# Upper bounds on the minimum size of Hamilton saturated hypergraphs

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**Abstract**

For $1 \leqslant \ell < k$, an *$\ell$-overlapping $k$-cycle* is a $k$-uniform hypergraph in which, for some cyclic vertex ordering, every edge consists of $k$ consecutive vertices and every two consecutive edges share exactly $\ell$ vertices.

A $k$-uniform hypergraph $H$ is *$\ell$-Hamiltonian saturated* if $H$ does not contain an $\ell$-overlapping Hamiltonian $k$-cycle but every hypergraph obtained from $H$ by adding one edge does contain such a cycle. Let $\mathrm{sat}(n,k,\ell)$ be the smallest number of edges in an $\ell$-Hamiltonian saturated $k$-uniform hypergraph on $n$ vertices. In the case of graphs Clark and Entringer showed in 1983 that $\mathrm{sat}(n,2,1) = \lceil \frac{3n}{2} \rceil$. The present authors proved that for $k \geqslant 3$ and $\ell = 1$, as well as for all $0.8k \leqslant \ell \leqslant k-1$, $\mathrm{sat}(n,k,\ell) = \Theta(n^\ell)$. In this paper we prove two upper bounds which cover the remaining range of $\ell$. The first, quite technical one, restricted to $\ell \geqslant \frac{k+1}{2}$, implies in particular that for $\ell = \frac{2}{3}k$ and $\ell = \frac{3}{4}k$ we have $\mathrm{sat}(n,k,\ell) = O(n^{\ell+1})$. Our main result provides an upper bound $\mathrm{sat}(n,k,\ell) = O(n^{(k+\ell)/2})$ valid for all $k$ and $\ell$. In the smallest open case we improve it further to $\mathrm{sat}(n,4,2) = O(n^{14/5})$.

## 1 Introduction

A hypergraph $H$ is a pair $H = (V,E)$ where $V$ is a set of elements called *vertices*, and $E$ is a set of non-empty subsets of $V$ called *edges*. If every edge of $H$ has exactly $k$ vertices, then $H$ is called a *$k$-uniform hypergraph* or a *$k$-graph*. In what follows we will often identify $H$ with its set of edges.

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Given integers $1 \leqslant \ell < k$, we define an *$\ell$-overlapping $k$-cycle* as a *$k$-graph* in which, for some cyclic ordering of its vertices, every edge consists of $k$ consecutive vertices, and every two consecutive edges (in the natural ordering of the edges induced by the ordering of the vertices) share exactly $\ell$ vertices. The notion of an *$\ell$-overlapping $k$-path* is defined similarly, that is, with vertices ordered $v_1, \ldots, v_s$, the edges of the path are $\{v_1, \ldots, v_k\}$, $\{v_{k-\ell+1}, \ldots, v_{k+\ell}\}$, $\ldots$, $\{v_{s-k+1}, \ldots, v_s\}$, Note that the number of edges of an $\ell$-overlapping $k$-cycle with $s$ vertices is $s/(k-\ell)$ (and thus, $s$ is divisible by $k-\ell$). Similarly, it can be easily seen that the number of vertices $s$ of an $\ell$-overlapping $k$-path equals $\ell$ modulo $k-\ell$.

We denote an *$\ell$-overlapping $k$-cycle* on $s$ vertices by $C_s^{(k,\ell)}$. We further denote by $g := g(k,\ell)$ the number of vertices between any two consecutive *disjoint* edges belonging to an $\ell$-overlapping path (or cycle) and notice that

$$0 \leqslant g = \left\lceil \frac{k}{k-\ell} \right\rceil (k-\ell) - k < k - \ell < k, \tag{1}$$

and that $g = 0$ if and only if $k - \ell$ divides $k$.

*An $\ell$-overlapping Hamiltonian $k$-cycle* in a $n$-vertex $k$-graph $H$ is defined as any sub-hypergraph of $H$ isomorphic to $C_n^{(k,\ell)}$. If $H$ contains an $\ell$-overlapping Hamiltonian $k$-cycle then $H$ itself is called *$\ell$-Hamiltonian*.

Given a $k$-graph $H$ and a $k$-element set $e \in H^c$, where $H^c = \binom{V}{k} \setminus H$ is the complement of $H$, we denote by $H + e$ the hypergraph obtained from $H$ by adding $e$ to its edge set. A $k$-graph $H$ is *$\ell$-Hamiltonian saturated*, $1 \leqslant \ell \leqslant k-1$, if $H$ is not $\ell$-Hamiltonian but for every $e \in H^c$ the $k$-graph $H + e$ is such. The largest number of edges in an $\ell$-Hamiltonian saturated $k$-graph on $n$ vertices is called the Turán number for the cycle $C_n^{(k,\ell)}$. In [2] this number has been determined in terms of the Turán number of a $(k-1)$-uniform path with a constant number of vertices.

In this paper we are interested in the other extreme. For $n$ divisible by $k - \ell$, let $\mathrm{sat}(n,k,\ell)$ be the *smallest* number of edges in an $\ell$-Hamiltonian saturated $k$-graph on $n$ vertices. In the case of graphs, Clark and Entringer proved in 1983 that $\mathrm{sat}(n,2,1) = \lceil \frac{3n}{2} \rceil$ for $n \geqslant 52$.

For $k$-graphs with $k \geqslant 3$ the problem was first mentioned in [3, 4]. It seems to be quite hard to obtain such precise results as for graphs. Therefore, the emphasis has been put on the order of magnitude of $\mathrm{sat}(n,k,\ell)$. The present authors proved in [5] that for $k \geqslant 3$ and $\ell = 1$, as well as for all $0.8k \leqslant \ell \leqslant k-1$,

$$\mathrm{sat}(n,k,\ell) = \Theta(n^\ell), \tag{2}$$

see also [6] for the case $\ell = k-1$. On the other hand, we have the easy lower bound ([5, Prop. 2.1])

$$\mathrm{sat}(n,k,\ell) = \Omega\left(n^\ell\right).$$

The facts that (2) holds for very small and very large (with respect to $k$) values of $\ell$ and that no better lower bound is known suggest, as conjectured already in [5], that (2) holds for all $1 \leqslant \ell \leqslant k-1$ and $k \geqslant 2$.

**Conjecture 1.** *For all $k \geqslant 2$ and $1 \leqslant \ell \leqslant k-1$,*

$$\text{sat}(n,k,\ell) = O\left(n^{\ell}\right).$$

Our first result provides an upper bound on $\text{sat}(n,k,\ell)$ higher than the conjectured $O(n^{\ell})$, but for a broader range of $\ell$ than in [5].

**Theorem 1.** *For all $k \geqslant 3$ and $\ell \geqslant \frac{k+1}{2}$*

$$\text{sat}(n,k,\ell) = O\left(n^{\ell+2g+1}\right).$$

Of course, this bound is good only when $g$ is small, and when $g=0$ it is only by a factor of $n$ worse than the conjectured optimum. All cases of Theorem 1 which are not covered by the result from [5], but for which $g=0$, are given in the following corollary.

**Corollary 2.** *For every $k$ divisible by three and $\ell = \frac{2}{3}k$, as well as for every $k$ divisible by four and $\ell = \frac{3}{4}k$, we have $\text{sat}(n,k,\ell) = O(n^{\ell+1})$.*

In the remaining range of $\ell$, that is, for $2 \leqslant \ell \leqslant k/2$, nothing else than the trivial upper bound

$$\text{sat}(n,k,\ell) = O(n^{k})$$

have been known. Our main result in this paper provides a first, non-trivial, general upper bound on $\text{sat}(n,k,\ell)$.

**Theorem 3.** *For all $k \geqslant 3$ and $2 \leqslant \ell \leqslant k-1$,*

$$\text{sat}(n,k,\ell) = O\left(n^{(k_\ell)/2}\right).$$

One consequence of Theorem 3, combined with the case $\ell = k-1$ of (2), is that for all $\ell$ and $k$ we have

$$\text{sat}(n,k,\ell) = O\left(n^{k-1}\right).$$

In view of Theorem 3, the bound in Theorem 1 is not overwritten only when $\ell + 2g + 1 \leqslant \frac{k+\ell-1}{2}$, equivalently, when $g \leqslant (k-\ell-1)/4$. Theorems 1 and 3 are proved, respectively, in Sections 3 and 4. In the smallest open case, $k=4$, $\ell=2$, we improve Theorem 3 a bit by showing the following result in Section 5.

**Theorem 4.** $\text{sat}(n,4,2) = O\left(n^{14/5}\right)$.

Our proofs expand and refine a general approach to this type of problems first developed in [6] and modified in [5]. In short, we begin with constructing two $k$-graphs, $H'$ and $H''$, such that $H'$ is not $\ell$-Hamiltonian, while $H'' \supset H'$ contains some "trouble-making" edges. Then we define $H$ as a maximal non-$\ell$-Hamiltonian $k$-graph satisfying $H' \subseteq H \subseteq H''$. It then remains to show that for every $e \notin H$, $H+e$ is $\ell$-Hamiltonian, but, what is crucial, in doing so we may restrict ourselves to $e \notin H''$.

In [6] the constructions of $H'$ and $H''$ were based on a special partition of the vertex set, while in [5] we used blow-ups of sparse Hamiltonian saturated graphs. In this paper we return to both these ideas: we use the approach from [5] in the proof of Theorem 1, and the approach from [6] in the proofs of Theorems 3 and 4.

## 2 Preliminaries

Our proofs utilize the following special construction of a $k$-graph. Given a partition of the vertex set $V = \bigcup_{i=1}^{h} U_i$, for a subset $S \subseteq V$, let

$$tr(S) = \{i : U_i \cap S \neq \varnothing\}$$

and

$$\min(S) = \min\{i : i \in tr(S)\} = \min\{i : U_i \cap S \neq \varnothing\}.$$

Let

$$H_{k,\ell}(U_1, \ldots, U_h) := H_{k,\ell} = \left\{ e \in \binom{V}{k} : |e \cap U_{\min(e)}| \geqslant k - \ell + 1 \right\}.$$

For further use, note that

$$|tr(e)| \leqslant \ell \qquad \text{for every } e \in H_{k,\ell}. \tag{3}$$

For $i = 1, \ldots, h$, let

$$C_i = \{e \in H_{k,\ell} \, : \, \min(e) = i\}.$$

Obviously, $H_{k,\ell} = C_1 \cup \cdots \cup C_h$.

