Ann. Comb.
DOI 10.1007/s00026-017-0345-1
© Springer International Publishing 2017

Annals of Combinatorics

On the Hamiltonicity of Triple Systems with High Minimum Degree*

Vojtěch Rödl[1], Andrzej Ruciński[2], Mathias Schacht[3], and Endre Szemerédi[4]

[1]Department of Mathematics and Computer Science, Emory University, Atlanta, USA
rodl@mathcs.emory.edu

[2]A. Mickiewicz University, Department of Discrete Mathematics, Poznań, Poland
rucinski@amu.edu.pl

[3]Fachbereich Mathematik, Universität Hamburg, Hamburg, Germany
schacht@math.uni-hamburg.de

[4]Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, Hungary
szemered@cs.rutgers.edu

Received May 03, 2016

Mathematics Subject Classification: 05C65, 05C45

Abstract. We show that every 3-uniform hypergraph with minimum vertex degree at least $0.8\binom{n-1}{2}$ contains a tight Hamiltonian cycle.

Keywords: hypergraphs, Hamiltonian cycles, Dirac's theorem

1. Introduction

In 1952 Dirac [6] proved that every graph $G = (V, E)$ with $|V| \geq 3$ and minimum vertex degree $\delta(G)$ at least $|V|/2$ contains a Hamiltonian cycle. Moreover, this is optimal as there are graphs G with $\delta(G) = \lceil |V|/2 \rceil - 1$ not containing a Hamiltonian cycle. We study an analogous Dirac-type problem for 3-uniform hypergraph, i.e., what minimum vertex degree in a 3-uniform hypergraph guarantees the existence of a (tight) Hamiltonian cycle? A lot of recent research concerning Dirac-type problems for hypergraphs originated in the work of Katona and Kierstead [16] (see also [26] for an overview).

A *k-uniform hypergraph* $H = (V, E)$ (or *k-graph*) consists of a finite set $V = V(H)$ of *vertices* together with a family $E = E(H)$ of k-element subsets of V, the so-called (*hyper*)*edges*. Whenever convenient we identify H with $E(H)$. In particular, we denote by $|H| = |E(H)|$ the number of edges in H. For k-graphs with $k \geq 3$, a cycle

* V. Rödl was supported by NSF grant DMS 1301698. A. Ruciński was supported by the Polish NSC grant 2014/15/B/ST1/01688. M. Schacht was supported through the *Heisenberg-Programme* of the DFG. Part of this research was carried out during research stays of A. Ruciński and M. Schacht at Emory University.

might be defined in several ways (see, e.g., [1, 2, 16, 20]). Here we restrict ourselves to $k = 3$. For $l = 1$ or 2 and an integer n with $(3-l)|n$, define an *l-overlapping cycle* C_n^l as an n-vertex 3-graph with $\frac{n}{3-l}$ edges, whose vertices can be ordered cyclically in such a way that the edges are segments of that cyclic ordering and every two consecutive edges share exactly l vertices. For $l = 2$, we call the cycle *tight* and for $l = 1$ we call it *loose*. A *tight* (*loose, respectively*) *Hamiltonian cycle* in a 3-graph H is a spanning tight (loose, respectively) cycle in H, that is, a subhypergraph of H which is a tight (loose, respectively) cycle that contains all vertices of H.

For a 3-graph $H = (V, E)$, in addition to having two types of cycles, there are also two natural notions of minimum vertex degree (see $\delta_1(H)$ and $\delta_2(H)$ below). For a vertex $v \in V$ we define $\deg_H(v)$ as the number of edges of H containing v and for every pair of distinct vertices $u, v \in V$ we define the *co-degree/pair degree* of that pair, $\deg_H(u, v)$, by the number of edges of H containing both u and v. Clearly, for an n-vertex 3-graph we have $\deg_H(v) \le \binom{n-1}{2}$ while $\deg_H(u, v) \le n-2$. For a 3-graph $H = (V, E)$ we denote by

$$\delta_1(H) = \delta(H) = \min_{v \in V} \deg_H(v)$$

the *minimum vertex degree of H* and by

$$\delta_2(H) = \min_{\substack{u, v \in V \\ u \neq v}} \deg_H(u, v)$$

the *minimum co-degree of H*.

We are now ready to define a crucial Dirac-type extremal parameter.

Definition 1.1. *Let d, l, and n be integers satisfying $1 \le l, d \le 2$, and $(3-l)|n$. The function $h_d^l(n)$ equals the smallest integer h such that every n-vertex 3-graph H with $\delta_d(H) \ge h$ contains a spanning l-overlapping cycle, that is, a loose Hamiltonian cycle for $l = 1$ and a tight Hamiltonian cycle for $l = 2$. In other words,*

$$h_d^l(n) = \min\left\{h \in N\colon\ \delta_d(H) \ge h \implies H \supseteq C_n^l\right\}.$$

For large n and the following choices of d and l the function $h_d^l(n)$ is well understood. The case $d = l = 2$ (co-degree forcing Hamiltonian tight cycles) was solved approximately and exactly (for large n) in [28, 31], while the case $d = 2$ and $l = 1$ (co-degree forcing Hamiltonian loose cycles) was solved approximately in [20]. In [4], an approximate formula for $h_1^1(n)$ (vertex degree forcing Hamiltonian loose cycles) was found, while an exact form of this result was obtained in [13]. The related problem concerning minimum degree conditions for perfect matchings was resolved for co-degrees approximately and exactly in [28, 30] and similarly for vertex degrees in [11, 18, 21].

In particular, the results mentioned above resolve the asymptotic behaviour for all possible values of d and l with the exception of $d = 1$ and $l = 2$. It seems that more difficulties arise in that case, since $d < l$ and, hence, we are not in control of co-degrees, while it seems that large co-degrees are instrumental in building long tight paths and cycles. We will derive new bounds for $h_1^2(n)$ and for simplicity we set

$$h(n) = h_1^2(n).$$

Some estimates on $h(n)$ were obtained over the last few years. While proving a more general result, Glebov, Person, and Weps [9] showed that

$$h(n) \le (1-\varepsilon)\binom{n-1}{2},$$

where the numerical value of ε is close to 5×10^{-7}. In [27] the first two authors improved upon that bound by showing that for every $\gamma > 0$ there exists n_0 such that if $n \ge n_0$ then

$$h(n) \le \left(\frac{5-\sqrt{5}}{3}+\gamma\right)\binom{n-1}{2} \approx .92\binom{n-1}{2}.$$

Here we make a further improvement.

Theorem 1.2. *There exists $n_{1.2}$ such that if $n \ge n_{1.2}$ then*

$$h(n) \le .8\binom{n-1}{2}.$$

This upper bound on $h(n)$ seems to be far from optimal. Indeed, the best known constructions yield

$$h(n) \ge \left(\frac{5}{9}+o(1)\right)\binom{n-1}{2}$$

and we briefly mention three constructions achieving this bound.

(i) Consider a partition $X \mathbin{\dot\cup} Y = V$ of the vertex set V of size n with $|X| = \lceil (n+1)/3 \rceil$ and let H be the 3-graph containing all edges e such that $|e\cap X| \ne 2$. It is not hard to show that H contains no tight Hamiltonian cycle, since two consecutive vertices in X cannot be connected to Y (see, e.g., [27]). Moreover, we have $\delta(H) \ge (5/9+o(1))\binom{n-1}{2}$.

(ii) Similarly, one may consider a partition $X \mathbin{\dot\cup} Y = V$ with $|X| = \lceil 2n/3 \rceil$ and let H be the 3-graph consisting of all hyperedges e such that $|e\cap X| \ne 2$. Again H has $\delta(H) \ge (5/9+o(1))\binom{n-1}{2}$ and it contains no tight Hamiltonian cycle.

(iii) The last example utilises the fact that every tight Hamiltonian cycle contains a matching of size $\lfloor n/3 \rfloor$. Again we consider a partition $X \mathbin{\dot\cup} Y = V$ this time with $|X| = \lfloor n/3 \rfloor - 1$ and let H consist of all hyperedges having at least one vertex in X. Consequently, H contains no matching of size $\lfloor n/3 \rfloor$ and, hence, no tight Hamiltonian cycle. On the other hand, $\delta(H) \ge (5/9+o(1))\binom{n-1}{2}$.

It might be possible that these constructions give the right asymptotic lower bound for $h(n)$, which leads to the following conjecture.

Conjecture 1.3. $h(n) = \left(\frac{5}{9}+o(1)\right)\binom{n-1}{2}$.

However, we remark that recently Han and Zhao [14] showed that for $k \ge 4$ some Dirac-type thresholds for tight Hamiltonian cycle are strictly larger than the corresponding thresholds for perfect matchings, which may put some doubt on Conjecture 1.3. However, it seems unlikely that the upper bound given by Theorem 1.2 is optimal and we shall return to the problem of determining the asymptotic behaviour of $h(n)$ in the near future.

