An improved upper bound on the density of universal random graphs

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Abstract. We we give a polynomial time randomized algorithm that proves that, for every integer $d \geq 3$ and suitable constant $C = C_d$, as $n \to \infty$, asymptotically almost all graphs with n vertices and $\lfloor Cn^{2-\frac{1}{d}} \log^{\frac{1}{d}} n \rfloor$ edges contain as subgraphs all graphs with n vertices and maximum degree at most d.

1 Introduction

Given graphs H and G, an *embedding* of H into G is an injective edge-preserving map $f\colon V(H) \to V(G)$, that is, for every $e = \{u, v\} \in E(H)$, we have $f(e) = \{f(u), f(v)\} \in E(G)$. We shall say that a graph H *is contained in* G *as a subgraph* if there is an embedding of H into G. Given a family of graphs $\mathcal{H}$, we say that G is *universal with respect to* $\mathcal{H}$, or $\mathcal{H}$-*universal*, if every $H \in \mathcal{H}$ is contained in G as a subgraph.

The construction of sparse universal graphs for various graph families has received a considerable amount of attention; see, e.g., [1,3,4,5,6,7,8,10] and the references therein. One is particularly interested in (almost) tight $\mathcal{H}$-universal graphs, i.e., graphs whose number of vertices is (almost) equal to $\max_{H\in\mathcal{H}} |V(H)|$.

Let $d \in \mathbb{N}$ be a fixed constant and let $\mathcal{H}(n, d) = \{H \subset K_n : \Delta(H) \leq d\}$ denote the class of (pairwise non-isomorphic) n-vertex graphs with

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maximum degree bounded by d and $\mathcal{H}(n,n;d) = \{H \subset K_{n,n} : \Delta(H) \leq d\}$ be the corresponding class for balanced bipartite graphs.

By counting all unlabeled d-regular graphs on n vertices one can easily show that every $\mathcal{H}(n,d)$-universal graph must have

$$\Omega(n^{2-2/d}) \tag{1}$$

edges (see [3] for details). This lower bound was almost matched by a construction from [4], which was subsequently improved in [2] and [1]. Those constructions were designed to achieve a nearly optimal bound and as such they did not resemble a "typical" graph with the same number of edges. To pursue this direction, in [3], the $\mathcal{H}(n,d)$-universality of random graphs was also investigated.

For random graphs a slightly better lower bound than (1) is known. Indeed, any $\mathcal{H}(n,d)$-universal graph must contain as a subgraph the union of $\lfloor \frac{n}{d+1} \rfloor$ vertex-disjoint copies of K_{d+1}, and, in particular, all but at most d vertices must each belong to a copy of K_{d+1}. Therefore, recalling the threshold for the latter property (see, e.g., [14, Theorem 3.22 (i)]), we conclude that the expected number of edges needed for the $\mathcal{H}(n,d)$-universality of $G_{n,p}$ must be

$$\Omega\left(n^{2-2/(d+1)}(\log n)^{1/\binom{d+1}{2}}\right), \tag{2}$$

a quantity bigger than (1).

We say that $G_{n,p}$ possesses a property $\mathcal{P}$ *asymptotically almost surely* (**a.a.s.**) if $\mathbf{P}[G_{n,p} \in \mathcal{P}] = 1 - o(1)$. In [3], it was proved that for a sufficiently large constant C:

- (almost spanning universality) $G_{(1+\varepsilon)n,p}$ is **a.a.s.** $\mathcal{H}(n,d)$-universal if $p = Cn^{-\frac{1}{d}} \log^{\frac{1}{d}} n$;
- (bipartite universality) $G_{n,n,p}$ is **a.a.s.** $\mathcal{H}(n,n,d)$-universal if $p = Cn^{-\frac{1}{2d}} \log^{\frac{1}{2d}} n$.

Note that the first result above deals with embeddings of n-vertex graphs into random graphs with larger vertex sets, which makes the embedding somewhat easier. On the other hand, the second result deals with tight universality at the cost of requiring the graphs to be bipartite and with a less satisfactory bound.