Define *an $\ell$-component* of a $k$-graph $H$ as a minimal subset of edges $C \subseteq H$ such that for all $e \in C$ and $f \in H \setminus C$, we have $|e \cap f| < \ell$.

**Proposition 5.** *For each $i = 1, \ldots, h$, the set $C_i$ is an $\ell$-component of $H_{k,\ell}$.*

*Proof.* By the definition of $H_{k,\ell}$, for every $e \in C_i$ and $f \in C_j$, where $i < j$, we have $|e \cap U_i| \geqslant k - \ell + 1$ and $f \cap U_i = \varnothing$, and so $|e \cap f| < \ell$. Moreover, for every $e \in C_i$ there is an $f \in C_i$, $f \neq e$ such that $|e \cap f| \geqslant k - 1 \geqslant \ell$ (just switch one vertex without violating the membership in $C_i$), so that $C_i$ satisfies the minimality condition in the definition of an $\ell$-component. □

Since every $\ell$-overlapping $k$-path in a $k$-graph $H$ must be entirely contained in one the $\ell$-components of $H$, we have the following corollary of Proposition 5.

**Corollary 6.** *For every $\ell$-overlapping $k$-path $P$ in $H_{k,\ell}$ there is an $i \in \{1, \ldots, h\}$ such that $P \subseteq C_i$, or equivalently, for every edge $e$ of $P$, we have $\min(e) = i$.*

We now investigate the maximum length of an $\ell$-overlapping $k$-path in $C_i$, $i < h$, which traverses through exactly $x$ vertices of $U_i$. Our next, purely combinatorial, result provides an easy upper bound, independent of $\ell$. Given a positive integer $x$, let $A$ and $B$ be two disjoint sets, with $|A| = x$ and $|B| = \infty$. Let $\nu(x) = \max_P |V(P)|$, where the maximum is taken over all $\ell$-overlapping paths $P$ with $A \subset V(P) \subset A \cup B$ and $|e \cap A| \geqslant k - \ell + 1$ for all $e \in P$.

**Proposition 7.** *For every $x \geqslant k - 2$, we have $\nu(x) \leqslant kx$.*

*Proof.* Suppose there is a path $P$ with $A \subset V(P) \subset A \cup B$, $|e \cap A| \geqslant k - \ell + 1$ for all $e \in P$, and $|V(P)| \geqslant kx + 1$. Let us view $V(P)$ as a binary sequence, where each vertex of $A$ is replaced by symbol $a$ and each vertex of $V(P) \cap B$ is replaced by symbol $b$. If there is a pair of consecutive symbols $a$ in the sequence then, by averaging, there is a run (=a sequence of consecutive symbols) of at least

$$\frac{(k-1)x+1}{x} > k-1,$$

that is, of at least $k$ symbols $b$. But then there is an edge of $P$ with at most $k - \ell$ vertices of $A$ – a contradiction. If, on the other hand, there are no consecutive symbols $a$ in the sequence then, again by averaging, there is a run of at least

$$\frac{(k-1)x+1}{x+1} > k-2,$$

that is, of at least $k-1$ symbols $b$ (here we use the assumption $x \geqslant k-2$). Thus, there is a segment $b \cdots bab$ where the run of $b's$ is of length $k-1$. The first (from the left) edge of $P$ whose leftmost end is in this run may have at most $k - \ell$ symbols $a$ – a contradiction, again. □

We also have the following lower bound on $\nu(x)$.

**Proposition 8.** *For every $x \geqslant (k-3)(k-1)$*

$$\nu(x) \geqslant x + \left\lfloor \frac{x}{k-1} \right\rfloor + 3 - k.$$

*Proof.* Let a sequence $Q$ begin with a vertex in $B$ and then traverse, alternately, groups of $k-1$ vertices of $A$ followed by one vertex of $B$ until fewer than $k-1$ vertices of $A$ are left. The remaining vertices of $A$ are placed all at one end of $Q$. Clearly, every $k$-tuple of consecutive vertices of $Q$ contains $k-1 \geqslant k-\ell+1$ vertices of $A$. To turn $Q$ into an $\ell$-overlapping path, the number of vertices of $Q$ must equal $\ell$ modulo $k-\ell$. Therefore, we may be forced to drop up to $k-\ell-1 \leqslant k-2$ vertices of $B$ from $Q$. This is possible as

$$|Q \cap B| = \left\lfloor \frac{x}{k-1} \right\rfloor + 1 \geqslant k-2,$$

by our assumption on $x$. The obtained path has the required properties and the claimed number of vertices. □

Note that $\nu(x)$ is a nondecreasing function of $x$ (just replace any vertex of $B$ with a new vertex of $A$). Our next observation shows that it cannot increase too fast.

**Proposition 9.** *For all $x \geqslant 1$ we have $\nu(x-1) \geqslant \nu(x) - k$.*

*Proof.* Consider a longest path $P$ of length $\nu(x)$ and remove its first (from the left) $s$ vertices, where $\ell \leqslant s \leqslant k$ and $s = \nu(x) \mod k-\ell$. As there must be a vertex of $A$ among the first $\ell$ vertices of any edge, the remaining path $P'$ satisfies $x' := |V(P') \cap A| \leqslant x - 1$ and, by the monotonicity of $\nu(x)$ we have

$$\nu(x) - k \leqslant \nu(x) - s \leqslant \nu(x') \leqslant \nu(x-1). \qquad \square$$

Returning to the hypergraph $H_{k,\ell}$, Propositions 7-9 imply the following corollary.

**Corollary 10.** *Let $i < h$, $k^2 \leqslant x \leqslant |U_i|$, $A \subset U_i$, $|A| = x$, and $B \subset \bigcup_{j>i} U_j$, $|B| \geqslant (k-1)x$. Then the length of a longest path $P$ in $C_i$ such that $A \subset V(P) \subset A \cup B$ equals $\nu(x)$. Moreover, we have $\nu(x) - k \leqslant \nu(x-1) \leqslant \nu(x)$ and*

$$\frac{k}{k-1}x - k < \nu(x) \leqslant kx.$$

In addition to the basic construction $H_{k,\ell}$, the proof of Theorem 1 relies on the notion of a (hypergraph) blow-up of a graph which will be defined soon. First, however, we recall a simple fact about graphs proved in [5, Fact 2.2]. For a graph $G$, let $c(G)$ denote the number of components of $G$. Given a subset $T \subseteq V(G)$, let $G[T]$ be the subgraph of $G$ induced by $T$.

**Fact 11** ([5])**.** *Let $k$, $\ell$, and $\Delta$ be constants, and for $h = 1, 2, \ldots,$ let $G_h$ be a graph with $h$ vertices and $\Delta(G_h) \leqslant \Delta$. Then the number of $k$-element subsets $T \subseteq V(G_h)$ with $c(G[T_h]) \leqslant \ell$ is $O(h^\ell)$.*

Given a graph $G$ and an integer sequence $\mathbf{a} = (a_1, \ldots, a_h)$, the $\mathbf{a}$-*blow-up* of $G$ is the $k$-graph $H := H[G]$ with

$$V(H) = \bigcup_{i=1}^{h} U_i, \quad |U_i| = a_i,$$
$$H = \bigcup_{ij \in G} K^{(k)}(U_i \cup U_j)$$

where $K^{(k)}(U)$ is the complete $k$-graph on $U$ and the sets $U_i$ are pairwise disjoint. For a subset $S \subset V(H)$, let

$$tr(S) = \{i \in V(G) : U_i \cap S \neq \varnothing\}.$$

Furthermore, set

$$c(S) = c\left(G[tr(S)]\right).$$

The following immediate corollary of Fact 11 has been already noted in [5, Cor. 2.3].

**Corollary 12** ([5])**.** *Let $a_1, \ldots, a_h$, $k$, $\ell$, and $\Delta$ be constants. If $\Delta(G_h) \leqslant \Delta$ and $H_h = H[G_h]$ is the $\mathbf{a}$-blow-up of $G_h$ then the number of $k$-element subsets $S \subseteq V(H_h)$ with $c(S) \leqslant \ell$ is $O(h^\ell)$.* $\square$

In order to facilitate the reading of the paper, the most frequent notation has been summarized in Table 1.

| | |
|---|---|
| $g(k,\ell)$ | $= \left\lceil \frac{k}{k-\ell} \right\rceil (k-\ell) - k$ |
| $H$ | a $k$-graph |
| $G$ | an auxiliary graph |
| $V(H)$ | $= \bigcup_{i=1}^{h} U_i$ |
| $V(G)$ | $= \{1, \ldots, h\}$ |
| $n$ | $= \lvert V(H) \rvert$ |
| $tr(S)$ | $= \{i : U_i \cap S \neq \varnothing\}$ |
| $\min(S)$ | $= \min\{i : S \cap U_i \neq \varnothing\}$ |
| $\min_2(S)$ | $= \min\{i : (S \setminus U_{\min(S)}) \cap U_i \neq \varnothing\}$ |
| $c(G)$ | the number of components of $G$ |
| $c(S)$ | $= c\,(G[tr(S)])$ |
| $H_{k,\ell}$ | $= \left\{e \in \binom{V}{k} : \lvert e \cap U_{\min(e)} \rvert \geqslant k - \ell + 1\right\}.$ |
| $C_i$ | $= \{e \in H_{k,\ell} \, : \, \min(e) = i\}.$ |
| $\nu(x)$ | $= \max\{\lvert V(P) \rvert$: $P$ is an $\ell$-overlapping path with $\lvert V(P) \cap A \rvert = x$ and $\lvert e \cap A \rvert \geqslant k - \ell + 1$ for all $e \in P\}$. |

Table 1: Notation

## 3 Proof of Theorem 1

In this section we prove Theorem 1, where the construction of an $\ell$-Hamiltonian saturated $k$-graph is based on a blow-up of a suitably chosen Hamiltonial saturated graph.