2. Outline of Proof and Preliminaries

2.1. Outline

Our proof follows the absorbing path method developed in [28, 29, 31]. We begin with building an absorbing path A and putting aside a small reservoir set R selected randomly so that $H[R]$ preserves the degree properties of H. Then a long cycle C containing A is created in the remaining hypergraph (by first building a family of disjoint paths and then connecting them, as well as A, together via the reservoir R). Finally, utilising the absorbing property of A, the cycle C is extended to a Hamiltonian cycle in H.

Our proof is founded on four pillars: the Connecting Lemma, the Absorbing Lemma, the Reservoir Lemma, and the Cover Lemma (replacing the Path Cover Lemma used earlier in [27, 29]), and we will prove them all in the next section. The Connecting Lemma and the Absorbing Lemma are the bottlenecks here. In [27] we 'shaved' the hypergraph H from edges containing pairs of small degree until all pairs of positive degree were large, that is, of degree a little bigger than $n/2$. In the obtained subhypergraph H' proving a connecting lemma was easy, however we paid a high price for that: to prevent H' from becoming empty, we had to raise the minimum vertex degree of H to about $.92\binom{n-1}{2}$. Here we refine that approach: we only dispose of the edges of H with all three pairs of small degree and, at the same time, we lower the notion of "small" to only $n/3$. Then both, the Connecting Lemma and the Absorbing Lemma, are a bit harder to prove, yet we manage to do so, keeping $\delta(H)$ at around $.8\binom{n-1}{2}$. The Reservoir Lemma, as usual, can be proved by a standard application of the probabilistic method. Finally, the proof of the Cover Lemma follows the lines of the approach from [27, 29] in that it relies on the Weak Regularity Lemma. Once the four lemmas are proved, the actual proof of Theorem 1.2 consists of five simple steps (stated below). For any $S \subset V(H)$, let $H - S$ denote the induced subhypergraph $H[V(H) \setminus S]$, that is, a subhypergraph obtained from H by deleting all vertices in S together with the edges they belong to.

(1) Find an absorbing path A in H.
(2) Find a reservoir set R in $H - V(A)$.
(3) Applying the Cover Lemma to a suitable selected sub-3-graph H' of H, find a collection of disjoint paths $\mathcal{P}$, covering most of the vertices of $H - (V(A) \cup R)$.
(4) Connect the paths in $\mathcal{P}$ and the absorbing path A, using vertices of R, to form a cycle C which contains most of the vertices of H.
(5) Using the absorbing property of A, put all the remaining vertices on the cycle to form a Hamiltonian cycle in H.

2.2. Preliminaries

Here we collect basic tools needed in the subsequent proofs. We begin with a lower bound on the number of triangles in an n-vertex graph in terms of the number of its edges. Although more refined results are available (see Razborov [25]), for us it will be sufficient to use an old bound of Nordhaus and Stewart [24] which is also attributed to Goodman [10] and Moon and Moser [23] (see [3, Corollary 1.6 in Chapter VI]).

Lemma 2.1. *Every graph G with n vertices and m edges contains at least $\frac{m}{3n}(4m-n^2)$ triangles. In particular, if for some $\rho > 0$ we have $m \geq \rho\frac{n^2}{2}$ then the number of triangles in G is at least $\rho(2\rho-1)\frac{n^3}{6}$.*

We will also need the following version of a result of Erdős [7]. A 3-graph H is 3-*partite* if there is a partition $V(H) = V_1 \cup V_2 \cup V_3$ such that every edge of H intersects each set V_i in precisely one vertex. A 3-partite 3-graph with $|V_1||V_2||V_3|$ edges is called *complete* and denoted by K_{h_1,h_2,h_3}, where $h_i = |V_i|$, $i = 1, 2, 3$.

Lemma 2.2. *For every $d > 0$ and an integer $h \geq 1$, there exist $c > 0$ and $n_{2.2}$ such that every 3-uniform hypergraph H on $n \geq n_{2.2}$ vertices and with at least dn^3 edges contains at least cn^{3h} copies of $K_{h,h,h}$.*

In the proof of the Cover Lemma we will also need the so-called *weak hypergraph regularity lemma*, which is a straightforward extension of Szemerédi's regularity lemma [33] from graphs to hypergraphs (see, e.g., [5, 8, 32]).

Given a 3-graph H and three non-empty, disjoint subsets $A_i \subset V(H)$, $i = 1, 2, 3$, by $H[A_1, A_2, A_3]$ we denote the 3-partite 3-graph with vertex set $A_1 \cup A_2 \cup A_3$ which consists of all edges in H with one vertex in each A_i. We set $e_H(A_1, A_2, A_3)$ for the number of edges of $H[A_1, A_2, A_3]$ and define *the density* of H with respect to (A_1, A_2, A_3) as

$$d_H(A_1, A_2, A_3) = \frac{e_H(A_1, A_2, A_3)}{|A_1||A_2||A_3|}.$$

We say that a 3-partite 3-graph H with 3-partition (V_1, V_2, V_3) is *ε-regular* if for all $A_i \subseteq V_i$ with $|A_i| \geq \varepsilon|V_i|$, $i = 1, 2, 3$,

$$|d_H(A_1, A_2, A_3) - d_H(V_1, V_2, V_3)| \leq \varepsilon.$$

Lemma 2.3. (Weak Regularity Lemma for 3-graphs) *For all $\varepsilon > 0$ and every integer t_0 there exist T_0 and $n_{2.3}$ such that the following holds. For every 3-graph H on $n \geq n_{2.3}$ vertices there is for some t, with $t_0 \leq t \leq T_0$, a partition $V(H) = V_1 \cup \cdots \cup V_t$ such that $|V_1| \leq |V_2| \leq \cdots \leq |V_t| \leq |V_1| + 1$ and for all but less than $\varepsilon\binom{t}{3}$ triplets of partition classes $\{V_{i_1}, V_{i_2}, V_{i_3}\}$, the 3-partite 3-graph $H[V_{i_1}, V_{i_2}, V_{i_3}]$ is ε-regular.*

Any partition guaranteed by Lemma 2.3 will be referred to as *ε-regular*.

For brevity, we will often write uv instead of $\{u, v\}$. *The link graph* of a vertex u in a 3-graph H is defined as

$$H(u) = \{vw\colon \{u, v, w\} \in H\}.$$

Note that for every $v \neq u$

$$\deg_{H(u)}(v) = \deg_H(u, v). \tag{2.1}$$

For each real $\alpha \in (0, 1)$ we define

$$G_\alpha = \{uv\colon \deg_H(u, v) \geq \alpha(n-2)\}$$

and call a pair α-*large* if it is in G_α. The 1/3-large pairs play a special role in our proof. However, also $G_{.33}$ will appear in our proof and should not be confused with $G_{1/3}$. Let

$$H_0 = \left\{ e \in H \colon \binom{e}{2} \cap G_{1/3} = \varnothing \right\} \quad \text{and} \quad H' = H \smallsetminus H_0, \tag{2.2}$$

that is, H' is a spanning subhypergraph of H with all edges of H_0 removed. Note that every edge of H' contains at least one pair from $G_{1/3}$.

We build a tight Hamiltonian cycle in H from several small pieces. *Tight paths* are defined in the same way as tight cycles, but with respect to a *linear* ordering of the vertices. From now on we will refer to tight paths and cycles as paths and cycles, respectively. If P is a path with $t \geq 3$ vertices $v_1, \dots, v_t$ and $t-2$ edges $\{v_1, v_2, v_3\}, \dots, \{v_{t-2}, v_{t-1}, v_t\}$, then we call the ordered pairs (v_1, v_2) and (v_t, v_{t-1}) the *endpairs* of P, and we say that P *connects* its endpairs. The *length* of a path is defined as the number of its edges and the *order* denotes its number of vertices.

3. The Four Pillars

In this section we prove the four crucial lemmas: the Connecting Lemma, the Absorbing Lemma, the Reservoir Lemma, and the Cover Lemma.

3.1. The Connecting Lemma

The connecting lemma in [29] assumes that $\delta_2(H)$ is large and guarantees a short path between any two ordered pairs of vertices. There is no hope for such a result here, as some pairs may have very small degree, even zero. So, we must be content with connecting just the pairs with large degrees. As a first step we establish a numerical relation between $\delta(H)$ and $\delta(G_\alpha)$. To this end, for all $0 < \alpha < c < 1$, define

$$g_c(\alpha) = \frac{c-\alpha}{1-\alpha}.$$

Claim 3.1. Let $0 < \alpha < c < 1$. If $\delta(H) \geq c\binom{n-1}{2}$ then $\delta(G_\alpha) \geq g_c(\alpha)(n-1)$.

Proof. Set $G := G_\alpha$. Let $u_0 \in V(H)$ satisfy $\deg_G(u_0) = \delta(G)$. Then, by (2.1) we have

$$2\delta(H) \leq 2|H(u_0)| = \sum_{u \neq u_0} \deg_{H(u_0)}(u) = \sum_{u \neq u_0} \deg_H(u, u_0).$$

Breaking the latter sum into two parts: over $uu_0 \in G$ and over $uu_0 \notin G$, and recalling that $|\{u \colon uu_0 \in G\}| = \delta(G)$, we obtain the inequality

$$c(n-1)(n-2) \leq 2\delta(H) \leq \sum_{u \neq u_0} \deg_H(u, u_0) \leq \delta(G)(n-2) + (n-1-\delta(G))\alpha(n-2),$$

from which the required bound follows. ∎

We also need a simple combinatorial inequality which was observed already in [27, Fact 1]).