Those results were improved by the authors in [9,11], where it was shown that $G_{n,n,p}$ is **a.a.s.** $\mathcal{H}(n,n,d)$-universal if $p = Cn^{-\frac{1}{d}} \log^{\frac{1}{d}} n$, and $G_{n,p}$ is **a.a.s.** $\mathcal{H}(n,d)$-universal if $p = Cn^{-\frac{1}{2d}} \log^{\frac{1}{2d}} n$. In this paper, making use of an additional randomization step in the embedding algorithm involved, we improve the latter result, by establishing a density threshold for $\mathcal{H}(n,d)$-universality of $G_{n,p}$ which matches (up to the log factor) the best previous bounds for both the bipartite tight universality and the almost tight universality in the general case.

Theorem 1. *Let $d \geq 3$ be fixed and $p = p(n) = C n^{-\frac{1}{d}} \log^{\frac{1}{d}} n$ for some sufficiently large constant C. Then the random graph $G_{n,p}$ is **a.a.s.** $\mathcal{H}(n,d)$-universal.*

Observe that there is still a gap between the lower bound (2) and the upper bound given by Theorem 1.

Remark 1. In Theorem 1 we assume that $d \geq 3$ since for $d = 2$ our proof would require a few modifications. On the other hand, we feel that the true bound is much lower. Possibly as low as (2), which, as proved by Johannson, Kahn, and Vu [16], is also a threshold for the triangle-factor in $G(n,p)$. The case $d = 2$ will be dealt with elsewhere.

This paper is organized as follows. In the next section we describe a randomized algorithm that seeks, for any $H \in \mathcal{H}(n,d)$ and any n-vertex graph G, an embedding $f\colon V(H) \to V(G)$. Crucially, at the beginning of our algorithm, a collection of pairwise vertex-disjoint d-cliques is sampled from a certain subset of vertices of G, uniformly at random. This randomization allows us to verify a Hall-type condition that we use to embed the final group of vertices. This is formally stated in Lemma 4, which is proved in the appendix (Section 5.3).

In Section 4, we prove that our algorithm succeeds with high probability for every $H \in \mathcal{H}(n,d)$ when run on $G_{n,p}$, as long as $p = Cn^{-\frac{1}{d}} \log^{\frac{1}{d}} n$ and $C = C_d$ is a large enough constant. Several relevant properties of $G_{n,p}$ for such a p is singled out in Section 3.

Throughout the paper we will use the following notation. For $v \in V$, let $G(v)$ denote the neighborhood of the vertex v in G. For $T \subset V$, let

$$G(T) = \{v \in V \setminus T \colon G(v) \cap T \neq \emptyset\} = \bigcup_{u \in T} G(u) \setminus T$$

denote the neighborhood of the set T in G. For $T \subset V$, let $G[T]$ denote the subgraph of G induced by T.

2 The embedding

Let

$$\varepsilon = \varepsilon(d) = \frac{1}{100d^4} \tag{3}$$

be fixed and $n = n(d)$ be a sufficiently large integer. Given an n-vertex graph G, set $V := V(G)$ and let

$$V = V_0 \cup R_1 \cup \cdots \cup R_{d^2+2}, \quad \text{where } |R_i| = \varepsilon n \text{ for all } i, \tag{4}$$

be a fixed partition of V.

Without loss of generality, we will assume that H is a maximal graph from $\mathcal{H}(n, d)$ in the sense that adding any edge to H increases its maximum degree beyond d. Since in such a graph the vertices with degrees smaller than d must form a clique, there are at most d of them.

We set $X := V(H)$ and $n := |X|$, and fix an integer $t = \tau n$, where

$$\tau = 2\varepsilon = \frac{1}{50d^4}. \tag{5}$$

In the embedding algorithm we will use the following procedure of preprocessing H with a given t.

THE PRE-PROCESSING OF H: Select vertices $x_1, \ldots, x_t \in X$ in such a way that they all have degree d and form a 3-*independent set* in H, that is every pair of distinct vertices x_i, x_j is at distance at least four. (Owing to our choice of t, we may find these t vertices by a simple greedy algorithm.) Let $S_i = H(x_i)$ for all $i = 1, \ldots, t$, and set

$$X_0 := \bigcup_{j=1}^{t} S_j.$$

Note that by the 3-independence, for all $i \neq j$ not only $S_i \cap S_j = \emptyset$, but also there is no edge between S_i and S_j in H, that is, $e_H(S_i, S_j) = 0$.