Our proof is a substantial modification of the proof of Theorem 1.1 in [5]. Specifically, we have made the range of $\ell$ in (7) broader (it used to be $2k - \ell + 1 \leqslant a_i \leqslant 4\ell - 2k + 1$) and, at the same time, we altered the definition of $H_2$ (by introducing the cores $\overline{U}_i$). In what follows, we assume that

$$g \leqslant \frac{k - \ell - 1}{4}, \tag{4}$$

since otherwise $\ell + 2g + 1 \geqslant (k + \ell)/2$ and Theorem 1 follows from Theorem 3.

We begin with a technical inequality.

**Proposition 13.** *If* $\frac{k+1}{2} \leqslant \ell \leqslant k - 1$ *then* $2k - \ell - 2g - 2 \leqslant 2\ell - 2$.

*Proof.* The inequality in question is equivalent to

$$3\ell + 2g \geqslant 2k, \tag{5}$$

To prove (5), note that, by the assumptions on $\ell$, there exists some integer $a \geqslant 1$ such that

$$\frac{ak + 1}{a + 1} \leqslant \ell < \frac{(a+1)k + 1}{(a+1) + 1} \leqslant \frac{2ak + 1}{2a + 1}.$$

Then, by the lower bound on $\ell$,

$$g = \left\lceil \frac{k}{k-\ell} \right\rceil (k-\ell) - k \geqslant \left\lceil \frac{k}{k-(ak+1)/(a+1)} \right\rceil (k-\ell) - k$$
$$= \left\lceil \frac{k}{k-1}(a+1) \right\rceil (k-\ell) - k \geqslant (a+2)(k-\ell) - k.$$

Hence, by the upper bound on $\ell$, we finally have

$$3\ell + 2g \geqslant (2a+2)k - (2a+1)\ell > 2k-1,$$

which implies (5). □

It follows from Proposition 13, as in [5], that every sufficiently large integer $n$ can be expressed as a sum

$$n = a_1 + \cdots + a_h, \tag{6}$$

for some $h$, where

$$2k - \ell - 2 - 2g \leqslant a_i \leqslant 2\ell - 1, \qquad i = 1, \ldots, h. \tag{7}$$

(This is because the range of $a_i$ in (7) has at least two consecutive values.)

Fix a large integer $n$ which is divisible by $(k-\ell)$ and let $\mathbf{a} = (a_1, \ldots, a_h)$, where the $a_i$'s and $h$ are as in (7). Note that $n = \Theta(h)$. Let $G_h$ be an $h$-vertex Hamiltonian saturated graph with $\Delta(G_h) = O(1)$, and let

$$H_1 = H[G_h]$$

be the $\mathbf{a}$-blow-up $k$-graph of $G_h$ (see the definition in Section 2) with

$$V = V(H_1) = \bigcup_{i=1}^{h} U_i, \text{ where } |U_i| = a_i, \quad i = 1, \ldots, h.$$

Thus, by (6),

$$|V| = n = \sum_{i=1}^{h} a_i.$$

It is easy to check that (4) implies that $a_i \geqslant k-\ell$, for all $i = 1, \ldots, h$. Fix a $(k-\ell)$-subset $\overline{U}_i$ of $U_i$, $i = 1, \ldots, h$, and let

$$H_2 = \left\{ e \in \binom{V}{k} : |e \cap U_{\min(e)}| \geqslant k - l + 1, e \supset \overline{U}_{\min(e)} \text{ and } c(e) \geqslant g+2 \right\}.$$

Since $H_2 \subseteq H_{k,\ell}$, by (3), for every $e \in H_2$ we have, in fact,

$$2 \leqslant g + 2 \leqslant c(e) \leqslant |tr(e)| \leqslant \ell. \tag{8}$$

(Note that (4) implies that, indeed, $g \leqslant \ell - 2$, which guarantees that $H_2$ is nonempty.) We have the following immediate consequence of the definition of $H_2$ and Corollary 6.

**Corollary 14.** *If $P$ is a path in $H_2$, then there is $i \in \{1, \ldots, h\}$ such that for every $e \in P$ we have $|e \cap U_i| \geqslant k - \ell + 1$ and $e \supset \overline{U}_i$. In particular, each path in $H_2$ has at most $\lfloor \frac{k}{k-\ell} \rfloor$ edges.* □

Observe also that for each $e \in H_1$, the set $tr(e)$ is either a vertex or an edge of $G$. Consequently, $c(e) = 1$ and the $k$-graphs $H_1$ and $H_2$ are edge-disjoint. Set $H' = H_1 \cup H_2$

**Lemma 15.** *$H'$ is not $\ell$-Hamiltonian.*

*Proof.* Suppose that $H'$ contains an $\ell$-Hamiltonian $k$-cycle $C_H = (e_1, \ldots, e_m)$. Unlike in [5], the proof breaks only into two cases:
**Case 1. $\boldsymbol{C_H \subseteq H_1}$:** We omit the proof in this case, as it is identical to Case 1 of the proof of Lemma 4.1 in [5] (Indeed that proof relied only on the assumption that $a_i \leqslant 2\ell - 1$.)

**Case 2. $\boldsymbol{H_2 \cap C_H \neq \varnothing}$:** Let (w.l.o.g.) $e_1, \ldots, e_{s-1}$ be a maximal segment in $C_H$ of consecutive edges from $H_2$. By Corollary 14, $s - 1 \leqslant \lfloor \frac{k}{k-\ell} \rfloor$ and there exists an index $i \in \{1, \ldots, h\}$ such that

$$e_1 \cap e_{s-1} \supseteq \overline{U}_i, \quad \text{and thus} \quad |e_1 \cap e_{s-1}| \geqslant |\overline{U}_i| = k - \ell. \tag{9}$$

Let $Z$ be the set of vertices that lie between $e_m$ and $e_s$ on $C_H$. Formally,

$$Z = \left( \bigcup_{t=1}^{s-1} e_t \right) \setminus (e_m \cup e_s).$$

Then $e_1 \subseteq e_m \cup Z \cup e_s$ and, consequently,

$$\{i\} \subseteq tr(e_1) \subseteq tr(e_m) \cup tr(Z) \cup tr(e_s). \tag{10}$$

What is more, $e_m \cap U_i \neq \varnothing$ and $e_s \cap U_i \neq \varnothing$. Since $e_m \in H_1$ and $e_s \in H_1$, by the definition of $H_1$, each of $tr(e_m)$ and $tr(e_s)$ is either the singleton $\{i\}$ or an edge of $G$ containing vertex $i$. Hence, by (10), $c(e_1) \leqslant 1 + |Z|$, which combined with the bound $g + 2 \leqslant c(e_1)$ from the definition of $H_2$, yields

$$|Z| \geqslant g + 1. \tag{11}$$

This further implies that $e_m$ and $e_s$ are disjoint, but more importantly, that $e_1$ and $e_s$ are disjoint too (since $e_m$ and $e_s$ cannot be consecutive disjoint edges). Thus, $s \geqslant 3$ and

$$|Z| \leqslant 2(k - \ell) - |e_1 \cap e_{s-1}| \leqslant k - \ell, \tag{12}$$

by (9). Note, however, that due to the structure of $\ell$-overlapping $k$-paths,

$$|Z| = g + t(k - \ell) \text{ for some } t \geqslant 0. \tag{13}$$

Therefore, by (13), (12) and (11), $|Z| = k - \ell$ (and $g = 0$). Consequently, by (12), $|e_1 \cap e_{s-1}| = k - \ell$, implying that, in fact, $e_1 \cap e_{s-1} = Z = \overline{U}_i$. But then (10) becomes

$$\{i\} \subseteq tr(e_1) \subseteq tr(e_m) \cup tr(e_s),$$

and hence, $c(e_1) = 1$ – a contradiction with the definition of $H_2$. □

Let

$$H'' = \left\{ e \in \binom{V}{k} : c(e) \leqslant \ell + 2g + 1 \right\}.$$

Recall that $H_1 = H[G_h]$ is the **a**-blow-up $k$-graph of a Hamiltonian saturated $h$-vertex graph $G_h$. It means that for all $e \in H_1$ we have $c(e) = 1$, while, by (8), for all $e \in H_2$ we have $c(e) \leqslant |tr(e)| \leqslant \ell$. Thus, $H' = H_1 \cup H_2 \subseteq H''$.

Finally, let $H$ be a maximal non-$\ell$-Hamiltonian $k$-graph on $V$ such that $H' \subseteq H \subseteq H''$. In view of Lemma 15, $H$ does exist. By Corollary 12,

$$|H| \leqslant |H''| = O(n^{\ell+2g+1}). \tag{14}$$

Thus, to complete the proof of Theorem 1, it remains to show the following lemma.

**Lemma 16.** *For every $e \in H^c$, $H + e$ is $\ell$-Hamiltonian.*

*Proof.* By the maximality of $H$, $H + e$ is $\ell$-Hamiltonian for each $e \in H'' \setminus H$. Hence, we may restrict ourselves only to $e \in (H'')^c$, that is, such that $c(e) \geqslant \ell+2g+2$. Let us fix one such $e$. Let $j_1, j_2, \ldots, j_{\ell+2g}, y$, and $x = \min(e)$ belong to $\ell + 2g + 2$ different components of $G[tr(e)]$ and satisfy

$$\min\{j_1, j_2, \ldots, j_{\ell+2g}\} > y > x. \tag{15}$$

Let $r_x = |e \cap U_x|$ and $r_y = |e \cap U_y|$. Note that, since $|tr(e)| \geqslant c(e) \geqslant \ell + 2g + 2$,

$$\max\{r_x, r_y\} \leqslant \max_{1 \leqslant i \leqslant n} |e \cap U_i| \leqslant k - (|tr(e)| - 1) \leqslant k - \ell - 2g - 1. \tag{16}$$

We will build an $\ell$-overlapping Hamiltonian cycle $C_H$ in $H + e$ using the Hamiltonian saturation of $G_h$. Let $(u_1, \ldots, u_n)$ be the vertices of $V$ in the order as they will appear on the $C_H$ under construction. Our goal is to define this ordering so that each segment of $k$ consecutive vertices which begins at $u_i$, where $i \equiv 1 \pmod{k - \ell}$, is an edge of $H + e$. We will denote by $e_1$ the edge beginning at $u_1$, by $e_2$ – the edge beginning at $u_{1+k-\ell}$ and so on, until the last edge $e_m$ of $C_H$ which begins at $u_{n-k+\ell+1}$, where $m = \frac{n}{k-\ell}$.