Claim 3.2. For any two finite sets B and R, with $|B| \leq |R|$, the set

$$\Pi(B,R) = \{\{b,r\} : b \in B, r \in R, b \neq r\} = \left\{e \in \binom{B \cup R}{2} : e \cap B \neq \varnothing \text{ and } e \cap R \neq \varnothing\right\}$$

has size

$$|\Pi(B,R)| \geq \binom{|B|}{2}.$$

Proof. Let $c = |B \cap R|$. Then, as $|R| \geq |B| \geq c$,

$$\begin{aligned}
|\Pi(B,R)| &= \binom{|B|+|R|-c}{2} - \binom{|B|-c}{2} - \binom{|R|-c}{2} \\
&= |B| \cdot |R| - \binom{c+1}{2} \\
&\geq |B|^2 - \binom{|B|+1}{2} \\
&= \binom{|B|}{2}.
\end{aligned}$$

∎

We are now ready to prove the Connecting Lemma.

Lemma 3.3. (Connecting Lemma) *There exists $n_{3.3}$ such that for all $n \geq n_{3.3}$ the following holds. Let H be an n-vertex 3-graph with $\delta(H) \geq .799\binom{n-1}{2}$. Then, for all $e = u_0u_1 \in G_{.33}$ and $f = v_0v_1 \in G_{.33}$ with $e \cap f = \varnothing$ there exists a path in H of length 12 connecting the endpairs (u_0, u_1) and (v_0, v_1).*

Proof. We build a connecting path by making our way from each side, increasing the degrees of the pairs as we go, until they both reach $.65(n-2)$, a quantity that guarantees an immediate connection of the two paths. In doing so we will use a sequence of numbers $\alpha_1, \dots, \alpha_5$ such that $\alpha_1 = .33$, $\alpha_5 = .65$, and for each $i = 2, \dots, 5$, with $c = .799$, we have

$$\alpha_i + g_c(\alpha_{i+1}) > 1.$$

By inspection we found that $\alpha_2 = .39$, $\alpha_3 = .48$, and $\alpha_4 = .58$ satisfy these requirements. Below we use Claim 3.1 repeatedly. For brevity, we write $g(\alpha)$ for $g_{.799}(\alpha)$. Moreover, for a graph G and a vertex v we denote by $N_G(v)$ the neighbourhood of v in G and, similarly, we denote by $N_H(u,v) = \{x \in V(H) : \{x,u,v\} \in E(H)\}$ the neighbours of the pair uv in the 3-graph H.

- Observe that, in view of Claim 3.1,

  $$\left|N_{G_{.39}}(u_1)\right| \geq \delta(G_{.39}) \geq g(.39)(n-1) = \tfrac{409}{610}(n-1) \geq .67(n-2) + 20,$$

  where the last inequality holds for sufficiently large n. Thus,

  $$|N_H(u_0,u_1)| + \left|N_{G_{.39}}(u_1)\right| \geq .33(n-2) + .67(n-1) + 20 > n + 18,$$

implying that

$$\left|N_H(u_0,u_1)\cap N_{G_{.39}}(u_1)\right| \geq |N_H(u_0u_1)| + \left|N_{G_{.39}}(u_1)\right| - (n-1) \geq 20.$$

Consequently, there exists a vertex $u_2 \notin \{v_0, v_1\}$ such that $\{u_0, u_1, u_2\} \in H$ and $u_1u_2 \in G_{.39}$.

- Next, since

$$\delta(G_{.48}) \geq g(.48)(n-1) = \tfrac{319}{520}(n-1) \geq .61(n-2)+20,$$

a similar argument yields that

$$\left|N_H(u_1,u_2)\cap N_{G_{.39}}(u_2)\right| \geq 20,$$

implying the existence of a vertex $u_3 \notin \{u_0, v_0, v_1\}$ such that $\{u_1, u_2, u_3\} \in H$ and $u_2u_3 \in G_{.48}$.

- Analogously, since

$$\delta(G_{.58}) \geq g(.58)(n-1) = \tfrac{219}{420}(n-1) \geq .52(n-2)+20,$$

there exists a vertex $u_4 \notin \{u_0, u_1, v_0, v_1\}$ such that $\{u_2, u_3, u_4\} \in H$ and $u_3u_4 \in G_{.58}$.

- Finally, since

$$\delta(G_{.65}) \geq g(.65)(n-1) = \tfrac{149}{350}(n-1) \geq .42(n-2)+20,$$

there exists a vertex $u_5 \notin \{u_3, u_4, v_0, v_1\}$ such that $\{u_3, u_4, u_5\} \in H$ and $u_4u_5 \in G_{.65}$.

Hence, we have created a 4-edge path $P_u = u_0u_1\cdots u_5$ in H with $V(P_u)\cap f = \varnothing$ and $u_4u_5 \in G_{.65}$.

In a similar fashion we build a path $P_v = v_0\cdots v_5$ which avoids all vertices of P_u and such that also $\{v_4, v_5\} \in G_{.65}$. The additional $+20$ guarantees, with a margin, that even when choosing the last vertex, v_5, we can still avoid the already selected vertices. To connect the two paths together, let us consider the intersection of link graphs of u_5 and v_5

$$I = H(u_5)\cap H(v_5).$$

Owing to the assumption $\delta(H) \geq .799\binom{n-1}{2}$ we have

$$\begin{aligned} |I| &\geq |H(u_5)| + |H(v_5)| - |H(u_5)\cup H(v_5)| \\ &\geq 2\delta(H) - \binom{n}{2} \\ &\geq .598\binom{n}{2} + O(n). \end{aligned} \tag{3.1}$$

Set

$$B = N_H(u_4, u_5) \quad \text{and} \quad R = N_H(v_4, v_5)$$

and assume as we may that $|B| \le |R|$. Let F be the set of pairs $xy \in \binom{V(H)}{2}$ such that $\{u_4, u_5, x\} \in H$ and $\{v_4, v_5, y\} \in H$, that is, the set of pairs of vertices with one vertex belonging to B and the other to R. By Claim 3.2, $|F| \ge \binom{|B|}{2}$ and, since $u_4u_5 \in G_{.65}$, we have $|B| \ge .65(n-2)$. Consequently,

$$|F| \ge \binom{|B|}{2} \ge (.65)^2 \binom{n}{2} + O(n).$$

Since $.598 + (.65)^2 > 1.02$, it follows from (3.1) and the above bound that

$$|F \cap I| \ge |F| + |I| - \binom{n}{2} > .02 \binom{n}{2} + O(n),$$

which, for sufficiently large n, is greater than $8n$, which is an upper bound on the number of pairs $\{x, y\} \in F \cap I$ with $\{x, y\} \cap V(P_u \cup P_v) \neq \varnothing$. We conclude that there exist two vertices $x = u_6$ and $y = v_6$ different from all $u_0, \ldots, u_5, v_0 \ldots, v_5$, such that all four triples $\{u_4, u_5, u_6\}$, $\{u_5, u_6, v_6\}$, $\{u_6, v_6, v_5\}$, $\{v_6, v_5, v_4\}$ are edges of H. Hence, a path between the endpairs (u_0, u_1) and (v_0, v_1) of length 12 can be completed (see Figure 1). ∎

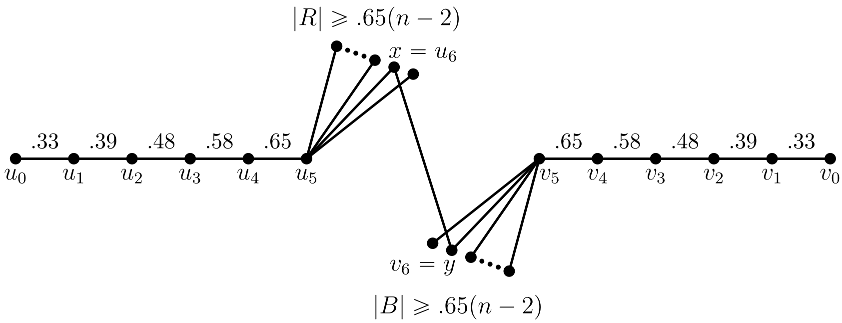

Figure 1: Underlying pairs of the tight path of length 12 in H connecting the pairs u_0u_1 with v_1v_0 from $G_{.33}$.

3.2. The Absorbing Lemma

We first define absorbing path and introduce the notion of an absorber in this context.

Definition 3.4. *We call a path A in H m-*absorbing *if for every subset $U \subset V(H) \setminus V(A)$ of size $|U| \le m$ there is a path A_U in H with $V(A_U) = V(A) \cup U$ and with the same endpairs as A.*

Recall that a pair of vertices in H is called *α-large* if it belongs to G_α.