Next, consider the square H^2 of the graph H obtained from H by adding edges between all pairs of vertices at distance two. Since the maximum degree of H^2 is at most d^2, by the Hajnal–Szemerédi Theorem (see [12]) applied to H^2, there is a partition

$$X = X_1' \cup X_2' \cup \cdots \cup X_{d^2+1}',$$

such that each set X_i', $1 \leq i \leq d^2 + 1$, is independent in H^2, and thus, 2-independent in H, and has roughly the same size, that is, $\big||X_i'| - |X_j'|\big| \leq 1$ for all i, j. (In fact, we apply here an algorithmic version from [17] (see also [18]) which yields a polynomial time algorithm.) Finally, set

$$X_i = X_i' \setminus \{x_1, \ldots, x_t\} \setminus X_0, \quad i = 1, \ldots, d^2 + 1,$$

and $X_{d^2+2} = \{x_1, \ldots, x_t\}$. Hence, we obtain a partition

$$X = X_0 \cup X_1 \cup \cdots \cup X_{d^2+2}, \tag{6}$$

where, for $i = 1, \dots, d^2+1$, the sets X_i are 2-independent and

$$|X_i| \geq \frac{n}{d^2+1} - 1 - t(d+1) \geq \frac{n}{2d^2} > t, \tag{7}$$

while X_{d^2+2} is 3-independent, $|X_{d^2+2}| = t$, and X_0 is a (disjoint) union of the d-element neighborhoods of the vertices in X_{d^2+2}. (See Figure 1 for an illustration of this partition.) The numbering of the sets $X_0, \dots, X_{d^2+2}$

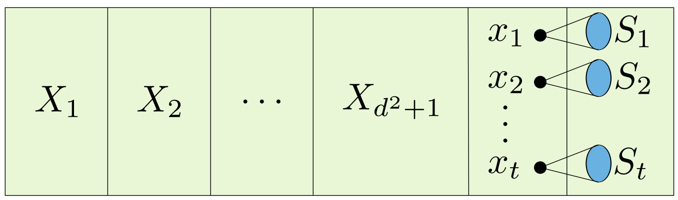

Fig. 1. The partition of $V(H)$

corresponds to the order in which these sets will be embedded into a graph G by the embedding algorithm.

Another building block of our embedding algorithm is a procedure which, given a partial embedding f_{i-1} of $H[X_0 \cup \dots \cup X_{i-1}]$ into G, constructs an auxiliary graph A_i displaying information about current candidates among selected vertices of G for images of the vertices from X_i.

The auxiliary graph A_i: For $i = 1, \dots, d^2+2$ and a partial embedding $f_{i-1} \colon X_0 \cup \dots \cup X_{i-1} \to V$, let A_i be a bipartite graph with classes X_i and

$$W_i := V \setminus \mathrm{im}(f_{i-1}) \setminus \bigcup_{j=i+1}^{d^2+2} R_j$$

and the edge set

$$\big\{(x, v) \in X_i \times W_i \ : \ f_{i-1}\big(H(x)\big) \subset G(v)\big\}. \tag{8}$$

Observe that $A_i(x)$ is the set of all vertices $v \in W_i$ for which $x \mapsto v$ is a valid extension of the embedding f_{i-1}, while $A_i(v)$ is the set of all vertices $x \in X_i$ for which v is a valid image.

Since the set X_i is independent, the embedding of X_i can be done at once and it corresponds to a matching in A_i saturating X_i. (The 2-independence of X_i's will only be used in the analysis of the algorithm for random-like graphs as inputs.) Note that $|W_{d^2+2}| = |X_{d^2+2}|$, while for $i \le d^2+1$, the set W_i is much bigger than the set X_i. Indeed,

$$|W_i| = n - \sum_{j<i} |X_j| - \sum_{j>i} |R_j| = |X_i| + \sum_{j>i} (|X_j| - |R_j|) \ge |X_i| + \varepsilon n. \quad (9)$$

The embedding will be done in d^2+2 rounds split into three phases:

- **Phase 1**: The sets $S_1, \dots, S_t$ are mapped *randomly* onto disjoint cliques of $G[V_0]$.
- **Phase 2**: The sets X_i, $i = 1, \dots, d^2+1$, are embedded, one by one, into sets W_i.
- **Phase 3**: The set X_{d^2+2} is mapped one-to-one onto the set W_{d^2+2} of t remaining vertices of G.