To achieve our goal, we will first construct an $\ell$-overlapping path $P \subseteq H_2+e$, extending $e$ in both directions, and using only the vertices of $U_x$ and $U_y$, one type at each end of $e$. Then, we will connect the endsets of $P$ by an $\ell$-overlapping path $P' \subseteq H_1$, covering all the remaining vertices and, thus, creating, together with $P$, an $\ell$-overlapping Hamiltonian cycle in $H + e$. The construction of $P'$ will be facilitated by tracing a Hamiltonian path in $G$ connecting $x$ and $y$.

To construct $P$, let $e_1 := e$ and order the vertices of $e_1 = (u_1, \ldots, u_k)$ so that the first $r_x$ vertices belong to $U_x$, the last $r_y$ vertices belong to $U_y$, and the $\ell - r_y$ vertices immediately preceding the $r_y$ vertices of $U_y \cap e_1$ all belong to sets $U_j$ with $j > y$. (We know from (15) that there are more than enough such vertices in $e_1$.) In other words, we request that

$$\{u_1, \ldots, u_{r_x}\} \subset U_x, \tag{17}$$
$$\{u_{k-r_y+1}, \ldots, u_k\} \subset U_y, \tag{18}$$
$$\min\left(\{u_{k-\ell+1}, \ldots, u_{k-r_y}\}\right) > y. \tag{19}$$

The remaining vertices of $e_1$ are labeled arbitrarily by $u_{r_x+1}, \ldots, u_{k-\ell}$.

Our plan is to extend $e_1$ in either direction, but only for as long as the new edges still intersect $e_1$. This means that we will have in $P$ precisely

$$\kappa := \left\lceil \frac{l}{k-\ell} \right\rceil$$

new edges, and thus, precisely

$$\kappa(k-\ell) = g + \ell$$

new vertices on each side of $e_1$, where the last equality follows from (1).

Formally, we set

$$V(P) = \{u_{n-\ell-g+1}, \ldots, u_n, u_1, \ldots, u_k, u_{k+1}, \ldots, u_{k+g+\ell}\}$$

and

$$E(P) = \{e_1\} \cup \{e_{m+1-i} : i = 1, \ldots, \kappa\} \cup \{e_{1+i} : i = 1, \ldots, \kappa\},$$

where, recall, the edge $e_j$ begins at the vertex $u_{1+(j-1)(k-\ell)}$.

We request that all vertices of $P$ to the left of $e_1$ belong to $U_x$ and all vertices to the right of $e_1$ belong to $U_y$, that is,

$$\{u_{n-\ell-g+1}, \ldots, u_n, u_1, \ldots, u_{r_x}\} \subseteq U_x \quad \text{and} \quad \{u_{k-r_x+1}, \ldots, u_k, u_{k+1} \ldots, u_{k+g+\ell}\} \subseteq U_y, \tag{20}$$

This is possible, since, by (16) and (7).

$$\min\left(|U_x \setminus e|, |U_y \setminus e|\right) \geqslant 2k - \ell - g - 2 - (k - \ell - 2g - 1) = k + g - 1 \geqslant \ell + g.$$

We also request that

$$\{u_{n-k+\ell+1}, \ldots, u_{r_x}\} \supseteq \overline{U}_x \quad \text{and} \quad \{u_{k-r_y+1}, \ldots, u_{2k-\ell}\} \supseteq \overline{U}_y. \tag{21}$$

This can be easily accommodated, as each of these sets contains precisely $k-\ell$ vertices from outside of $e_1$. Note that $P$ is, trivially, an $\ell$-overlapping path *in the complete $k$-graph on $V$*. We will show that, in fact, $P \subseteq H_2 + e$.

Suppose first that $m + 1 - \kappa \leqslant j \leqslant m$. Then, by the definition of $x$, $\min(e_j) = x$. By our construction (see (17), (20), and (21)), $|e_j \cap U_x| \geqslant k - \ell + 1$ and $e_j \supseteq \overline{U}_x$. The same is true for $e_j$ with $j = 2, \ldots, \kappa + 1$, if we replace $x$ by $y$ (see (18), (19),(20), and (21)).

To conclude that $P \subseteq H_2 + e$, it remains to show that $c(e_j) \geqslant g + 2$ for each $e_j$, $j \neq 1$. As, clearly, $|e_j \setminus e_1| \leqslant \ell + g$, we also have

$$|e_1 \setminus e_j| \leqslant \ell + g. \tag{22}$$

Trivially, $c(e_1) \leqslant c(e_1 \setminus e_j) + c(e_1 \cap e_j)$. Moreover, $tr(e_j) = tr(e_1 \cap e_j)$. Therefore, by the choice of $e = e_1$ and (22),

$$c(e_j) = c(e_1 \cap e_j) \geqslant c(e_1) - c(e_1 \setminus e_j) \geqslant c(e_1) - |e_1 \setminus e_j| \geqslant \ell + 2g + 2 - (\ell + g) = g + 2.$$

Thus $e_j \in H_2$ for each $e_j \in P$, $j \neq 1$.

Now we will build the rest of $C_H$ using only the edges of $H_1$. Recall that $x$ and $y$ belong to different components of $tr(e)$ and, hence, $xy \notin G$. Therefore, by the Hamiltonian saturation of $G$, there is a Hamiltonian path $Q = (v_1 = y, v_2, \ldots, v_{h-1}, v_h = x)$ from $y$ to $x$ in $G$. We connect the two $\ell$-element endsets of $P$ by an $\ell$-overlapping path $P' = (e_{\kappa+2}, \ldots, e_{m-\kappa})$ in $H_1 \subseteq H$ which, by tracing $Q$, "swallows" all the remaining $n - |V(P)|$ vertices of $V$.

Set $U'_v = U_v \setminus V(P)$, $v \in V(G)$, and

$$R := \bigcup_{v \in V(G)} U'_v.$$

Observe that

$$|R| = n - |V(P)| = n - 2\kappa(k - \ell) - k = n - 2(g + \ell) - k.$$

Let us order the elements $R$ so that all elements of $U'_{v_i}$ precede all elements of $U'_{v_{i+1}}$, for $i = 1, \ldots, h-1$, and denote this ordering by $(u_{k+g+\ell+1}, \ldots, u_{n-g-\ell})$. The vertex set of $P'$ is then defined as

$$V(P') = \{u_{k+g+1}, \ldots, u_{k+g+\ell}, u_{k+g+\ell+1}, \ldots, u_{n-g-\ell}, u_{n-g-\ell+1}, \ldots, u_{n-g}\}.$$

Note that for $v \notin \{x, y\}$, by (7) and (16),

$$|U'_v| \geqslant |U_v| - (k - \ell - 2g - 1) \geqslant k - 1.$$

Hence, every edge of $P'$ stretches over at most two sets $U_v$ and each such two sets are always indexed by adjacent vertices of $G$. This implies that $P' \subseteq H_1$. □

## 4 Proof of Theorem 3

In this section we prove Theorem 3, where the construction of an $\ell$-Hamiltonian saturated $k$-graph is based on a special partition of the vertex set into $q + 1$ sets $U_1, \ldots, U_{q+1}$ ($q$ to be chosen), and the associated with it notion of the hypergraph $H_{k,\ell}(U_1, \ldots, U_{q+1})$, introduced at the beginning of Section 2.

Recall that the function $\nu(x)$ has been defined in Section 2. Given a large integer $n$ divisible by $k - \ell$, choose integers $\alpha = \Theta\left(n^{1/2}\right)$, $\beta = \Theta\left(n^{1/2}\right)$, $p = \Theta\left(n^{1/2}\right)$, and

$$q = \left\lfloor \frac{p(k + 2g) + (p - 1)\nu}{\alpha} \right\rfloor + 2, \tag{23}$$

where $g = g(k, \ell)$ is given by (1) and $\nu := \nu(\alpha)$, such that

$$\alpha \geqslant 10k^3 p, \tag{24}$$

$$\beta \geqslant k\alpha,$$

and

$$n = (q-1)\alpha + \beta + p(k-2) + k - 3. \tag{25}$$

To see that such a choice is feasible, one may set, for instance, $\alpha = \lceil 2k^2\sqrt{n}\rceil$. Recall that, by Proposition 7, $\alpha \leqslant \nu \leqslant k\alpha$. Next, choose $p = \lfloor n/\nu \rfloor - k - 1$. Then, first of all, (24) holds. Furthermore, using (23) and the estimates $g \leqslant k$, $2p \geqslant k-3$, and $4kp \leqslant \alpha$ among others, we can sandwich the quantity

$$n - \beta = (q-1)\alpha + p(k-2) + k - 3$$

as follows:

$$n - (k+3)\nu \leqslant \nu(p-1) \leqslant n - \beta \leqslant 4kp + \alpha + n - (k+2)\nu \leqslant n - k\alpha.$$

Thus, there exists an integer $\beta$, $k\alpha \leqslant \beta \leqslant (k+3)\alpha$, which satisfies (25). Note that, in particular, by (23) and Proposition 8,

$$q \geqslant p + 2k + 1. \tag{26}$$

Let

$$V = \bigcup_{i=1}^{q+1} U_i,$$

where

$$|U_i| = \alpha \quad \text{for} \quad i = 1, \ldots, q-1, \quad |U_q| = \beta \quad \text{and} \quad |U_{q+1}| = p(k-2) + k - 3,$$

and all sets $U_i$, $i = 1, \ldots, q+1$, are pairwise disjoint.

We begin our construction of the required $\ell$-Hamiltonian saturated $k$-graph $H$, by letting

$$H_1 = H_{k,\ell}(U_1, \ldots, U_{q+1}).$$

Recall from Section 2 that $H_1$ breaks naturally into $q+1$ $\ell$-components, that is, $H_1 = C_1 \cup \cdots \cup C_{q+1}$. Thus, every path in $H_1$ is entirely contained in some $C_i$, and, by Corollary 10, for all $i \leqslant q-1$ such paths are no longer than $k\nu \leqslant k^2\alpha$. On the other hand, by the definition of $C_i$, the vertex set of every path contained in $C_q \cup C_{q+1}$ must be a subset of $U_q \cup U_{q+1}$. Therefore, in view of our assumptions on $\beta$, $p$ and $\alpha$, we have the following conclusion.