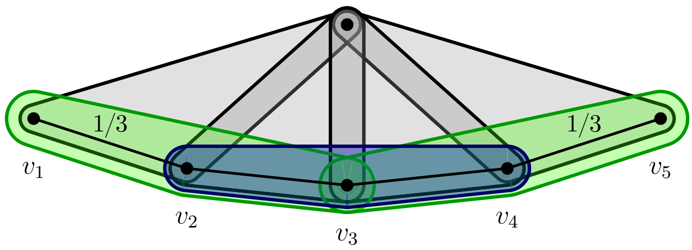

Figure 2: An x-absorber of order 5 on $v_1,\dots,v_5$ with v_1v_2 and v_4v_5 from $G_{1/3}$.

Definition 3.5. *Given a vertex $x \in V(H)$, an x-*absorber of order *i is a path $P = v_1 \cdots v_i$ in H such that the graph path $Q = v_1 \cdots v_i$ is a subgraph of the link graph $H(x)$ and the endpairs v_1v_2 and v_iv_{i-1} of P belong to $G_{1/3}$ (see Figure 2).*

The Absorbing Lemma states that every 3-graph H with sufficiently large minimum degree contains a relatively short absorbing path which may absorb a small fraction of the vertices of H.

Lemma 3.6. (Absorbing Lemma) *There exists $\gamma_0 > 0$ and $n_{3.6}$ such that for every $0 < \gamma < \gamma_0$ and every $n \geq n_{3.6}$ the following is true. If $\delta(H) \geq .8\binom{n-1}{2}$, then there exists in H a $\gamma^2 n$-absorbing path A with $(1/3)$-large endpairs and with $|V(A)| \leq \gamma n$.*

We will build an absorbing path A by connecting, via Lemma 3.3, disjoint absorbers of order 4 or 5, by paths of length 12. For the proof of Lemma 3.6 we need Claims 3.7 and 3.9 stated below.

Recall the definition of the subhypergraph $H' \subseteq H$ in (2.2). Fix a vertex $x \in V$ and let T^x stand for the family of the vertex sets of all triangles in the link graph $H'(x)$.

Claim 3.7. There exists some $c > 0$ such that for every $x \in V$ with $|V| = n$ sufficiently large we have

(a) $|T^x| \geq .296262\binom{n-1}{3}$,
(b) $|T^x \cap H'| \geq 0.005\binom{n-1}{3}$, and
(c) $T^x \cap H'$ contains at least cn^6 copies of $K_{2,2,2}$.

Proof. By the definition of $H' \subseteq H$ and by Claim 3.1,

$$\begin{aligned}
\delta(H') &\geq \delta(H) - \binom{n-1-\delta(G_{1/3})}{2} \\
&\geq \delta(H) - \binom{n-1-g_{.8}(1/3)(n-1)}{2} \\
&\geq .8\binom{n-1}{2} - \binom{.3(n-1)}{2} \\
&\geq .709\frac{n^2}{2}, \qquad (3.2)
\end{aligned}$$

for large n. The link graph $H'(x)$ has at least $\delta(H') \geq .709n^2/2$ edges and by Lemma 2.1,

$$|T^x| \geq .709(2\cdot(.709)-1)\frac{n^3}{6} \geq .296262\binom{n}{3}.$$

As

$$|H'| \geq \delta(H')\frac{n}{3} \geq .709\frac{n^3}{6} \geq .709\binom{n}{3},$$

we have

$$\begin{aligned}|T^x \cap H'| &\geq |T^x| + |H'| - |T^x \cup H'| \\ &\geq (.296262 + .709 - 1)\binom{n}{3} \\ &> 0.005\binom{n}{3}.\end{aligned}$$

Therefore, by Lemma 2.2 with $d = 0.005$ and $h = 2$, there exists a constant c such that for $n \geq n_{2.2}$ there are at least cn^6 copies of $K_{2,2,2}$ in $T^x \cap H'$. ∎

Recall that $e \in H'$ if $\binom{e}{3} \cap G_{1/3} \neq \varnothing$, and $e \in T^x$ if the vertices of e form a triangle in $H'(x)$. We will need the following notions.

Definition 3.8. *Let x be a vertex of H.*

- *Every copy of $K_{2,2,2}$ contained in the 3-graph $T^x \cap H'$ is called* x-friendly.
- *For a copy K of $K_{2,2,2}$ in H', let S_K denote the set of all vertices $x \in V(H)$ for which K is x-friendly.*

Claim 3.9. Every copy K of $K_{2,2,2}$ in H' contains a path of order 4 or 5, which is an x-absorber for every $x \in S_K$.

Proof. Let the three partition classes of K are $\{u_1, u_2\}$, $\{v_1, v_2\}$, and $\{w_1, w_2\}$. We have to find a path of order 4 or 5 in K whose endpairs are in $G_{1/3}$. Consider the two disjoint hyperedges $\{u_1, v_1, w_1\}$ and $\{u_2, v_2, w_2\}$ of K. By definition of H' each of these two hyperedges must contain at least one pair from $G_{1/3}$. By symmetry, it suffices to consider the following two cases.

Case 1 $\big(\{u_1, v_1\} \in G_{1/3}$ and $\{u_2, v_2\} \in G_{1/3}\big)$. Then $u_2v_1w_1u_2v_2$ is an x-absorber of order 5 for all $x \in S_K$.

Case 2 $\big(\{u_1, v_1\} \in G_{1/3}$ and $\{u_2, w_2\} \in G_{1/3}\big)$. Then $u_1v_1w_2u_2$ is an x-absorber of order 4 for all $x \in S_K$. ∎

With these preparations we can prove the Absorbing Lemma.

Proof of Lemma 3.6. Set

$$\gamma \leq \gamma_0 = \min\left\{\frac{c}{61}, 0.001\right\},$$

where c is given by Claim 3.7. Let $\gamma \leq \gamma_0$ and n be sufficiently large. Select randomly a family $\mathcal{F}'$ of copies of $K_{2,2,2}$ in H', independently and with probability $p = \frac{1}{30}\gamma n^{-5}$. By Markov's inequality, with probability at least $1 - 0.5 - 0.4 = 0.1$, $\mathcal{F}'$ satisfies:

- $|\mathcal{F}'| \le 2n^6 p = \frac{\gamma}{15} n$,
- there are at most $\frac{5}{2} \times n^6 \times 6 \times n^5 p^2 = 15 n^{11} p^2$ pairs of vertex-intersecting copies of $K_{2,2,2}$ in $\mathcal{F}'$,

and, for large n, by Chernoff's inequality and using Claim 3.7, with probability greater than 0.9,

- for every vertex x, there are at least $\frac{1}{2} c n^6 p$ x-friendly copies of $K_{2,2,2}$ in $\mathcal{F}'$.

Hence, there exists a family $\mathcal{F}'$ of copies of $K_{2,2,2}$ in H' satisfying all three above conditions. By removing from $\mathcal{F}'$ one copy of each intersecting pair, we obtain a subfamily $\mathcal{F}$ such that

- $|\mathcal{F}| \le \frac{\gamma}{15} n$,
- $\mathcal{F}$ consists of disjoint copies of $K_{2,2,2}$ in H',
- for every vertex x, there are at least
$$\tfrac{1}{2} c n^6 p - 15 n^{11} p^2 \ge \tfrac{c}{60} \gamma n - \tfrac{15}{900} \gamma^2 n \ge \gamma^2 n$$
x-friendly copies of $K_{2,2,2}$ in $\mathcal{F}$,

where for the last inequality we used the bound $\gamma \le c/61$.

Recall that by Claim 3.9 each copy K of $K_{2,2,2}$ in H' contains a path of order 4 or 5 which is an x-absorber for all $x \in S_K$. We now select one absorber from each copy of $K_{2,2,2}$ in $\mathcal{F}$. Let us denote the resulting family of paths by $\mathcal{P}$ and note that $|\mathcal{P}| = |\mathcal{F}| \le \gamma n/15$. Using Lemma 3.3, we will connect all paths in $\mathcal{P}$ into one path A of order at most
$$(5+10)|\mathcal{P}| - 10 \le 15 \times \frac{\gamma}{15} n \le \gamma n.$$
(There are at most 5 vertices on a path in $\mathcal{P}$ and the number of new vertices connecting this path with another one is, by Lemma 3.3, $14 - 4 = 10$, since 4 of the 14 vertices belong to the paths in $\mathcal{P}$.)

Let $P_1, \dots, P_t$, $t \le \frac{\gamma}{15} n$, be the paths (of order 4 or 5) in $\mathcal{F}$. Assume that, for some $i = 1, \dots, t-1$, we have already connected $P_1, \dots, P_i$ into a path A_i of order at most $15i$. Let H_i be the subhypergraph of H obtained by removing from H all vertices of A_i along with all vertices of $P_{i+1} \cup \dots \cup P_t$, except for one endpair $e_{i+1} = w_{i+1} w'_{i+1}$ of P_{i+1} and the endpair $e_i = w_i w'_i$ of P_i (which is also an endpair of A_i). Since $\gamma \le \gamma_0 \le 0.001$, assuming $n_{3.6} \ge n_{3.3}/(1-\gamma_0)$,
$$|V(H_i)| \ge n - 15t \ge (1-\gamma_0) n \ge n_{3.3},$$
$$\delta(H_i) \ge \delta(H) - 15tn \ge \delta(H) - \gamma n^2 \ge .799 \binom{n-1}{2} \ge .799 \binom{|V(H_i)|-1}{2},$$
and
$$\min\left\{ \deg_{H_i}(w_i, w'_i), \deg_{H_i}(w_{i+1}, w'_{i+1}) \right\} \ge \frac{1}{3} n - 15tn \ge .33n \ge .33 |V(H_i)|.$$

Hence, the assumptions of Lemma 3.3 are satisfied, and so there is a path Q_i of length 12 connecting e_i and e_{i+1} in H_i. The concatenation of the paths A_i, Q_i, and P_{i+1} constitutes the path A_{i+1}. Finally, set $A = A_t$.