A potential problem for our proposed embedding scheme is that the candidate set for a given vertex $x \in X = V(H)$ may be depleted before we have a chance to embed x. If that happens, there is no hope to complete the embedding. Similarly, a vertex $v \in V = V(G)$ may lose all of its neighbors in the auxiliary graph as a result of an unfortunate sequence of extensions. In other words, v can be excluded from all candidate sets and thus cannot be used in the embedding. Since we have to use all vertices of $v \in V$ in the embedding, we must prevent this event as well. Our algorithm incorporates two devices that help to address these problems.

Buffer vertices in G (used in Phases 2 and 3). We will make sure that for each $i = 1, \dots, d^2+2$, $\mathrm{im}(f_{i-1}) \cap R_i = \emptyset$ (see line 5 of Algorithm 1). This way the vertices of R_i will be reserved as a *buffer* to help embed the set X_i, provided the sets R_i will satisfy certain properties in G – see Section 3.)

Buffer vertices in H (used in Phase 3). Since the neighborhoods S_j of the vertices x_j from X_{d^2+2} are embedded during Phase 1, the sets $A_i(v) \cap X_{d^2+2}$, $v \in V$, remain the same throughout Phase 2. This will help to ensure the existence of a perfect matching in A_{d^2+2} in Phase 3, provided the random choices of $f(S_j)$ satisfy certain properties – see Lemma 4.

Now we present our embedding algorithm.

Algorithm 1: The embedding algorithm

Input : A graph H with n vertices and $\Delta(H) \le d$ and a graph G together with a vertex partition (4).

Output: An embedding $f\colon V(H) \to V(G)$ (or the algorithm fails).

```
// Phase 1
```

1 Pre-process H, obtaining a partition $X = X_0 \cup \cdot \cup X_{d^2+2}$ as in (6), where $X_0 = S_1 \cup \cdots \cup S_t$, $X_{d^2+2} = \{x_1, \dots, x_t\}$, and $H(x_j) = S_j$, $j = 1, \dots, t$.

2 Randomly select from V_0 a sequence of pairwise disjoint d-element sets $T_1, \dots, T_t$ such that, for each $i = 1, \dots, t$, $G[T_i]$ is a clique, with all such sequences equiprobable.

3 Define a map $f_0\colon X_0 \to \bigcup_{i=1}^{t} T_i$ in such a way that $f_0(S_i) = T_i$ for each $i = 1, \dots t$.

```
// Phase 2
```

4 **for** $i = 1$ *to* $i = d^2 + 1$ **do**

5   Set $W_i = V \setminus \mathrm{im}(f_{i-1}) \setminus \displaystyle\bigcup_{j=i+1}^{d^2+2} R_j$;

6   Construct the auxiliary bipartite graph A_i between the sets X_i and W_i, and find therein a matching M_i of size $|M_i| = |X_i|$.

7   Define the extension f_i of f_{i-1} by setting $f_i(x) = v$ for all $x \in X_i$, where $(x, v) \in M_i$, and $f_i(x) = f_{i-1}(x)$ for all $x \in X_0 \cup \cdots \cup X_{i-1}$.

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// Phase 3
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8 Set $W_{d^2+2} = V \setminus \mathrm{im}(f_{d^2+1})$ $(\supset R_{d^2+2})$.

9 Construct the auxiliary bipartite graph A_{d^2+2} between sets X_{d^2+2} and W_{d^2+2}, and find therein a perfect matching M_{d^2+2}.

10 Define the output embedding f by setting $f(x) = v$ for all $x \in X_{d^2+2}$, where $(x, v) \in M_{d^2+2}$, and $f(x) = f_{d^2+1}(x)$ for all $x \in X \setminus X_{d^2+2}$.