**Corollary 17.** *The length of a longest path in $H_1$ is $O(\sqrt{n})$. In particular, $H_1$ is not $\ell$-Hamiltonian.* □

Following the outline described in the Introduction, we build a $k$-graph $H'$ by slightly enriching $H_1$, but so that it still remains non-$\ell$-Hamiltonian. Let

$$H_2 = \left\{ e \in \binom{V}{k} : |e \cap U_{q+1}| \geqslant k-2 \right\} \tag{27}$$

and $H' = H_1 \cup H_2$.

**Lemma 18.** *$H'$ is not $\ell$-Hamiltonian.*

*Proof.* Suppose that $C$ is an $\ell$-overlapping Hamiltonian cycle in $H'$. Let $M$ be a maximal set of disjoint edges in $C \cap H_2$. By Corollary 17, $M \neq \varnothing$. Set $t := |M|$. Since

$$|U_{q+1}| = p(k-2) + k - 3 < (p+1)(k-2),$$

we have $t \leqslant p$.

From $C$ we now extract $t$ vertex disjoint paths, all contained in $H_1$, as follows. For every $e \in M$, denote by $N(e)$ the union of the set of vertices of $e$, the set of $g$ consecutive vertices lying just before $e$, and the set of $g$ consecutive vertices lying just after $e$ (here, 'before' and 'after' refer to an arbitrarily fixed direction of traversing $C$). Let $W = \bigcup_{e \in M} N(e)$. Then $C[V \setminus W]$ consists of at most $t$ paths (we treat a nonempty set of fewer than $k$ consecutive isolated vertices as a single trivial path). Observe that

$$|W| \leqslant t(k+2g). \tag{28}$$

Since each obtained path $P$ is contained in $H_1$, either $\min(V(P)) \leqslant q-1$ or $V(P) \subseteq U_q \cup U_{q+1}$. If all $t$ paths are of the former kind, then their total number of vertices is at most $t\nu$, and otherwise, it is at most $(t-1)\nu + |U_q| + |U_{q+1}|$. Note that, since $|U_q| = \beta \geqslant k\alpha \geqslant \nu$, we have

$$\max\{t\nu, (t-1)\nu + |U_q| + |U_{q+1}|\} \leqslant (t-1)\nu + |U_q| + |U_{q+1}|. \tag{29}$$

Finally, by (23), (28), and (29), and using $t \leqslant p$, we get

$$\begin{aligned} n = |V(C)| &\leqslant |W| + (t-1)\nu + |U_q| + |U_{q+1}| \\ &\leqslant p(k+2g) + (p-1)\nu + |U_q| + |U_{q+1}| \\ &< (q-1)\alpha + |U_q| + |U_{q+1}| = n, \end{aligned}$$

which is a contradiction. Hence, there is no $\ell$-overlapping Hamiltonian cycle in $H'$. □

Before we finalize our construction, we need one more piece of notation. For each $e \in \binom{V}{k}$ with $|tr(e)| \geqslant 2$, let

$$\min_2(e) = \min\{i : (e \setminus U_{\min(e)}) \cap U_i \neq \varnothing\}. \tag{30}$$

Finally, set

$$H_3 = \left\{ e \in \binom{V}{k} : |tr(e)| \geqslant 2 \quad \text{and} \quad \min_2(e) \geqslant q - 2k \right\},$$

$$H'' = H_1 \cup H_2 \cup H_3,$$

and let $H$ be a maximal non-$\ell$-Hamiltonian $k$-graph such that $H' \subseteq H \subseteq H''$. By Lemma 18, such a $k$-graph $H$ exists.

**Fact 19.**

$$|H| = O(n^{(k+\ell)/2})$$

*Proof.* By the definitions of $H$ and $H''$,

$$|H| \leqslant |H''| \leqslant |H_1| + |H_2| + |H_3|.$$

Now, noticing that $\max_{1 \leqslant i \leqslant q+1} |U_i| = \beta$, we have

$$|H_1| \leqslant \sum_{i=1}^{q+1} \binom{|U_i|}{k-\ell+1} \cdot \binom{n}{\ell-1} \leqslant (q+1) \cdot \beta^{k-\ell+1} \cdot n^{\ell-1} = O\left(n^{(k+\ell)/2}\right),$$

$$|H_2| \leqslant \binom{|U_q|}{k-2} \cdot \binom{n}{2} \leqslant \beta^{k-2} \cdot n^2 = O\left(n^{(k+2)/2}\right), \text{ and}$$

$$|H_3| \leqslant \sum_{i=1}^{q} \sum_{t=1}^{k-1} \binom{|U_i|}{t} \cdot \binom{|U_{q-2k}| + \cdots + |U_{q+1}|}{k-t} = O\left(q \cdot \alpha^t \cdot \beta^{k-t}\right) = O\left(n^{(k+1)/2}\right),$$

where $i = \min(e)$ and $t = |e \cap U_{\min(e)}|$. □

To complete the proof of Theorem 3, it remains to show the following lemma.

**Lemma 20.** *For every $e \in \binom{V}{k} \setminus H$ the $k$-graph $H + e$ is $\ell$-Hamiltonian.*

*Proof.* Fix $e \in \binom{V}{k} \setminus H$. If $e \in H''$, then, by the definition of $H$, $H + e$ is $\ell$-Hamiltonian. Therefore, we may assume that $e \notin H''$. This implies that $|tr(e)| \geqslant 2$, since otherwise $e \in H_1$. Define

$$x = \min(e) \quad \text{and} \quad y = \min{}_2(e).$$

Since $e \notin H_1 \cup H_3$, we have $|U_x \cap e| \leqslant k - \ell$ and $x < y \leqslant q - 2k - 1$.

Our ultimate goal is to construct in $H$ an $\ell$-overlapping Hamiltonian cycle $C$. Recalling (26), let $J = \{j_1, \ldots, j_{p-2}\}$ be the set of the $p-2$ smallest indices in the set $\{1, \ldots, q - 2k - 1\} \setminus \{x, y\}$. Further, let

$$r_i = |e \cap U_i|, \qquad i = 1, \ldots, q+1.$$

Since $e \notin H_2$, we have $r_{q+1} \leqslant k - 3$. Thus $|U_{q+1} \setminus e| \geqslant p(k-2)$. Let us now set aside $p$ disjoint $(k-2)$-element subsets $B_1, \ldots, B_p$ of $U_{q+1} \setminus e$ and let

$$B = \bigcup_{i=1}^{p} B_i.$$

Note that

$$|U_{q+1} \setminus (B \cup e)| = k - 3 - r_{q+1} \leqslant k. \tag{31}$$

Furthermore, let us also put aside a set $Q = A_q \cup A'_q$ of $2(g+1)$ elements of $U_q \setminus e$, where $|A_q| = |A'_q| = g+1$. The vertices in $B$ and $Q$ will be used later in our construction.

First, however, we construct $p$ vertex disjoint paths $P_{j_1}, \ldots, P_{j_{p-2}}$, $P_{xy}$ and $P_q$. Together, these $p$ paths will contain all elements of $V$, except for some $k-\ell+g+1$ vertices of $U_x$, the same number of vertices of $U_y$, twice as many vertices of each $U_j$, $j \in J$, and except for the vertices in $B \cup Q$. Using these exceptional vertices, the paths will be connected by $p$ 'bridges', made mostly of the edges of $H_2$, to form an $\ell$-overlapping Hamiltonian cycle $C$ in $H$.

**Construction of $\boldsymbol{P_{xy}}$.** Order the vertices of $e$ so that the set $e \cap U_x$ constitutes the leftmost segment of $e$, while the rightmost vertex of $e$ belongs to $U_y$. Next, we will extend $e$ in both directions (see Fig. 1). Let $A'_x$ be a set of arbitrary $k-\ell+g$ vertices of $U_x \setminus e$ and $A_y$ be a set of arbitrary $k-\ell+g$ vertices of $U_y \setminus e$ (the reader should not worry, we will later construct sets $A_x$ and $A'_y$ too). Let

$$R = \bigcup_{i=q-2k}^{q-1} U_i \setminus e.$$

Further, for each $z \in \{x, y\}$, let $P_z \subseteq C_z$ be a path containing precisely

$$\alpha_z := \alpha - r_z - (2k - 2\ell + 2g + 1)$$

vertices of $U_z \setminus (e \cup A'_x \cup A_y)$ and $\nu(\alpha_z) - \alpha_z$ vertices of $R$, where $V(P_x) \cap V(P_y) = \varnothing$. Since, by Proposition 7, each of $P_x$ and $P_y$ requires no more than $(k-1)\alpha$ vertices of $R$, while $|R| \geqslant 2k\alpha - k$, we will not run out of the vertices of $R$.

To finish the construction of $P_{xy}$, we extend $e$

- to the left, by adding the set $A'_x$, followed by $P_x$, and
- to the right, by adding the set $A_y$, followed by $P_y$.

Thus,

$$V(P_{xy}) = V(P_x) \cup A'_x \cup e \cup A_y \cup V(P_y) \subset U_x \cup U_y \cup e \cup R\ .$$

Set

$$A_x = U_x \setminus V(P_{xy}) \qquad \text{and} \qquad A'_y = U_y \setminus V(P_{xy})$$

and observe that

$$|A_x| = |A'_y| = k - \ell + g + 1\ . \tag{32}$$

**Fact 21.**

$$P_{xy} \subseteq H_1 + e$$

*Proof.* The path $P_{xy}$ consists, besides the edges of $P_x$, $P_y$, and $e$ itself, also of a set $A$ of $2\lceil \frac{k}{k-\ell} \rceil$ additional edges, $\lceil \frac{k}{k-\ell} \rceil$ on each side of $e$. These are precisely those edges of $P_{xy}$ which intersect the set $A'_x \cup A_y$. Thus, to prove that $P_{xy} \subseteq H_1 + e$, it remains to show that each edge from $A$ belongs to $H_1$.