To see that A is indeed a $\gamma^2 n$-absorbing path in H, consider an arbitrary subset $U \subseteq V \setminus V(A)$ of size $|U| \leq \gamma^2 n$. Since for every $x \in U$ there are at least $\gamma^2 n$ x-absorbers P_i in A, there is a one-to-one mapping $f\colon U \to \{1, \dots, t\}$ such that for every $x \in U$, $P_{f(x)}$ is an x-absorber. Let $\left(v_1^x, \dots, v_i^x\right)$ be the vertices of the path $P_{f(x)}$ $(4 \leq i \leq 5)$. Then the path obtained from A by replacing, for each $x \in U$, the edges $\left\{v_1^x, v_2^x, v_3^x\right\}$ and $\left\{v_2^x, v_3^x, v_4^x\right\}$ with $\left\{v_1^x, v_2^x, x\right\}$, $\left\{v_2^x, x, v_3^x\right\}$, and $\left\{x, v_3^x, v_4^x\right\}$, is the desired path A_U. ∎

3.3. The Reservoir Lemma

The next preparatory step toward the proof of Theorem 1.2 is to put aside a reservoir set R which should be small, quickly reachable from any pair in $G_{1/3}$, and, moreover, the induced subhypergraph $H[R]$ should satisfy the assumption of Lemma 3.3 with some margin. We state this lemma in a general form.

Lemma 3.10. *Let $U_1, \dots, U_s$ be subsets of an n-element set V and let $L_1, \dots, L_g$ be graphs on V, where s and g are both polynomials in n and such that for constants $\alpha_i, \beta_j \in (0,1)$ for $i = 1, \dots, s$ and $j = 1, \dots, g$ we have $|U_i| \geq \alpha_i n$ and $|L_j| \geq \beta_j \binom{n}{2}$, $j = 1, \dots, g$.*

Then for every constant p, $0 < p < 1$ there is $n_{3.10} = n_{3.10}(p)$ such that if $n \geq n_{3.10}$ then there exists a subset $R \subset V$ satisfying

(a) $\big||R| - pn\big| \leq pn^{2/3}$,

(b) *for all $i = 1, \dots, s$, we have $|U_i \cap R| \geq \left(\alpha_i - 2n^{-1/3}\right)|R|$, and*

(c) *for all $i = 1, \dots, g$, we have $|L_j[R]| \geq \left(\beta_j - 3n^{-1/3}\right)\binom{|R|}{2}$.*

Proof. Select a binomial random subset R of V by including to R every element of V, independently, with probability p. The random variable $|R|$ has the binomial distribution with expectation np. By Chebyshev's inequality, with probability tending to 1 as $n \to \infty$, part (a) holds.

For every i, the random variable $X_i = |U_i \cap R|$ is also binomially distributed, with expectation

$$\mathbb{E}[X_i] = |U_i| p \geq \alpha_i np.$$

Thus, by a standard application of Chernoff's bound (see [15, Inequality (2.6)]) we have

$$\begin{aligned} \mathbb{P}\left(\exists i\colon X_i \leq np\left(\alpha_i - n^{-1/3}\right)\right) &\leq \mathbb{P}\left(\exists i\colon X_i \leq |U_i|p - pn^{2/3}\right) \\ &\leq s \cdot \exp\left(-(p/2)n^{1/3}\right) \\ &= o(1), \end{aligned}$$

where in the last step we used, in passing, the trivial bound $|U_i| \le n$. Hence, using the estimate in (a), for large n, we have

$$\begin{aligned}
&\mathbb{P}\left(\exists i\colon X_i \le \left(\alpha_i - 2n^{-1/3}\right)|R|\right) \\
&= o(1) + \mathbb{P}\left(\exists i\colon X_i \le \left(\alpha_i - 2n^{-1/3}\right)|R|,\ |R| \le np\left(1+n^{-1/3}\right)\right) \\
&\le o(1) + \mathbb{P}\left(\exists i\colon X_i \le \left(\alpha_i - 2n^{-1/3}\right)np\left(1+n^{-1/3}\right)\right) \\
&\le o(1) + \mathbb{P}\left(\exists i\colon X_i \le np\left(\alpha_i - n^{-1/3}\right)\right) \\
&= o(1).
\end{aligned}$$

Consequently, the randomly chosen set R satisfies condition (b) with probability tending to 1 as $n \to \infty$.

For part (c), fix i and consider a random variable $Y_i = |L_i[R]|$ counting the number of edges $\{u, w\} \in L_i$ with $\{u, w\} \subseteq R$. Note that

$$\mathbb{E}[Y_i] = |L_i| p^2 \ge \beta_i \binom{n}{2} p^2.$$

We apply to Y_i Janson's inequality (see, e.g., [15, Theorem 2.14]), which states that

$$\mathbb{P}\left(Y_i \le \mathbb{E}[Y_i] - t\right) \le \exp\left\{-t^2/\overline{\Delta}\right\}.$$

Here $\overline{\Delta} = \sum\sum \mathbb{E}[I_e I_f]$, where the summation runs over all ordered pairs of not necessarily distinct edges of L_i which share at least one vertex, while $I_e = 1$ when $e \subset R$ and $I_e = 0$ otherwise. Observe that, up to the order of magnitude, $\overline{\Delta}$ is equal to the expected number of pairs of edges of L_i, sharing a vertex, whose all three vertices are included in R. Thus, $\overline{\Delta} = \Theta(n^3)$, and, consequently, with $t = n^{-1/3}\binom{n}{2}p^2$, we have

$$\begin{aligned}
&\mathbb{P}\left(\exists i\colon Y_i \le \left(\beta_i - n^{-1/3}\right)\binom{n}{2}p^2\right) \\
&\le \mathbb{P}\left(\exists i\colon Y_i \le \left(\mathbb{E}[Y_i] - t\right)\right) \\
&\le g \exp\left\{-\Theta\left(n^{1/3}\right)\right\} \\
&= o(1).
\end{aligned}$$

Using part (a) again, for large n, we obtain

$$\begin{aligned}
&\mathbb{P}\left(\exists i\colon Y_i \le \left(\beta_i - 3n^{-1/3}\right)\binom{|R|}{2}\right) \\
&= o(1) + \mathbb{P}\left(\exists i\colon Y_i \le \left(\beta_i - 3n^{-1/3}\right)\binom{|R|}{2},\ |R| \le np\left(1+n^{-1/3}\right)\right) \\
&\le o(1) + \mathbb{P}\left(\exists i\colon Y_i \le \left(\beta_i - 3n^{-1/3}\right)\binom{n}{2}p^2\left(1+n^{-1/3}\right)^2\right)
\end{aligned}$$

$$\leq o(1)+\mathbb{P}\left(\exists i\colon Y_i \leq \left(\beta_i - n^{-1/3}\right)\binom{n}{2}p^2\right)$$
$$= o(1),$$

which means that the random set R satisfies the condition of part (c) with probability tending to 1 as $n \to \infty$. In summary, for sufficiently large n, the probability that at least one of conditions (a), (b), or (c) fails is less than 1, and thus, there exists a set $R \subset V$ satisfying all three properties (a), (b), and (c). ∎

3.4. The Cover Lemma

Recall that $K_{L,L,L}$ denotes the complete 3-partite 3-graph on vertex classes of size L.

Lemma 3.11. (Cover Lemma) *For every $\rho > 0$, $\lambda > 0$, and an integer L, there exists an integer $n_{3.11}$ such that every 3-graph H with $n \geq n_{3.11}$ vertices and $\delta(H) \geq \left(\frac{5}{9}+\lambda\right)\binom{n-1}{2}$, contains a family of vertex-disjoint copies of $K_{L,L,L}$, which together cover at least $(1-\rho)n$ vertices of H.*

In the proof of Lemma 3.11 we will need the following result from [11].