This algorithm finds a desired embedding of H into G as long as it is successful in lines 2, 6 and 9. The sets S_i are embedded into V_0 by uniformly sampling a sequence of pairwise disjoint d-subsets $T_1, \dots, T_t \subset V_0$ such that every set T_i induces a clique in G. Thus, one (trivial) necessary condition for the success of the algorithm is that G contains at least t disjoint cliques K_d. Notice that the map f_0 is an embedding, since the edges within S_i are clearly preserved ($G[T_i]$ is a clique), while $e_H(S_i, S_j) = 0$ holds for all $j \ne i$ by construction.

Two more demanding conditions are that the auxiliary bipartite graphs A_i from lines 6 and 9 do possess the required matchings. Superficially, we could have combined the last two phases by including round $d^2 + 2$ into the loop, however we chose not to do so, because of the much more involved analysis of the last round. Indeed, it is a lot harder to prove the existence of a perfect matching in A_{d^2+2} than the existence of a matching saturating one side of A_i when the other side is much bigger.

It is worth pointing out that the success of Phase 3 relies entirely on the (random) outcome of Phase 1. The algorithm's goal in Phase 3 is to find a perfect matching in the auxiliary bipartite graph A_{d^2+2} (which has classes X_{d^2+2} and W_{d^2+2}). Recall that the neighborhoods $S_j = H(x_j)$ of the vertices $x_j \in X_{d^2+2}$ are completely embedded in Phase 1. Since f_{d^2+1} is an extension of f_0, for each $x_j \in X_{d^2+2}$ we have $f_{d^2+1}(S_j) = f_0(S_j) = T_j$. This implies that for every $v \in W_{d^2+2}$, by definition, $\{x_j, v\} \in A_{d^2+2}$ if and only $T_j \subset G(v)$ if and only if $\{x_j, v\} \in A_1$. Thus,

$$A_{d^2+2} = A_1[X_{d^2+2} \cup W_{d^2+2}]. \tag{10}$$

This observation will be utilized in the analysis of Algorithm 1 in Section 4.

3 Random graphs

In this section we show that a random graph $G_{n,p}$ with $p = p(n)$ as in Theorem 1 **a.a.s.** satisfies several properties with respect to the distribution of edges and cliques. These properties are selected in order to jointly guarantee tight $\mathcal{H}(n,d)$-universality. More specifically, in Section 4 we will show that Algorithm 1 is **a.a.s.** successful on all pairs of input graphs (H, G), where $H \in \mathcal{H}(n,d)$ and G satisfies all these properties. But first we need some more notation.

– Given a graph G, $V(G) = V$, and a subset of vertices $U \subset V$, denote by

$$\binom{U}{K_d}$$

the family of all d-element sets $T \subset V$ such that the subgraph of G induced by T is complete, that is, $G[T] \cong K_d$.

– Given a family $\mathcal{X} = \{J_1, \ldots, J_r\}$ of pairwise disjoint subsets of V and a set $U \subset V$, let $B = B(\mathcal{X}, U)$ be a bipartite graph with vertex classes $\mathcal{X}$ and $U_{\mathcal{X}} := U \setminus \bigcup_{i=1}^r J_i$, where an edge (J_i, v) is included whenever $G(v) \supset J_i$. Furthermore, let

$$\alpha(\mathcal{X}, U) = |\{v \in U_{\mathcal{X}} : \deg_B(v) \geq 1\}|.$$

If all sets J_i are singletons, then we write $B(Y, U)$ instead of $B(\mathcal{X}, U)$ and $\alpha(Y, U)$ instead of $\alpha(\mathcal{X}, U)$, where $Y = \bigcup J_i$. Note that in this special case $\alpha(Y, U) = |G(Y) \cap U|$.

– We write $a = (1 \pm \delta)b$ whenever $(1 - \delta)b \leq a \leq (1 + \delta)b$.

– Set

$$\omega = C \log n. \tag{11}$$

Let $\varepsilon = \varepsilon(d) > 0$ be as in (3). Set $V = [n]$ and fix a partition

$$V = V_0 \cup R_1 \cup \cdots \cup R_{d^2+2}$$

satisfying (4). By (3),

$$|V_0| = n - (d^2+2)\varepsilon n \geq \frac{3n}{4}. \tag{12}$$

The following lemma, proved in the appendix (Section 5.1), summarizes several relevant properties of $G_{n,p}$.

