Rachunek Prawdopodobieństwa 2

Zestaw zadań nr 5

Termin realizacji: 28 XI 2008

1. Let Z_n be a branching process with $Z_0 = 1$, $E(Z_1) = \mu$, and $var(Z_1) > 0$. Show that $E(Z_n Z_m) = \mu^{n-m} E(Z_m^2)$ for $m \leq n$. Hence find the correlation coefficient $\rho(Z_m, Z_n)$ in terms of μ. (Hint: use conditional expectation.)

2. Udowodnić wzór na $G_n(s)$ dla geometrycznego procesu gałązkowego podany na wykładzie. Korzystając z niego, znaleźć rozkład zmiennej losowej $T = \min\{n : Z_n = 0\}$. (Gdy $\{n : Z_n = 0\} = \emptyset$, przyjmujemy $T = \infty$.) Kiedy $E(T) < \infty$?

3. Show that the generating function H_n of the total number of individuals in the first n generations of a branching process satisfies $H_n(s) = sG(H_{n-1}(s))$.

4. Show that the number Z_n of individuals in the nth generation of a branching process satisfies

$$P(Z_n > N \quad | \quad Z_m = 0) \leq G_m(0)^N$$

for $n < m$.

5. A die is rolled repeatedly. Which of the following are Markov chains? For those that are, supply the transition matrix.

   (a) The largest number X_n shown up to the nth roll.

   (b) The number N_n of sixes in n rolls.

   (c) At time r, the time C_r since the most recent six.

   (d) At time r, the time B_r until the next six.

6. Let X be a Markov chain. Which of the following are Markov chains? (a) X_{m+r}, $r \geq 0$; (b) X_{2m}, $m \geq 0$; (c) the sequence of pairs (X_n, X_{n+1}), $n \geq 0$.

7. Show that if P is stochastic (doubly stochastic) then so is P^n for all n.

8. Spacerując losowo po cyklu C_4 (jak na rys. 5) wyznaczyć rozkład $\mu^{(n)}$. To samo pytanie dla pogody w Geteborgu. W obu przypadkach znaleźć rozkład stacjonarny.

9. Let X be a Markov chain with $S = \{s_1, s_2, s_3\}$,

$$P = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix},$$

   and $\mu^{(0)} = (1/3, 1/3, 1/3)$. Define $Y_n = 0$ if $X_n = s_1$ and $Y_n = 1$ otherwise. Show that Y is not a Markov chain. (Hint: this might be a hard problem.)