Theorem 3.12. *For every $\beta > 0$ there exists t_1 such that every 3-graph H with $t \geq t_1$ vertices, $3|t$, and with $\delta(H) \geq \left(\frac{5}{9}+\beta\right)\binom{t-1}{2}$ contains a perfect matching.*

The proof of Lemma 3.11 consists of several short steps. We begin by applying the Weak Regularity Lemma (Lemma 2.3) to H. Let

$$\varepsilon = \min\left\{\frac{1}{4}\rho^2, \frac{1}{400}\lambda^2\right\}, \quad t_0 \geq \max\left\{\frac{13}{\lambda}, 2t_1\right\}, \quad n_{3.11} \geq \max\left\{n_{2.3}, T_0 n_{2.2}\right\}, \quad (3.3)$$

where T_0 is given by Lemma 2.3. We apply Lemma 2.3 to H, obtaining an ε-regular partition $(V_1,\dots,V_t)$, where $t_0 \leq t \leq T_0$. Let us call the sets V_i *clusters* and below we consider the *cluster 3-graph* $K = K(\lambda/12, \varepsilon)$ on the vertex set $[t] = \{1,\dots,t\}$. First, we define two auxiliary 3-graphs on $[t]$:

- $D(\lambda/12)$ consisting of all triples $\{i_1, i_2, i_3\} \subset [t]$ such that $d_H\left(V_{i_1}, V_{i_2}, V_{i_3}\right) \geq \lambda/12$, and
- $R(\varepsilon)$ consisting of all triples $\{i_1, i_2, i_3\} \subset [t]$ such that $H[V_{i_1}, V_{i_2}, V_{i_3}]$ is ε-regular.

Having defined D and R we define $K = K(\lambda/12, \varepsilon)$ as the intersection

$$K = D(\lambda/12) \cap R(\varepsilon). \quad (3.4)$$

So, the edges of K are all triples of indices $\{i_1, i_2, i_3\}$ such that $d_H\left(V_{i_1}, V_{i_2}, V_{i_3}\right) \geq \lambda/12$ and $H\left[V_{i_1}, V_{i_2}, V_{i_3}\right]$ is ε-regular.

Claim 3.13.
$$\delta(D) \geq \left(\frac{5}{9}+\frac{2}{3}\lambda\right)\frac{t^2}{2}.$$

Proof. Assume for simplicity that $t|n$ so that $|V_j| = n/t$ for every j. Fix $i \in [t]$ and set N_i for the number of edges in $\bigcup_{j,l} |H[V_i, V_j, V_l]|$, where the union runs over all pairs j, l such that $\{i, j, l\} \in D$. Then, on one hand,

$$N_i \le \deg_D(i) \cdot (n/t)^3,$$

while on the other hand, we can bound N_i from below as follows. We have

$$\sum_{u \in V_i} \deg_H(u) \ge \left(\frac{5}{9} + \lambda\right)(n/t)\binom{n-1}{2},$$

but this sum counts the edges within V_i three times and the edges with two vertices in V_i twice. Thus, the difference

$$\left(\frac{5}{9} + \lambda\right)(n/t)\binom{n-1}{2} - 3\binom{n/t}{3} - 2\binom{n/t}{2}(t-1)(n/t)$$

sets a lower bound on the number of edges of H with exactly one vertex in V_i. To get a lower bound on N_i we need to further exclude the at most $\binom{t-1}{2}(\lambda/12)(n/t)^3$ edges belonging to sub-3-graphs $H[V_i, V_j, V_l]$ with $\{i, j, l\} \notin D$, as well as, the at most $\binom{n/t}{2}(t-1)(n/t)$ edges with two vertices in the same set V_j, $j \ne i$. Altogether, we arrive at the inequality

$$N_i \ge \left(\frac{5}{9} + \lambda\right)(n/t)\binom{n-1}{2} - \frac{\lambda}{12}\binom{t-1}{2}(n/t)^3 - 3\binom{n/t}{3} - 3\binom{n/t}{2}(t-1)(n/t),$$

whose right-hand side, for large n and using the bound on t_0, can be further bounded from below by

$$\begin{aligned}
&\left(\frac{5}{9} + \lambda - \frac{\lambda}{12} - \frac{1}{t^2} - \frac{3}{t}\right)\frac{n^3}{2t} + O\left(n^2\right) \\
&\ge \left(\frac{5}{9} + \frac{11\lambda}{12} - \frac{\lambda^2}{169} - \frac{3\lambda}{13} + o(1)\right)\frac{n^3}{2t} \\
&\ge \left(\frac{5}{9} + \frac{2}{3}\lambda\right)\frac{n^3}{2t}.
\end{aligned}$$

Comparing the upper and lower bound on N_i, we obtain the desired estimate. ∎

The just established lower bound on the minimum degree in D is essentially valid for the 3-graph K as well, and thus, allows one to find in K an almost perfect matching.

Claim 3.14. There exists a matching M in K with $|V(M)| \ge \left(1 - \sqrt{\varepsilon}\right)t$.

Proof. We will find a sub-3-graph K' of K with $|V(K')| := t' \ge \left(1 - \sqrt{\varepsilon}\right)t$ and

$$\delta(K') \ge \left(\frac{5}{9} + \frac{1}{2}\lambda\right)\binom{t'-1}{2}. \tag{3.5}$$

Once we are done with this task, the claim will follow from Theorem 3.12 with $\beta = \frac{1}{2}\lambda$ (note that, by (3.3), $\varepsilon \le 1/2$ and $t' \ge t/2 \ge t_0/2 \ge t_1$).

Since the number of ε-irregular triples is less than $\varepsilon\binom{t}{3}$, the set W of vertices $i \in [t]$ incident in D to more than $\sqrt{\varepsilon}\binom{t-1}{2}$ of them has size $|W| \le \sqrt{\varepsilon}t$. For every vertex $i \in [t] \setminus W$,

$$\deg_K(i) \ge \left(\frac{5}{9} + \frac{2}{3}\lambda\right)\frac{t^2}{2} - \sqrt{\varepsilon}\binom{t-1}{2}.$$

Let $W' \supset W$ be such that $t - |W'|$ is divisible by 3 and $|W'| \le |W| + 2$. As for every $i \in [t] \setminus W'$ and $j \in W'$ there are at most $t-2$ edges in K containing both these vertices, the induced sub-3-graph $K' = K - W'$ has minimum degree at least

$$\left(\frac{5}{9} + \frac{2}{3}\lambda\right)\frac{t^2}{2} - \sqrt{\varepsilon}\binom{t-1}{2} - \left(\sqrt{\varepsilon}t + 2\right)(t-2) \ge \left(\frac{5}{9} + \frac{\lambda}{2}\right)\binom{t-1}{2},$$

where the first inequality follows from the bound $\varepsilon \le \lambda^2/400$. Since $t \ge t'$, we have also (3.5) which, as explained above, completes the proof of Claim 3.14. ∎

Claim 3.15. For every $\{i, j, l\} \in M$, the 3-graph $H[V_i, V_j, V_l]$ contains a family $\mathcal{Q}_{ijl}$ of vertex-disjoint copies of $K_{L,L,L}$ such that $|\mathcal{Q}_{ijl}|L \ge \big(1-\varepsilon\big)n/t$.

Proof. The claim will follow if we show that for all $W_k \subset V_k$, $|W_i| = \varepsilon n/t$, $k = i, j, l$, the induced subhypergraph $H[W_i, W_j, W_l]$ contains a $K_{L,L,L}$. Indeed, then a maximal family of vertex disjoint copies of $K_{L,L,L}$ in $H[V_i, V_j, V_l]$ may miss only less than $\varepsilon n/t$ vertices in each set V_k for $k = i, j, l$.

By ε-regularity of $H[V_i, V_j, V_l]$ we have

$$\big|H[W_i, W_j, W_l]\big| \ge (\lambda/12 - \varepsilon)(n/t)^3 = \frac{1}{27}(\lambda/12 - \varepsilon)(3n/t)^3.$$

Recalling that $t \le T_0$, $n \ge n_{3.11}$, and in view of (3.3), observe that

$$n/t \ge n/T_0 \ge n_{3.11}/T_0 \ge n_{2.2}.$$

We apply Lemma 2.2 to $H[W_i, W_j, W_l]$ with $d = \frac{1}{27}(\lambda/12 - \varepsilon)$ and $h = L$, and conclude that there is in $H[W_i, W_j, W_l]$ a copy of $K_{L,L,L}$ (in fact, as many as $c(3n/t)^{3L} > 0$, for some $c > 0$). ∎

Proof of Lemma 3.11 (conclusion). Consider the union of all the families guaranteed by Claim 3.15, $\mathcal{Q} = \bigcup_{\{i,j,l\} \in M} \mathcal{Q}_{ijl}$. Since, clearly, $|M| \le t/3$ and, by Claim 3.14, at most $\sqrt{\varepsilon}t(n/t) = \sqrt{\varepsilon}n$ vertices of H are not covered by the clusters of M, we conclude that $\mathcal{Q}$ covers all but at most

$$|M| \times 3\varepsilon n/t + \sqrt{\varepsilon}n \le \left(\varepsilon + \sqrt{\varepsilon}\right)n \le \rho n$$

vertices of H, where the last inequality follows from the assumption that $\varepsilon \le \rho^2/4$. This completes the proof of Lemma 3.11. ∎

Remark 3.16. As it will become clear in the next section, for our purposes it would be sufficient to prove Lemma 3.11 under the stronger assumption $\delta(H) \ge .7\binom{n-1}{2}$ and then the proof of Claim 3.14 would be quite straightforward, in particular, we would not need Theorem 3.12. But with Theorem 3.12 at hand, the strengthening of Lemma 3.11 comes for free and may be useful in the future work towards Conjecture 1.3.