Lemma 1. *For every $\delta > 0$, there exists $C > 0$ such that the random graph $G = G_{n,p}$ with $p \geq Cn^{-1/d} \log^{1/d} n$* ***a.a.s.*** *satisfies Properties* (**I**)–(**V**) *below.*

(**I**) (*a*) *For all $y \in V$,*

$$|G(y) \cap V_0| = (1+o(1))p|V_0|.$$

(*b*) *For all $y \neq y' \in V$,*

$$|G(y) \cap G(y') \cap V_0| = (1+o(1))p^2|V_0|.$$

(**II**) (*a*) *For all $Y \subset V$, $|Y| \leq \delta p^{-1}$,*

$$|G(Y) \cap V_0| = (1 \pm 2\delta)p\,|Y|\,|V_0|. \tag{13}$$

(*b*) *For all $Y \subset V$ with $|Y| \geq \omega p^{-1}$ and $U \subset V \setminus Y$ with $|U| \geq \omega p^{-1}$,*

$$|B(Y,U)| = (1 \pm \delta)p\,|Y|\,|U|. \tag{14}$$

(**III**) (*a*) *For all $r \leq \delta p^{-d}$, every family $\mathcal{X} = \{J_1, \ldots, J_r\}$ of pairwise disjoint d-subsets of V, and for every set $U \in \{V_0, R_1, \ldots, R_{d^2+2}, V\}$, we have*

$$\alpha(\mathcal{X}, U) = (1 \pm \delta)p^d r\,|U|. \tag{15}$$

(*b*) *For all $r \geq \omega p^{-d}$, every family $\mathcal{X} = \{J_1, \ldots, J_r\}$ of pairwise disjoint d-subsets of V, and $U \subset V \setminus \bigcup_{i=1}^r J_i$ with $|U| \geq \omega p^{-d}$,*

$$|B(\mathcal{X}, U)| = (1 \pm \delta)p^d r\,|U|. \tag{16}$$

(**IV**) *Equation*

$$\left|\binom{U}{K_d}\right| = (1 \pm \delta)p^{\binom{d}{2}}\binom{|U|}{d}. \tag{17}$$

holds for all $U \subset V$ such that

(*a*) $U \subset G(v)$ *for some* $v \in V$*, and* $|U| \geq pn/3$*, or*

(*b*) $U = G(u) \cap G(v)$ *for some distinct* $u, v \in V$*, or*

(*c*) $|U| \geq |V|/4$.

(**V**) *For all* $v \in V_0$*, the number of* d*-cliques in* $G[V_0]$ *containing vertex* v *is*

$$(1 \pm \delta)\frac{d}{|V_0|}\left|\binom{V_0}{K_d}\right|.$$

4 The analysis of Algorithm 1

In this section we prove the following lemma that, together with Lemma 1, implies Theorem 1.

Lemma 2. *If* ε *and* τ *are as in* (3) *and* (5)*, and a graph* G *on vertex set* $V = [n]$ *together with a partition* $V = R_1 \cup \cdots \cup R_{d^2+2} \cup V_0$ *as in* (4) *satisfy Properties* (**I**)*–*(**V**) *from Lemma 1 with* $\delta = 0.01$ *and sufficiently large* C*, then Algorithm 1 with input* G *is* ***a.a.s.*** *successful, that is, for every* $H \in \mathcal{H}(n, d)$ *it* ***a.a.s.*** *outputs an embedding of* H *into* G.

As mentioned before, Algorithm 1 is successful if it does not terminate at lines 2, 6, or 9. To perform line 2 we need at least t disjoint d-cliques in $G[V_0]$. This follows from Property (**IV**)(c), since $t \leq \frac{1}{2d}n$. Lines 6 and 9 rely on the existence of saturating matchings in the auxiliary graphs A_i. The existence of such matchings will follow from the next two lemmas. In both, we implicitly assume that a fixed graph G satisfies Properties (**I**)–(**V**) from Lemma 1, and that (3)–(5) hold.

Lemma 3. *For* $i = 1, \ldots, d^2 + 2$ *and for every* $Q \subset X_i$ *we have*

$$|A_i(Q)| \geq \min\{|Q|, |W_i| - \omega p^{-d}\}. \tag{18}$$

In particular, if $|W_i| \geq |X_i| + \omega p^{-d}$*, then* $|A_i(Q)| \geq |Q|$ *for all sets* $Q \subset X_i$.