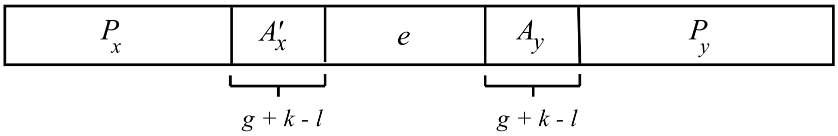


Figure 1: Construction of $P_{xy}$

Let us consider an edge $e'$ intersecting $A'_x$. Obviously, $\min(e') = x$. Also, $|e' \cap A'_x| \geqslant k - \ell$, and so $|e' \cap U_x| \geqslant k - \ell$. Furthermore, if $|e' \cap A'_x| = k - \ell$ then either $e'$ contains also the leftmost vertex of $e$ (which belongs to $U_x$), or $|e' \cap V(P_x)| = \ell$. In the latter case, recall that each edge of $P_x$ contains at least $k - \ell + 1$ vertices from $U_x$, and consequently there is always a vertex form $U_x$ among any $\ell$ vertices of such an edge. In either case, this implies that $|e' \cap U_x| \geqslant k - \ell + 1$, thus $e' \in H_1$. If an edge $e'$ intersects $A_y$ then, by the same argument, we also have $|e' \cap U_y| \geqslant k - \ell + 1$. Finally, note that $\min(e') = y$. Indeed, since $|U_x \cap e| \leqslant k - \ell$, none of the $\ell$ rightmost vertices of $e$ is in $U_x$, and hence, we have $e' \cap U_x = \varnothing$. ☐

**Construction of $\boldsymbol{P_q}$.** Let $P_q$ be a longest path with $V(P_q) \subset U_q \setminus (e \cup Q)$. Clearly, at most $k - \ell - 1$ vertices of $U_q$ will be left out, that is,

$$|U_q \setminus (V(P_q) \cup e \cup Q))| \leqslant k - \ell - 1 \leqslant k. \tag{33}$$

Trivially, $P_q \subset H_1$.

**Construction of $\boldsymbol{P_j}$, $\boldsymbol{j \in J}$.** Set

$$W := \left( \bigcup_{i \in \{1,\dots,q+1\} \setminus (J \cup \{x,y\})} U_i \right) \setminus \left( V(P_{xy}) \cup V(P_q) \cup B \cup Q \cup e \right),$$

and, for each $j \in J$, let $P_j \subseteq C_j \subseteq H_1$ be a path with $V(P_j) \subseteq U_j \cup W$ which uses precisely

$$\alpha_j := \alpha - r_j - (2k - 2\ell + 2g + 2)$$

vertices of $U_j \setminus e$ and *as many as possible* vertices from $W$ (we maintain that all paths $P_j$, $j \in J$, are pairwise vertex-disjoint). Since $i > j$ for every $i \in [q+1] \setminus (J \cup \{x, y\})$, we do have $\min(V(P_j)) = j$. Also,

$$|U_j \setminus (V(P_j) \cup e)| = 2(k - \ell + g + 1) \qquad \text{for each } j \in J\ . \tag{34}$$

Split arbitrarily the set $U_j \setminus (V(P_j) \cup e)$ into two sets $A_q$ and $A'_q$ of equal size $|A_q| = |A'_q| = k - \ell + g + 1$.

Next, we perform crucial calculations showing that we have, indeed, used all the vertices of $W$, that is, there are no vertices outside the constructed paths except for those listed in (32,34) and those put aside in $B \cup Q$.

**Fact 22.**

$$W \subseteq \bigcup_{j \in J} V(P_j)$$

*Proof.* We have, by the definition of $P_{xy}$, and by (31) and (33),

$$\begin{aligned} |W| &= (q-1-p)\alpha - |R \cap V(P_{xy})| + |U_q \setminus (V(P_q) \cup e \cup Q)| + |U_{q+1} \setminus (B \cup e)|\,, \\ &\leqslant (q-1-p)\alpha - (\nu(\alpha_x) - \alpha_x) - (\nu(\alpha_y) - \alpha_y) + 2k. \end{aligned}$$

Recall that each path $P_j$, $j \in J$, may have the maximum length $\nu(\alpha_j)$, and thus cover up to $\nu(\alpha_j) - \alpha_j$ vertices of $W$. Therefore, to complete the proof it suffices to show that

$$(q-1-p)\alpha - (\nu(\alpha_x) - \alpha_x) - (\nu(\alpha_y) - \alpha_y) + 2k \leqslant \sum_{j \in J} (\nu(\alpha_j) - \alpha_j),$$

or, equivalently,

$$\sum_{j \in J \cup \{x,y\}} (\nu(\alpha_j) - \alpha_j) \geqslant (q-1-p)\alpha + 2k.$$

Note that for each $j \in J \cup \{x, y\}$

$$r_j + 2k - 2\ell + 2g + 2 \leqslant 5k. \tag{35}$$

Hence, by the monotonicity of the function $\nu(\cdot)$ and by Proposition 9, we have

$$\nu(\alpha_j) - \alpha_j \geqslant \nu(\alpha - 5k) - \alpha \geqslant \nu - 5k^2 - \alpha,$$

and it remains to show that

$$p(\nu - 5k^2 - \alpha) \geqslant (q-1-p)\alpha + 2k. \tag{36}$$

To this end,

$$\begin{aligned} p(\nu - 5k^2) - p\alpha &\geqslant (p-1)\nu + (\alpha + \alpha/(k-1) - k) - 5k^2 p - p\alpha \quad \text{(by Corollary 10)} \\ &\geqslant (p-1)\nu + \alpha + p(k + 2g) + 2k - p\alpha \quad \text{(by (24))} \\ &\geqslant (q-1-p)\alpha + 2k \quad \text{( by (23))}. \end{aligned}$$

(Since there is some margin in the above estimates, it means that not all the paths $P_j$, $j \in J$, are of maximum length.) □

Now comes the final stage of our construction, where we glue together the paths $P_{j_1}, \ldots, P_{j_{p-2}}$, $P_q$, and $P_{xy}$, in this order, to form a Hamiltonian cycle $C$. We do it as indicated in Fig. 4, with the set $A_x$ placed at the left end of $P_{xy}$, that is, next to the end of the path $P_x$ (see Fig. 4).

Clearly, every edge of $\bigcup_{i=1}^{p-2} P_{j_i} \cup P_{xy} \cup P_q$ belongs to $H + e$. As the last ingredient of our proof of Theorem 3, we now show that every other edge of $C$ belongs to $H_1 \cup H_2 \subseteq H$.

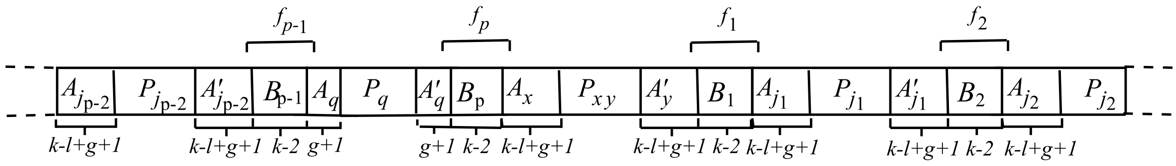


Figure 2: Construction of $C$

**Fact 23.**

$$C \setminus \left( \bigcup_{i=1}^{p-2} P_{j_i} \cup P_{xy} \cup P_q \right) \subseteq H_1 \cup H_2$$

*Proof.* Let

$$\mathcal{A} := \left\{ A_{j_i}, A'_{j_i} : i = 1, \ldots, p-2 \right\} \cup \left\{ A_q, A'_q, A_x, A'_y \right\}.$$

Note that each edge of $C \setminus \left( \bigcup_{i=1}^{p-2} P_{j_i} \cup P_{xy} \cup P_q \right)$ intersects some set $A \in \mathcal{A}$. recall that between any two disjoint edges of $C$ there are exactly $g + t(k - \ell)$ vertices on $C$, for some $t \geqslant 0$. In that case we say that the edge to the right (in some fixed ordering of $C$) *t-follows* the other edge. Let $f_1$, be the edge of $C$ which 1-follows the rightmost edge of $P_{xy}$. Similarly, for $i = 1, \ldots, p-2$, let $f_{i+1}$ be the edge of $C$ which 1-follows the rightmost edge of $P_{j_i}$. Finally, let $f_p$ be the edge of $C$ which 1-follows the rightmost edge of $P_q$, see Fig. 4. Note that for each $i = 1, \ldots, p$, we have $B_i \subset f_i$, and thus $f_i \in H_2$. Furthermore, these are the only edges of $C$ which intersect more than one set from $\mathcal{A}$.

Consider now some $f \in C$, $f \neq f_i$ intersecting $A_{j_i}$. Obviously $\min(f) = j_i$. Also $|f \cap A_{j_i}| \geqslant k - \ell$. However, if $|f \cap A_{j_i}| = k - \ell$, then $|f \cap V(P_{j_i})| = \ell$. Recall that each edge of $P_{j_i}$ contains at least $k - \ell + 1$ vertices of $U_{j_i}$, and consequently there is always a vertex of $U_{j_i}$ among any $\ell$ vertices of such an edge. This implies that $|f \cap U_{j_i}| \geqslant k - \ell + 1$ and so, $f \in H_1$. The same argument works for any $f \in C$ intersecting some set $A \in \mathcal{A}$. $\square$

Thus, we have constructed an $\ell$-overlapping Hamiltonian cycle $C$ in $H + e$, which completes the proof of Lemma 20, which together with Fact 19, implies Theorem 3.

## 5 The smallest open case: $k = 4$ and $\ell = 2$

In this section we prove Theorem 4. Our ultimate goal is, given large even integer $n$, to construct a maximally non-2-Hamiltonian 4-graph $H$. In doing so we refine the technique used in the proof of Theorem 3.