4. Proof of Theorem 1.2

Let H be a 3-graph with $\delta(H) \geq .8\binom{n-1}{2}$ and $n \geq n_{1.2}$. To find a Hamiltonian cycle in H we follow the five step outline presented in Section 2.1., and use Lemmas 3.6, 3.10, 3.11, and 3.3 along the way. To facilitate their application, we begin with setting up the constants.

Let γ_0 be given by Lemma 3.6 and let

$$\gamma = \min\{\gamma_0, 10^{-6}/3\}. \tag{4.1}$$

Further, let $n_{3.10} = n_{3.10}(\gamma^2/3)$ and $n_{3.11}$ come from Lemma 3.11 with $\rho = \gamma^3$, $L = \lceil \frac{1}{3}\gamma^{-3} \rceil$, and, say, $\lambda = 1/9$. Finally, set

$$n_{1.2} = \max\left(\frac{n_{3.3}}{\gamma^2/4 - 14\gamma^3}, n_{3.6}, \frac{n_{3.10}}{1-\gamma}, \frac{n_{3.11}}{1-\gamma-\gamma^2/2}, 10^{12}, \frac{2}{\gamma^3}\right). \tag{4.2}$$

4.1. Finding an Absorbing Path A in H

By Lemma 3.6 there exists in H a $\gamma^2 n$-absorbing path A with $1/3$-large endpairs and with $|V(A)| \leq \gamma n$. Recall that the powerful absorbing property of A asserts that for every subset $U \subset V(H) \setminus V(A)$ of size $|U| \leq \gamma^2 n$ there is a path A_U in H with $V(A_U) = V(A) \cup U$ and with the same endpairs as A (see Definition 3.4). We are going to use this property at the very end of the proof.

4.2. Finding a Reservoir Set R in $H - V(A)$

We need a small reservoir set R which can be quickly reached from any $1/3$-large pair of vertices in H and which satisfies the assumptions of Lemma 3.3. The next claim is a simple corollary of Lemma 3.10.

Claim 4.1. There exists a set $R \subset V(H) \setminus V(A)$ such that

(a) $\gamma^2 n/4 \leq |R| \leq \gamma^2 n/2$,
(b_1) every $1/3$-large pair e of H has at least $.333|R|$ neighbours in R,
(b_2) every vertex v of H has at least $.697|R|$ neighbours in $G_{1/3}$ which belong to R,
(c) $\delta(H[R]) \geq .7994\binom{|R|}{2}$.

Proof. Note that $|V(H) \setminus V(A)| \geq (1-\gamma)n \geq n_{3.10}$ and apply Lemma 3.10 to $V(H) \setminus V(A)$ with $p = \gamma^2/3$ and with the following choice of sets U_i and graphs L_j:

(b) The sets U_i are

- (b_1) the sets $N_H(e) \setminus V(A)$, over all $e \in G_{1/3}$,
- (b_2) the sets $N_{G_{1/3}}(v) \setminus V(A)$, over all $v \in V$.

(c) The graphs L_j are the graphs $H(v) - V(A)$, $v \in V$, obtained from the link graphs by removing the vertices on A.

The corresponding coefficients α_i for sets in group (b_1) are $\alpha_e = 1/3 - \gamma$, while in group (b_2) they are, by Claim 2.1, with $\alpha_v = h_{.8}(1/3) - \gamma = .7 - \gamma$. The coefficient β_j for graphs (part (c)) are $\beta_v = .8 - 3\gamma$, because every vertex v belongs to at most $|V(A)|(n-2) < 3\gamma\binom{n-1}{2}$ edges intersecting $V(A)$.

In the short argument below we use the bounds on γ and n stemming from (4.1) and (4.2). Because $n \geq 10^{12} \geq 364$, part (a) follows from Lemma 3.10 (a). Parts (b_1) and (b_2) follow from Lemma 3.10 (b), because $\gamma \leq 0.001$ and $n^{-1/3} \leq 0.001$. Finally, part (c) follows from Lemma 3.10 (c), using $\gamma \leq 0.0001$ and $n^{-1/3} \leq 0.0001$, and the relation $\delta(H[R]) = \min_{v \in R} |H(v)[R]|$. ∎

4.3. Finding a Collection of Disjoint Paths $\mathcal{P}$, Covering Most of the Vertices

Our goal here is to find a collection of disjoint paths in H which are disjoint from $V(A) \cup R$, cover almost all vertices in $V(H) \setminus (V(A) \cup R)$, and, most importantly, have $1/3$-large endpairs. This last condition is needed in the next step of the proof where we connect all these paths together.

Recall that H' is a sub-3-graph of H consisting of all edges containing at least one $1/3$-large pair, that is, an edge of the graph $G_{1/3}$ (see (2.2)). We have already shown that $\delta(H') \geq .709n^2/2$ (see (3.2)). Setting

$$H'' := H' - (V(A) \cup R)$$

and noting that, by (4.1), $\gamma < 0.0005$, we thus have

$$\delta(H'') \geq .709n^2/2 - |V(A) \cup R|n \geq .709n^2/2 - 2\gamma n^2 \geq .708n^2/2 \geq .708\binom{n-1}{2}.$$

By (4.2), we also have $|V(H'')| \geq n_{3.11}$. We apply Lemma 3.11 to H'' with $\rho = \gamma^3$, $L = \left\lceil \frac{1}{3}\gamma^{-3} \right\rceil$, and say, $\lambda = 1/9$, obtaining a family $\mathcal{Q}$ of vertex-disjoint copies of $K_{L,L,L}$ which together cover at least $\left(1-\gamma^3\right)|V(H'')|$ vertices of H''.

By the definition of H', every copy $Q \in \mathcal{Q}$ of $K_{L,L,L}$ contains a path P of length at least $3L-1$ with both endpairs in $G_{1/3}$. Let $\mathcal{P}$ be the family of all these paths. Note that

$$|\mathcal{P}| = |\mathcal{Q}| \leq n/3L < \gamma^3 n.$$

Unlike in [27], the path cover $\mathcal{P}$ we have gotten consists of $\Theta(n)$ paths of length $O(1)$. Yet, the number of these paths, $|\mathcal{P}|$, is much smaller than $|R|$ (compare the above bound with the lower bound in part (a) of Claim 4.1.) This allows us to glue them all together using the reservoir set.

4.4. Connecting the Paths in $\mathcal{P}$ and the Absorbing Path A into a Long Cycle

Our task is to connect all the paths in $\mathcal{P}$, as well as the absorbing path A, into one cycle. Let $m = |\mathcal{P}| + 1$ be the number of these paths and let $\mathcal{P} = \{P_1, \ldots, P_{m-1}\}$. Further, let e_i and f_{i+1} be the endpairs of P_i, $i = 1, \ldots, m$, where we set $P_m = A$ and $f_{m+1} = f_1$ for convenience. Recall that all these endpairs are ordered pairs of vertices and, treated as unordered pairs, they belong to $G_{1/3}$.

Then, the following claim is all what we need. It states that all pairs $\{e_i, f_i\}$, $i = 1, \ldots, m$, can be simultaneously connected by short, mutually vertex-disjoint paths whose all inner vertices (other than those in e_i and f_i) belong to R. Of course, this connecting scheme results in a cycle C in H containing all paths $P_1, \ldots, P_{m-1}$ and A as sub-3-graphs.

Claim 4.2. Let $\delta(H) \geq .8\binom{n-1}{2}$ and let R satisfy properties (a)–(c) of Claim 4.1. Further, let m be an integer, $m \leq \gamma^3 n + 1$, and let $e_1, \ldots, e_m$ and $f_1, \ldots, f_m$ be disjoint ordered pairs in $V \setminus R$ which belong to $G_{1/3}$. Then, there are disjoint paths $\Pi_1, \ldots, \Pi_m$, each of length 16, where, for $i = 1, \ldots, m$, Π_i has endpairs e_i and f_i and $V(\Pi_i) \setminus (e_i \cup f_i) \subset R$.

Proof. We show by induction on i that the paths $\Pi_1, \ldots, \Pi_m$ exist and that $|V(\Pi_i) \cap R| = 14$. Suppose that for some $0 \leq i \leq m-1$ we have already found paths Π_j for all $j = 1, \ldots, i$. Together these paths occupy $14i \leq 14(m-1) \leq 14\gamma^3 n$ vertices of R. Let R_i be the set of all the remaining vertices of R. Thus, by part (a) of Claim 4.1,

$$|R_i| \geq |R| - 14\gamma^3 n \geq n_{3.3}. \tag{4.3}$$

The properties (a)–(c) of R established in Claim 4.1 imply the following, a bit weaker, properties of every subset $R' \subseteq R$, with $|R'| \geq |R| - 15\gamma^3 n$:

(b_1') every $1/3$-large pair e of H has at least $.33|R'|$ neighbours in R',

(b_2') every vertex v of H has at least $.69|R'|$ neighbours in $G_{1/3}$ which belong to R',

(c') $\delta(H[R']) \geq .799\binom{|R'|-1}{2}$.