In the next lemma the probability space corresponds to the random choice of f_0.

Lemma 4. *The random embedding f_0 of the sets S_i, $i = 1, \dots, t$, is such that* **a.a.s.** *for every set $Y \subset V$, $|Y| \le \delta(4p)^{-d}$,*

$$|A_1(Y) \cap X_{d^2+2}| \ge \frac{1}{2}\Big(\frac{p}{5}\Big)^d t\,|Y|. \tag{19}$$

The proof of Lemma 3 is given in appendix (Section 5.2), as is the proof of Lemma 4 (Section 5.3), which is much more involved. The following corollary of the above two lemmas completes the proof of Lemma 2.

Corollary 1. (i) *For each $i = 1, \dots, d^2+1$, the graph A_i has a matching saturating set X_i.* (ii) *The graph A_{d^2+2} has a perfect matching* **a.a.s.**

Proof. (i) Fix $1 \le i \le d^2+1$ and recall that

$$W_i = V \setminus \mathrm{im}(f_{i-1}) \setminus \bigcup_{j=i+1}^{d^2+2} R_j$$

and, by (9), that $|W_i| \ge |X_i| + \varepsilon n$. For C sufficiently large, we have $\varepsilon n \ge C^{-d+1} n = \omega p^{-d}$. Thus, $|W_i| \ge |X_i| + \omega p^{-d}$, which, by Lemma 3, implies that $|A_i(Q)| \ge |Q|$ for all $Q \subset X_i$. Consequently, by Hall's theorem, there is a matching in A_i covering X_i.

(ii) Set $h = d^2+2$ for convenience. To prove that A_h has **a.a.s.** a perfect matching, recall that, as a consequence of (10), $A_h = A_1[X_h \cup W_h]$. By Lemma 4, **a.a.s.** for every $Y \subset W_h$ with $|Y| \le \delta(4p)^{-d}$, we have (see (19)),

$$|A_h(Y)| = |A_1(Y) \cap X_h| \ge \frac{1}{2}\Big(\frac{p}{5}\Big)^d t\,|Y| \ge \delta^{-1}4^d\omega\,|Y|, \tag{20}$$

provided C is large enough. We claim that the condition above ensures that there is a perfect matching in A_h. Recall that $|X_h| = |W_h| = t$. Let $Q \subset X_h$. If $|Q| \le t - \omega p^{-d}$ then Lemma 3 implies that $|A_h(Q)| \ge |Q|$. Assume then that $|Q| \ge t - \omega p^{-d} + 1$ (for simplicity, we assume that ωp^{-d} is an integer), and suppose, for the sake of contradiction, that $|A_h(Q)| \le |Q| - 1$, equivalently, that $|W_h \setminus A_h(Q)| \ge t - |Q| + 1$. If $|W_h \setminus A_h(Q)| \le \delta(4p)^{-d}$, take $Y = W_h \setminus A_h(Q)$. Otherwise, take any $Y \subset W_h \setminus A_h(Q)$ with $|Y| = \delta(4p)^{-d}$. By (20),

$$|A_h(Y)| \ge \delta^{-1}4^d\omega\,|Y| \ge t - |Q| + 1, \tag{21}$$

where the last inequality is clear if $Y = h \setminus A_h(Q)$, while otherwise we argue, using the definition of Y and our assumption on $|Q|$, that $\delta^{-1}4^d\omega\,|Y| = \omega p^{-d} \ge t - |Q| + 1$. Inequality (21) contradicts the fact that $A_h(Y) \cap Q = \emptyset$. Therefore, $|A_h(Q)| \ge |Q|$ for all $Q \subset X_h$ and Hall's condition guarantees the existence of a perfect matching in A_h. □

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5 Appendix

5.1 Proof of Lemma 1

Properties $(\mathbf{I})(a)$ **and** (b) follow easily from the Chernoff bound.

$(\mathbf{II})(a)$**:** Note that $B(Y, V_0)$ is a bipartite random graph with vertex classes Y and $V_0 \setminus Y$ and edge probability p. We will establishing Property $(\mathbf{II})(a)$ by counting how many vertices of $V_0 \setminus Y$ are not isolated in $B(Y, V_0)$.