Choose integers $\alpha = \Theta\left(n^{2/5}\right)$, $\alpha \equiv 1 \bmod 3$, $\beta = O\left(n^{3/5}\right)$, $p = \Theta\left(n^{3/5}\right)$, and

$$q = \left\lfloor \tfrac{4(\alpha-1)}{3\alpha}(p-1) \right\rfloor + 1 \tag{37}$$

such that

$$n = q\alpha + 3p + \beta. \tag{38}$$

To see that such a choice is feasible, one may set, for instance, $\alpha = \lceil n^{2/5} \rceil + \epsilon$ where $\epsilon \in \{0,1,2\}$ is such that $\alpha \equiv 1 \bmod 3$. Next choose $p = \lceil \frac{3n}{4\alpha+8} \rceil + 1$. Then, using (37,38) we have

$$\begin{aligned} n-\beta &> \frac{4}{3}(\alpha-1)(p-1) \geqslant n - \frac{3n}{\alpha+2} \quad \text{and} \\ n-\beta &\leqslant \frac{4}{3}(\alpha-1)(p-1)+\alpha+3p = (p-2)\left(\frac{4}{3}(\alpha-1)+4\right) - \left(p-\frac{7}{3}(\alpha-1)-9\right) \\ &\leqslant n - \left(p - \frac{7}{3}(\alpha-1)-9\right), \end{aligned}$$

which shows that a choice of an appropriate $\beta$ is possible.

Let $V = \bigcup_{i=1}^{q+1} U_i$, where $|U_i| = \alpha$, $i = 1,\dots,q$, while $|U_{q+1}| = 3p+\beta$, and all sets $U_i$, $i = 1,\dots,q+1$, are pairwise disjoint. Furthermore, let $G \cong pK_3 + \beta K_1$ be a graph with vertex set $V(G) = U_{q+1}$ consisting of $p$ vertex disjoint triangles and $\beta$ isolated vertices.

We define $H_1$ in the same way as in the general case, while $H_2$ is defined smaller:

$$\begin{aligned} H_1 &= \left\{ e \in \binom{V}{4} : |e \cap U_{\min(e)}| \geqslant 3 \right\}, \\ H_2 &= \left\{ e \in \binom{V}{4} : |e \cap U_{q+1}| = 2,\ |tr(e)| = 2 \text{ and } G[e \cap U_{q+1}] = K_2 \right\}. \end{aligned} \tag{39}$$

The improvement of the upper bound on $\mathrm{sat}(n,4,2)$ is possible mainly because in this particular case one can compute (quiet easily) the value of $\nu(x)$. Below we give only a (sharp) upper bound in some special case.

**Proposition 24.** *Let $x \equiv 0$ mod 3. Then*

$$\nu(x) \leqslant 4\frac{x}{3}.$$

*Proof.* Let $P = (e_1,\dots,e_r)$, $P \subseteq H_1$ and $|V(P) \cap U_{\min(V(P))}| = x$. Recall that each $e_i$, $i = 1,\dots,r$, contains at least 3 vertices from $U_{\min(V(P))}$. Since the $e_i$'s with odd indices are disjoint,

$$\lceil r/2 \rceil \leqslant \frac{x}{3}.$$

If $r$ is odd then

$$|V(P)| \leqslant 4\lceil r/2 \rceil \leqslant 4\frac{x}{3}$$

and the statement follows. Similarly, if $r$ is even and $r/2 \leqslant \frac{x}{3} - 1$ then

$$|V(P)| \leqslant 2r + 2 \leqslant 4\frac{x}{3} - 2$$

and the statement follows again. Suppose, finally, that $r/2 = \frac{x}{3}$, $r$ even. Since $e_r$ contains at least 3 vertices from $U_{\min(V(P))}$, at least one of them is not in $e_{r-1}$, however there are no more available vertices in $U_{\min(V(P))}$, meaning that this case is vacuous. $\square$

**Lemma 25.** *$H' = H_1 \cup H_2$ is not 2-Hamiltonian.*

*Proof.* Suppose that $C$ is a 2-overlapping Hamiltonian cycle in $H'$. As before (cf. Corollary 17), one can easily show that $H_1$ cannot be 2-Hamiltonian. Let $M$ be a maximal set of edges in $C \cap H_2$ with the property that if $e_1, e_2 \in M$ then $(e_1 \cap e_2) \cap U_{q+1} = \varnothing$. In view of the above remark $M \neq \varnothing$. Set

$$V_2 = \bigcup_{e \in M} e \cap U_{q+1}.$$

Clearly, $t := |M| \leqslant p$ and $|V_2| = 2t$. We divide $C$ into $t$ vertex disjoint paths $P_j$, $j = 1, \dots, t$, by cutting through the middle of every edge from $M$ (we treat a set ot 2 consecutive isolated vertices as a single trivial path). More precisely, we keep all vertices in and take the edge set $C - M$. We number the obtained paths so that, for some $1 \leqslant s \leqslant t$, we have $\min(V(P_j)) \leqslant q$ for all $j = 1, \dots, s$ and $V(P_j) \subseteq U_{q+1}$ for all $j = s+1, \dots, t$. Note that, because $M \neq \varnothing$, at least one path must be of the first kind, but possibly $s = t$. Let

$$V_2' = V_2 \cap \bigcup_{j=1}^{s} V(P_j).$$

Since $V(P_j) \subseteq U_{q+1}$ for all $j = s+1, \dots, t$, we have

$$\sum_{j=s+1}^{t} |V(P_j)| \leqslant |U_{q+1}| - |V_2'|. \tag{40}$$

**Claim** For every $j = 1, \dots, s$

$$|V(P_j) \setminus V_2'| \leqslant 4\frac{\alpha - 1}{3}.$$

*Proof.* If some $P_j$ consists of only two vertices then the claim obviously holds. Thus, we may assume that each $P_j$ is non-trivial. For $j \leqslant s$, consider the path $P_j = (e_1, \dots, e_r)$. Let $e_m \in M$ with $|e_m \cap e_1| = 2$. That is $e_m$ precedes $e_1$ on $C$. Similarly, let $e_{r+1} \in M$ with $|e_{r+1} \cap e_r| = 2$, which means that $e_{r+1}$ follows $e_r$ on $C$.

Note that the edges from $H_2$ can occur in $P_j$ only at the ends. Thus $(e_2, \dots, e_{r-1}) =: P_j' \subset H_1$. If $e_1 \in H_1$ then $|e_1 \cap U_{\min(V(P_j))}| \geqslant 3$, meaning that $|e_m \cap U_{\min(V(P_j))}| \geqslant 1$. Thus, by the definition of $H_2$, $|e_m \cap U_{\min(V(P_j))}| = 2$. If $e_1 \in H_2$ then, since $e_1 \notin M$, we have $|e_1 \cap V_2'| \in \{1, 2\}$. If $|e_1 \cap V_2'| = 1$ then $|e_m \cap U_{\min(V(P_j))}| \geqslant 1$ because $|e_m \cap e_1| = 2$ and $|tr(e_1)| = 2$. Thus, again, $|e_m \cap U_{\min(V(P_j))}| = 2$. To sum up

$$\text{if } e_1 \in H_1 \text{ or } |e_1 \cap V_2'| = 1 \text{ then } |e_m \cap U_{\min(V(P_j))}| = 2. \tag{41}$$

The same holds for $e_r$ and $e_{r+1}$

$$\text{if } e_r \in H_1 \text{ or } |e_r \cap V_2'| = 1 \text{ then } |e_{r+1} \cap U_{\min(V(P_j))}| = 2. \tag{42}$$

Suppose first that the assumptions on both $e_1$ and $e_r$ from (41,42), respectively, holds. Thus, $|V(P_j') \cap U_{\min(V(P_j))}| \leqslant \alpha - 4$. Since $\alpha - 4 \equiv 0 \mod 3$, by Proposition 24 and the monotonicity of the function $\nu$,

$$|V(P_j)| = |V(P_j')| + 4 \leqslant 4\frac{\alpha-4}{3} + 4 = 4\frac{\alpha-1}{3}$$

and the claim follows.

Suppose now that $e_1 \in H_2$ with $|e_1 \cap V_2'| = 2$, while $e_r$ satisfies the assumptions from (42). Let $P_j''$ be defined by $(e_3, \ldots, e_{r-1})$. By the definition of $H_2$, $|e_1 \cap U_{\min(V(P_j))}| = 2$. This together with (42) implies that $|V(P_j'') \cap U_{\min(V(P_j))}| \leqslant \alpha - 4$. Hence, by Proposition 24 and the assumption on $e_1$,

$$|V(P_j) \setminus V_2'| = (|V(P_j'')| + 6) - 2 \leqslant 4\frac{\alpha-4}{3} + 4 = 4\frac{\alpha-1}{3}$$

and the claim follows again.

The case when $e_1$ satisfies the assumption of (41) and $|e_r \cap V_2'| = 2$, is analogous (with $P_j'' = (e_2, \ldots, e_{r-2})$).

Finally, if $|e_1 \cap V_2'| = 2$ and $|e_r \cap V_2'| = 2$ then let $P_j'' = (e_3, \ldots, e_{r-2})$. Since $e_1, e_r \in H_2$ (and $e_2, e_{r-1} \in H_1$), we have $|e_1 \cap U_{\min(V(P_j))}| = 2$ and $|e_r \cap U_{\min(V(P_j))}| = 2$. Therefore,

$$|V(P_j) \setminus V_2'| = (|V(P_j'')| + 8) - 4 \leqslant 4\frac{\alpha-4}{3} + 4 = 4\frac{\alpha-1}{3}$$

and the claim follows. □

Returning to the proof of Lemma 25, notice that $|V_2'| \leqslant |V_2| = 2t \leqslant 2p$. Thus

$$|U_{q+1}| = 3p > |V_2'| + 4\frac{\alpha-1}{3}, \tag{43}$$

because $p >> \alpha$. Recalling that $q > \frac{4(\alpha-1)}{3\alpha}(p-1)$ and using the above claim as well as (40,43), we finally argue that

$$\begin{aligned}
n = |V(C_H)| &= \sum_{j=1}^{s} |V(P_j)| + \sum_{j=s+1}^{t} |V(P_j)| \\
&\leqslant \max\{|V_2'| + 4t\frac{\alpha-1}{3}, |V_2'| + 4(t-1)\frac{\alpha-1}{3} + |U_{q+1}| - |V_2'|\}, \\
&\text{(according to wheather } s = t \text{ or } s \leqslant t-1) \\
&= |V_2'| + 4(t-1)\frac{\alpha-1}{3} + |U_{q+1}| - |V_2'| \quad \text{by (43)} \\
&\leqslant 4(p-1)\frac{\alpha-1}{3} + 3p < q\alpha + 3p \leqslant n,
\end{aligned}$$

which is a contradiction. Hence, no 2-overlapping Hamiltonian cycle exists in $H_1 \cup H_2$. □

Let

$$H_3 = \left\{ e \in \binom{V}{4} : |tr(e)| \geqslant 2 \quad \text{and} \quad \min_2(e) \geqslant q \right\}$$

be the same as in the proof of Theorem 3. Finally, let $H'' = H_1 \cup H_2 \cup H_3$ and let $H$ be a maximal non-2-Hamiltonian hypergraph such that $H' \subseteq H \subseteq H''$. By Lemma 25, such a 4-graph exists.