Indeed, to see, for instance, that (c') holds, observe that

$$0.7994\binom{|R|}{2} - 15\gamma^3 n(|R|-2) \geq 0.799\binom{|R|-1}{2}$$

follows from $0.0004|R| \geq 30\gamma^3 n$, which, in turn, follows by the lower bound in (a) and by (4.1).

We shall use these properties to connect the pairs $e_{i+1} = (v, u)$ and $f_{i+1} = (y, x)$. Since $.33 + .69 > 1$ and $\{u, v\} \in G_{1/3}$, by (b_1') and (b_2') applied to $R' = R_i$, there exists a vertex $w \in R_i$ such that $uvw \in H$ and $vw \in G_{1/3}$. This, in turn, implies that there is also a vertex $w' \in R_i$ such that $vww' \in H$ and $ww' \in G_{1/3}$. Note that

$$\left|R_i \setminus \{w, w'\}\right| \geq |R| - 14\gamma^3 n - 2 \geq |R| - 15\gamma^3 n.$$

Similarly, this time applying (b_1') and (b_2') to $R' = R_i \setminus \{w, w'\}$, we argue that there exist two other vertices in R_i, z and z', such that $xyz, yzz' \in H$, while $zz' \in G_{1/3}$.

By (b_1') applied again to $R' = R_i$, both ww' and zz' are .33-large in $H[R_i]$, the 3-graph induced in H by R_i. Also, by (c'), we have $\delta(H[R_i]) \geq .799\binom{|R_i|-1}{2}$. Hence, recalling also (4.3), we are in position to apply Lemma 3.3 to $H[R_i]$ and conclude that there is a path π_{i+1} of length 12 between (w, w') and (z, z'). This path, together with the previously constructed four edges, forms a desired 18-vertex path Π_{i+1} between e_{i+1} and f_{i+1} (see Figure 3). ∎

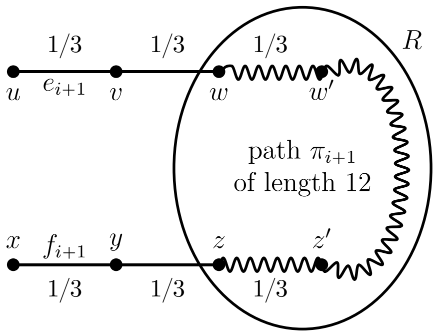

Figure 3: Path Π_{i+1} of length 16 from e_{i+1} to f_{i+1}.

4.5. Creating a Hamiltonian Cycle in H

Let us denote by T the set of vertices of H'', not covered by the paths in $\mathcal{P}$. It consists of up to $\gamma^3 n$ vertices not covered by the copies of $K_{L,L,L}$ in $\mathcal{Q}$ plus up to $|\mathcal{Q}|$ vertices dropped from the each Q_i whenever the path P_i had $3L-1$ vertices and not all $3L$. Therefore,

$$|T| \le \gamma^3 n + |\mathcal{Q}| \le 2\gamma^3 n.$$

Observe further that, since $C \supset A$, we have $V \setminus V(C) \subset R \cup T$ and thus

$$|V \setminus V(C)| \le \frac{1}{2}\gamma^2 n + 2\gamma^3 n \le \gamma^2 n.$$

Since the path A forms a segment of C, we can employ the $\gamma^2 n$-absorbing property of A to the set $U := V \setminus V(C)$. By replacing in C, the path A by a path A_U, we finally obtain a Hamiltonian cycle in H, which concludes the proof of Theorem 1.2. ∎

Acknowledgment. We thank the referee for her or his timely work and helpful comments.

References

1. Berge, C.: Graphs and Hypergraphs. North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York (1973)
2. Bermond, J.-C., Germa, A., Heydemann, M.-C., Sotteau, D.: Hypergraphes hamiltoniens. In: Problèmes Combinatoires et Théorie des Graphes, pp. 39–43. CNRS, Paris (1978)
3. Bollobás, B.: Extremal Graph Theory. Dover Publications, Inc., Mineola, NY (2004)
4. Buß, E., Hàn, H., Schacht, M.: Minimum vertex degree conditions for loose Hamilton cycles in 3-uniform hypergraphs. J. Combin. Theory Ser. B 103(6), 658–678 (2013)
5. Chung, F.R.K.: Regularity lemmas for hypergraphs and quasi-randomness. Random Structures Algorithms 2(2), 241–252 (1991)
6. Dirac, G.A.: Some theorems on abstract graphs. Proc. London Math. Soc. (3) 2, 69–81 (1952)
7. Erdős, P.: On extremal problems of graphs and generalized graphs. Israel J. Math. 2, 183–190 (1964)

8. Frankl, P., Rödl, V.: The uniformity lemma for hypergraphs. Graphs Combin. 8(4), 309–312 (1992)
9. Glebov, R., Person, Y., Weps, W.: On extremal hypergraphs for Hamiltonian cycles. European J. Combin. 33(4), 544–555 (2012)
10. Goodman, A.W.: On sets of acquaintances and strangers at any party. Amer. Math. Monthly 66, 778–783 (1959)
11. Hàn, H., Person, Y., Schacht, M.: On perfect matchings in uniform hypergraphs with large minimum vertex degree. SIAM J. Discrete Math. 23(2), 732–748 (2009)
12. Hàn, H., Schacht, M.: Dirac-type results for loose Hamilton cycles in uniform hypergraphs. J. Combin. Theory Ser. B 100(3), 332–346 (2010)
13. Han, J., Zhao, Y.: Minimum vertex degree threshold for loose Hamilton cycles in 3-uniform hypergraphs. J. Combin. Theory Ser. B 114, 70–96 (2015)
14. Han, J., Zhao, Y.: Forbidding Hamilton cycles in uniform hypergraphs. arXiv:1508.05623 (2015)
15. Janson, S., Łuczak, T., Ruciński, A.: Random Graphs. Wiley-Interscience, New York (2000)
16. Katona, G.Y., Kierstead, H.A.: Hamiltonian chains in hypergraphs. J. Graph Theory 30(3), 205–212 (1999)
17. Keevash, P. Kühn, D., Mycroft, R., Osthus, D.: Loose Hamilton cycles in hypergraphs. Discrete Math. 311(7), 544–559 (2011)
18. Khan, I.: Perfect matchings in 3-uniform hypergraphs with large vertex degree. SIAM J. Discrete Math. 27(2), 1021–1039 (2013)
19. Kühn, D., Mycroft, R., Osthus, D.: Hamilton ℓ-cycles in uniform hypergraphs. J. Combin. Theory Ser. A 117(7), 910–927 (2010)
20. Kühn, D., Osthus, D.: Loose Hamilton cycles in 3-uniform hypergraphs of high minimum degree. J. Combin. Theory Ser. B 96(6), 767–821 (2006)
21. Kühn, D., Osthus, D., Treglown, A.: Matchings in 3-uniform hypergraphs. J. Combin. Theory Ser. B 103(2), 291–305 (2013)
22. Markström, K., Ruciński, A.: Perfect matchings (and Hamilton cycles) in hypergraphs with large degrees. European J. Combin. 32(5), 677–687 (2011)
23. Moon, J.W., Moser, L.: On a problem of Turán. Magyar Tud. Akad. Mat. Kutató Int. Közl. 7, 283–286 (1962)
24. Nordhaus, E.A., Stewart, B.M.: Triangles in an ordinary graph. Canad. J. Math. 15, 33–41 (1963)
25. Razborov, A.A.: On the minimal density of triangles in graphs. Combin. Probab. Comput. 17(4), 603–618 (2008)
26. Rödl, V., Ruciński, A.: Dirac-type questions for hypergraphs — a survey (or more problems for Endre to solve). In: Bárány, I., Solymosi, J., Sági, G. (eds.) An Irregular Mind, pp. 561–590. János Bolyai Math. Soc., Budapest (2010)
27. Rödl, V., Ruciński, A.: Families of triples with high minimum degree are Hamiltonian. Discuss. Math. Graph Theory 34(2), 361–381 (2014)
28. Rödl, V., Ruciński, A., Szemerédi, E.: A Dirac-type theorem for 3-uniform hypergraphs. Combin. Probab. Comput. 15(1-2), 229–251 (2006)
29. Rödl, V., Ruciński, A., Szemerédi, E.: An approximate Dirac-type theorem for k-uniform hypergraphs. Combinatorica 28(2), 229–260 (2008)
30. Rödl, V., Ruciński, A., Szemerédi, E.: Perfect matchings in large uniform hypergraphs with large minimum collective degree. J. Combin. Theory Ser. A 116(3), 613–636 (2009)
31. Rödl, V., Ruciński, A., Szemerédi, E.: Dirac-type conditions for Hamiltonian paths and cycles in 3-uniform hypergraphs. Adv. Math. 227(3), 1225–1299 (2011)
32. Steger, A.: Die Kleitman-Rothschild methode. PhD thesis. Forschungsinstitut für Diskrete Mathematik, Rheinische Friedrichs-Wilhelms-Universität Bonn (1990)

33. Szemerédi, E.: Regular partitions of graphs. In: Problèmes Combinatoires et Théorie des Graphes, pp. 399–401. CNRS, Paris (1978)