges of G such that e and f are adjacent in H if e and f belong to a triangle in G. Clearly $|V(H)| = |E(G)|$ and $|E(H)| = 3t_\triangle(G)$. Moreover, observe that there is one to one correspondence between blue-red colorings of edges of G and bipartitions of vertices of H. Let C be a cut with the partition $V(H) = B \cup R$. Since the edges between B and R correspond to nonmonochromatic triangles in G, we conclude that the value corresponding to the cut C equals to

$$M_C(H) = \sum_{i \in V(G)} T_{BR}(i). \tag{5}$$

Counting the edges which lie entirely in B or in R yields

$$\sum_{i \in V(G)} \big(T_{BB}(i) + T_{RR}(i)\big) = |E(H)| - M_C(H) = \big(3t_\triangle - M_C(H)\big). \tag{6}$$

By (1) we have that

$$\sum_{i \in V(G)} T_{BR}(i) \le 2 \sum_{i \in V(G)} \big(T_{BB}(i) + T_{RR}(i)\big),$$

and by (2), $G \to \triangle$ if and only if the strict inequality holds for every edge coloring of G. Consequently, (5) and (6) yield that $G \to \triangle$ if and only if

$$M_C(H) < 2\big(3t_\triangle - M_C(H)\big),$$

for every cut of H. Consequently, the following holds.

Theorem 2.3. *Let G be a graph. Then, there exists a graph H of order $|E(G)|$ with $M(H) \le 2t_\triangle(G)$ such that $G \to \triangle$ if and only if $M(H) < 2t_\triangle(G)$.*

2.2 Approximating the maxcut

Since Theorem 2.3 requires an assumption regarding the maxcut of graph H we will approximate it with Proposition 2.4 below. The proof of this proposition for regular graphs can be found in a paper of M. Krivelevich and B. Sudakov [10]. Along the lines of their proof one can obtain the following easy generalization, which we present here.

Proposition 2.4. *Let $H = (V, E)$ be a graph of order n. Let $\lambda_{min} = \lambda_{min}(H)$ be the smallest eigenvalue of the adjacency matrix of H. Then*

$$M(H) \leq \frac{|E(H)|}{2} - \frac{\lambda_{min}|V(H)|}{4}.$$

Proof. Let $A = (a_{ij})$ be the adjacency matrix of $H = (V, E)$ with the average degree d and $V = \{1, 2, \ldots, n\}$. Let $\mathbf{x} = (x_1, \ldots, x_n)$ be any vector with coordinates ± 1. Then,

$$\sum_{\{i,j\}\in E} (x_i - x_j)^2 = \sum_{i=1}^{n} d_i x_i^2 - \sum_{i \neq j} a_{ij} x_i x_j = \sum_{i=1}^{n} d_i - \sum_{i \neq j} a_{ij} x_i x_j = nd - \mathbf{x}^T A \mathbf{x}.$$

By the Rayleigh-Ritz ratio (see, e.g., Theorem 4.2.2 in [9]), for any vector $\mathbf{z} \in \mathbb{R}^n$, $\mathbf{z}^T A \mathbf{z} \geq \lambda_{min} \|\mathbf{z}\|^2$, where by $\|.\|$ we denote the Euclidean norm. Therefore,

$$\sum_{\{i,j\}\in E} (x_i - x_j)^2 = nd - \mathbf{x}^T A \mathbf{x} \leq nd - \lambda_{min} \|\mathbf{x}\|^2 = nd - \lambda_{min} n. \qquad (7)$$

Let $V = V_1 \cup V_2$ be an arbitrary partition of V into two disjoint subsets and let $e(V_1, V_2)$ be the number of edges in the bipartite subgraph of H with bipartition (V_1, V_2). For every vertex $i \in V$ set $x_i = 1$ if $i \in V_1$ and $x_i = -1$ if $i \in V_2$. Note that for every edge $\{i, j\}$ of H, $(x_i - x_j)^2 = 4$ if this edge has its ends in the distinct parts of the above partition and is zero otherwise. Now using (7), we conclude that

$$e(V_1, V_2) = \frac{1}{4} \sum_{\{i,j\}\in E} (x_i - x_j)^2 \leq \frac{1}{4}(dn - \lambda_{min} n) = \frac{|E|}{2} - \frac{\lambda_{min}|V|}{4}.$$

□

2.3 Numerical results

Let G be a circulant graph defined as follows:

$$V(G_{941}) = \mathbb{Z}_{941},$$

and

$$E(G_{941}) = \big\{\{x, y\} \mid x - y = \alpha^5 \bmod 941\big\},$$

i.e. the set of edges consists of those pairs of vertices x and y which differ by a 5th residue of 941. Equivalently,

$$V(G_{941}) = \{0, 1, \dots, 940\},$$

and

$$E(G_{941}) = \big\{\{x, y\} \bigm| |x - y| \in D \text{ or } 941 - |x - y| \in D\big\},$$

where D is a distance set defined below,

$$\begin{aligned} D = \{&1, 12, 15, 32, 34, 37, 40, 42, 44, 46, 50, 52, 54, 55, 65, 73, 83, 93, 97, 112, 114,\\ &116, 118, 119, 122, 123, 131, 140, 142, 144, 145, 147, 153, 154, 161, 167, 172,\\ &175, 178, 180, 182, 189, 191, 198, 202, 207, 215, 218, 223, 225, 234, 243, 248,\\ &251, 254, 278, 281, 282, 293, 302, 304, 310, 311, 317, 318, 323, 328, 339, 341,\\ &380, 384, 386, 389, 392, 399, 402, 403, 406, 408, 410, 413, 418, 419, 427, 428,\\ &431, 437, 444, 447, 451, 454, 461, 466, 467\}. \end{aligned}$$