**Fact 26.**

$$|H| = O(n^{14/5})$$

*Proof.* By the definitions of $H$ and $H''$,

$$|H| \leqslant |H''| \leqslant |H_1| + |H_2| + |H_3|.$$

Furthermore,

$$\begin{aligned} |H_1| &= O\left(q \cdot \alpha^3 \cdot n + p^4\right) = O\left(n^{14/5}\right), \\ |H_2| &= O\left(3p \cdot n \cdot n^{2/5}\right) = O\left(n^2\right) \text{ and} \\ |H_3| &= O\left(n \cdot p^3\right) = O\left(n^{14/5}\right). \end{aligned}$$

□

To complete the proof of Theorem 4, it remains to show the following lemma.

**Lemma 27.** *For every $e \in \binom{V}{4} \setminus H$ the 4-graph $H + e$ is 2-Hamiltonian.*

Proof. Let $e = \{u_1, u_2, u_3, u_4\}$, where $u_j \in U_{i_j}$, $j = 1, 2, 3, 4$, and $i_1 \leqslant i_2 \leqslant i_3 \leqslant i_4$. As $e \notin H_1$, we have $|tr(e)| \geqslant 2$. Let $x$ and $y$ stand for the two smallest *different* indices among $i_1, i_2, i_3, i_4$. Note that by the definition of $H$, $e \notin H_3$, and thus $y \leqslant q - 1$.

Set $I = [q-1] \setminus \{x, y\}$, note that $p - 2$ is (much) smaller than $q - 3$, and let $J = \{j_1, \ldots, j_{p-2}\}$ be the set of the $p - 2$ smallest indices in $I$. We will construct $p$ paths $P_{j_1}, \ldots, P_{j_{p-2}}$, $P_{xy}$, and $P_{q+1}$, such that for each $j \in J$, we have $V(P_j) \supseteq U_j \setminus e$,

$$U_x \cup U_y \cup e \subseteq V(P_{xy}) \subset U_x \cup U_y \cup e \cup U_q,$$

and $V(P_{q+1}) \subset U_{q+1}$. Together, these paths will contain all vertices in $V$ except some $2p$ vertices of $U_{q+1}$. Using these exceptional vertices, the paths will be connected by $p$ 'bridges' made of the edges of $H_2$, to form a 2-Hamiltonian cycle in $H$.

For the ease of notation assume that $x = q - 2$ and $y = q - 1$. Then $J = [p - 2]$. To display the structure of each path we will use a shorthand notation $j$ for any element of $U_j$, $j = 1, \ldots, p-2, x, y, q, q+1$. Finally, we designate by * each of the two unknown elements of $e = \{u_1, u_2, u_3, u_4\}$ (other than $x$ and $y$); recall that $u_1 \in U_x$, while $\{u_2, u_3, u_4\} \subseteq \bigcup_{i=x}^{q+1} U_i$ and $|\{u_2, u_3, u_4\} \cap U_x| \leqslant 1$.

**Construction of $P_{xy}$.** We consider five cases with respect to the multiplicities of the vertices of $V_x$ and $V_y$ in $e$.

**Case 1.** In the case when $u_1 \in U_x$, $u_2 \in U_y$ and none of $u_3, u_4$ belongs to $U_y$, the path $P_{xy}$ is constructed as follows:

$$xx|xx|xx|qx|xx|qx|xx|\dots|qx|xx\underbrace{|x*|*y|}_{e}yy|yq|yy|yq|\dots|yy|yq|yy|yy|yy$$

(the sequence begins with 3 blocks $|xx|$ followed by $(\alpha-7)/3$ pairs $|qx|xx|$ and the edge $e$; the right side is constructed similarly with $y$ replacing $x$ and the blocks being arranged in the opposite order), where every element of $U_x \cup U_y$ appears exactly once, while $\frac{2}{3}(\alpha-7) \leqslant |V(P_{xy}) \cap U_q| \leqslant \frac{2}{3}(\alpha-7)+2$ or equivalently $\frac{2}{3}(\alpha-1)-4 \leqslant |V(P_{xy}) \cap U_q| \leqslant \frac{2}{3}(\alpha-1)-2$ (recall that $3|(\alpha-1)$). Note that each pair of consecutive blocks of size two forms an edge of $H_1$ (except the middle pair $x*\,|\,*y$, which is just the edge $e$) and $|V(P_{xy})| = 2\left(4\frac{\alpha-7}{3}+8\right) = \frac{8}{3}(\alpha-1)$.

**Case 2.** If $u_1 \in U_x$, $u_2 \in U_y$ and exactly one of $u_3, u_4$ belongs to $U_y$, the path $P_{xy}$ is constructed as follows:

$$xx|xx|xx|qx|xx|\dots|qx|xx\underbrace{|x*|yy|}_{e}yq|yy|yq|\dots|yy|yq|yy|yy.$$

Again, $|V(P_{xy})| = \frac{8}{3}(\alpha-1)$, while $\frac{2}{3}(\alpha-1)-3 \leqslant |V(P_{xy}) \cap U_q| \leqslant \frac{2}{3}(\alpha-1)-2$.

**Case 3.** If $u_1 \in U_x$ and $u_2, u_3, u_4 \in U_y$ then we form $P_{xy}$ as follows:

$$xx|xx|xx|qx|xx|\dots|qx|xx\underbrace{|xy|yy|}_{e}yq|yy|yq|\dots|yy|yq|yy|yy|yy.$$

This time $|V(P_{xy})| = \frac{8}{3}(\alpha-1)-2$ and $|V(P_{xy}) \cap U_q| = \frac{2}{3}(\alpha-1)-4$.

**Case 4.** If $u_1, u_2 \in U_x$, $u_3 \in U_y$ and $u_4 \notin U_y$, the path $P_{xy}$ is constructed as follows:

$$xx|xx|qx|xx|\dots|qx|xx|qx\underbrace{|xx|*y|}_{e}yy|yq|yy|\dots|yq|yy|yy|yy.$$

Now $|V(P_{xy})| = \frac{8}{3}(\alpha-1)$ and $\frac{2}{3}(\alpha-1)-3 \leqslant |V(P_{xy}) \cap U_q| \leqslant \frac{2}{3}(\alpha-1)-2$.

**Case 5.** If $u_1, u_2 \in U_x$ and $u_3, u_4 \in U_y$, we form the path $P_{xy}$ as follows:

$$xx|xx|qx|xx|\dots|qx|xx|qx\underbrace{|xx|yy|}_{e}yq|yy|yq|\dots|yy|yq|yy|yy.$$

We have again $|V(P_{xy})| = \frac{8}{3}(\alpha-1)$, while $|V(P_{xy}) \cap U_q| = \frac{2}{3}(\alpha-1)-2$.

Let us now set aside $p$ 2-element disjoint subsets $B_1, \dots, B_p$ of $U_{q+1}$ which correspond to disjoint edges of the graph $G$, one from each triangle of $G$. Set $B = \bigcup_{i=1}^{p} B_i$. These pairs will be used to glue together all $p$ paths into a Hamiltonian 2-cycle.

To describe the remaining paths, let symbol $w$ represent any element of the set

$$W := \bigcup_{i=p-1}^{q-3} U_i \cup U_q \cup (U_{q+1} \setminus B) \setminus V(P_{xy}).$$

**Construction of $P_j$, $j = 1, \ldots, p-2$.** For $j = 1, \ldots, p-2$, we build path $P_j$ by splitting $\alpha - 4$ vertices of $U_j$ into $(\alpha-4)/3$ blocks of length 3, separating them by arbitrary vertices from $W$ and putting the remaining 4 vertices of $U_j$ at the end. In a diagram form

$$P_j = jj|jw|jj|jw|\ldots|jj|jw|jj|jj.$$

Because $j < \min\{i : U_i \cap W \neq \varnothing\}$, each pair of consecutive blocks of size two forms an edge of $H_1$. Also, $|V(P_j)| = \frac{4}{3}(\alpha - 1)$, which means that $P_j$ can accommodate precisely $(\alpha - 4)/3$ vertices from $W$. As, by our choice of $q$,

$$(p-2)\frac{\alpha-4}{3} \geqslant (q-p-1)(\alpha-1) + \frac{\alpha-1}{3} + 3, \tag{44}$$

we have

$$\bigcup_{r=1}^{p-2} V(P_j) \supseteq \bigcup_{i=p-1}^{q-3} U_i \cup (U_q \setminus V(P_{xy}))\,.$$

On the other hand, the difference between the L-H-S and R-H-S of (44) is less than $4\frac{\alpha}{3} << p$, so that the surplus $w$-spots can be filled with some elements of $U_{q+1}$.

**Construction of $P_{q+1}$.** The last path, $P_{q+1}$, consists of all the remaining vertices of $U_{q+1}$ whose number is even, because $n$ is even and every so far built path, as well as the set $B$, consists of an even number of vertices.

The constructed paths $P_1, \ldots, P_{p-2}, P_{xy}$, and $P_{q+1}$ are now connected together, in arbitrary order, by the 2-element blocks $B_1, \ldots, B_p$. Note that each $B_j$ makes edges of $H_2$ with arbitrary 2-element sets from some $U_i$, $i = 1, \ldots, q$. This completes the construction of a 2-Hamiltonian cycle in $H + e$. □

The proof of Theorem 4 follows immediately from Lemma 27 and Fact 26.

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