One can check that G_{941} is K_4-free, 188-regular graph with $|V(G_{941})| = 941$, $|E(G_{941})| = 88\,454$ and $t_\triangle(G_{941}) = 707\,632$. Then, the graph H corresponding to G_{941} in Theorem 2.3 is 48-regular with $|V(H)| = 88\,454$, $|E(H)| = 3t_\triangle(G_{941}) = 2\,122\,896$. Moreover, using in MATLAB [12] the function `eigs` for real, symmetric and sparse matrices with option `sa`, we get $\lambda_{min}(H) \geq -15.196$. Thus, Proposition 2.4 implies,

$$M(H) \leq \frac{|E(H)|}{2} - \frac{\lambda_{min}(H)|V(H)|}{4} \leq 1\,397\,484.746 < 1\,415\,264 = 2t_\triangle(G_{941}).$$

Consequently, Theorem 2.3 yields the main result of this note.

Theorem 2.5. *The Folkman number $f(2, 3, 4) \leq 941$.*

Remark 2.6. *For given numbers n and r, let $G(n, r)$ be a circulant graph with the vertex set*

$$V(G(n, r)) = \mathbb{Z}_n,$$

$G(n,r)$	ρ
$G(127,3)$	0.030884
$G(281,4)$	0.042306
$G(313,4)$	0.040612
$G(337,4)$	0.034517
$G(353,4)$	0.037667
$G(457,4)$	0.030386
$G(541,5)$	0.049676
$G(571,5)$	0.044144
$G(701,5)$	0.029507
$G(769,6)$	0.044195
$G(937,6)$	0.048529
$\boldsymbol{G(941,5)}$	-0.012728

Table 1: Candidates for membership and one member of $\mathcal{F}(2,3,4)$.

and the edge set

$$E(G(n,r)) = \big\{\{x,y\} \mid x \neq y \textit{ and } x - y = \alpha^r \textit{ mod } n\big\}.$$

Note that $G(n,r)$ is well-defined, i.e. the graph is undirected, if -1 is an r-th residue of n. In particular, $G_{941} = G(941,5)$. By exhaustive search we found that G_{941} is the smallest graph, which belongs to the family $\mathcal{F}(2,3,4)$, among all graphs $G(n,r)$ for which our technique works.

For a given K_4-free graph $G(n,r)$ let H be a graph, which corresponds to $G(n,r)$ from Theorem 2.3. Let $\alpha = \frac{|E(H)|}{2} - \frac{\lambda_{min}(H)|V(H)|}{4}$ and $\beta = 2t_\triangle(G(n,r))$. In view of Theorem 2.3 and Proposition 2.4, if $\alpha < \beta$, then $G(n,r) \rightarrow \triangle$, and so, $G(n,r) \in \mathcal{F}(2,3,4)$. Obviously the converse is not true since α is only an approximation on $M(H)$. We define a parameter $\rho = \frac{\alpha-\beta}{\alpha}$ to get an estimate how "close" $G(n,r)$ is from property $\mathcal{F}(2,3,4)$. In Table 2.3 we listed all (up to isomorphism) K_4-free graphs $G(n,r)$ with $n \leq 941$ and $\rho < 0.05$.

3 Concluding remarks

Recently, S. P. Radziszowski and Xu Xiaodong suggested [15] that the graph $G_{127} = G(127,3)$, considered by R. Hill and R. W. Irving [8], belongs to the family $\mathcal{F}(2,3,4)$. One can check that $t_\triangle(G_{127}) = 9779$. Let H be a graph from Theorem 2.3 which corresponds to G_{127}. Using a semidefinite program with polyhedral relaxations [16, 17] we obtained an upper bound on $M(H) \leq 19558 = 2t_{\triangle}(G_{127})$. Note that $2t_{\triangle}(G_{127})$ is also the straightforward upper bound from Theorem 2.3. This coincidence between numerical and theoretical bounds may suggest that $G_{127} \nrightarrow \triangle$. However, the question whether $G_{127} \in \mathcal{F}(2, 3, 4)$, remains still open.

A related, interesting question is to find a reasonable upper for $f(3, 3, 4)$. We tried to find another argument that would ensure the existence of relatively small K_4-free graphs. Such a construction for 2-colors was considered in an earlier version of our paper (see, e.g., [2]). The existence of a reasonably small graph G that yields a monochromatic triangle under every 3-coloring is an open question which we are currently trying to address.

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