For each $v \in V_0 \setminus Y$, let $\mathbb{I}_v$ denote the indicator random variable of the event $\deg_B(v) \geq 1$. Set $y = |Y|$ and notice that $\mathbb{I}_v$ is a Bernoulli random variable with expectation

$$q = 1 - (1-p)^y = (1 \pm \delta)\, yp,$$

where the last equation follows from the inequality $1 + x \leq e^x$, provided $p = o(1)$. Also notice that the variables $\{\mathbb{I}_v \ : \ v \in V_0 \setminus Y\}$ are mutually independent. Therefore the distribution of

$$\mathbb{X} := \left| \left\{ v \in V_0 \setminus Y \ : \ \deg_B(v) \geq 1 \right\} \right|$$

is binomial with parameters $|V_0 \setminus Y| = (1 + o(1))|V_0|$ and q. The expectation of $\mathbb{X}$ is therefore

$$(1 + o(1))(1 \pm \delta) y |V_0| p.$$

By the Chernoff bound, we thus have $\mathbb{X} = (1 \pm 2\delta) y |V_0| p$ with probability at least

$$1 - \exp\{-cnyp\}$$

for some $c = c(\delta) > 0$ (recall that $|V_0| \geq \frac{3}{4}n$).

On the other hand, the number of choices of the set Y is less than n^y. Consequently, the probability Property $(\mathbf{II})$(a) fails for $G_{n,p}$ is at most

$$\sum_{y \geq 1} n^y \exp\{-cnyp\} = o(1)$$

because cnp is of a much bigger order than $\log n$.

$(\mathbf{II})(b)$**:** Here we are just counting the edges of the bipartite graph $B(Y, U)$ defined above. Setting, $y = |Y|$ and $u = |U|$, the expected number of edges in B is yup. Hence, again by the Chernoff bound, the probability that Property $(\mathbf{II})(b)$ fails for $G_{n,p}$ is at most

$$\sum_{y} \sum_{u} n^{y+u} \exp\{-cyup\} = o(1)$$

for $C > 0$ large enough, because $yp \geq \omega$ and $up \geq \omega$.

(III)(a) **and** (b)**:** These proofs go mutatis mutandis along the lines of the proofs of **(II)**(a) and **(II)**(b), respectively. The only differences are that the edge probability in B is p^d and, in part (a), the set U, besides V_0, could also be equal to V as well as to one of the sets R_i of size εn. Therefore, the Chernoff constant $c = c(\delta, \varepsilon)$ depends also on ε. Note that for C large enough, $cnp^d = cC^d \log n$ is still sufficiently bigger than $\log n$.

(IV): Let $\mathbb{X} := \mathbb{X}(d, m, p)$ be a random variable counting the number of copies of K_d in $G_{m,p}$ for some $m \leq n$ and $p = p(m)$. Let $\delta > 0$ be a fixed small constant. From the results of [13] and [15, Corollary 1.7], it follows that

$$\mathbf{P}\left[|\mathbb{X} - \mathbf{E}\mathbb{X}| \geq \delta\, \mathbf{E}\mathbb{X}\right] \leq \exp\left\{-c(\delta, d)\, m^2 p^{d-1}\right\}, \tag{22}$$

provided $p \geq m^{-2/(d-1)}$.

(a)**:** For $v \in V$, expose the random neighborhood $G(v)$. Let us condition on $|G(v)| \leq 1.01pn$ (which is an event occurring with probability at least $1 - e^{-\Theta(pn)}$). For any $U \subset G(v)$, $m = |U| \geq pn/3$, the graph $G[U]$ is an instance of $G_{m,p}$. In particular, the assumption on p is satisfied and the bound (22) applies to the random variable $\mathbb{X} = \binom{U}{K_d}$. Moreover, there are fewer than $n\, 2^{1.01pn} < e^{2pn}$ choices for v and the set $U \subset G(v)$. In view of (22) and the fact that $pn = o(m^2 p^{d-1})$, the union bound yields that with probability

$$1 - e^{-\Theta(pn)} - e^{2pn} \exp\left\{-c(\delta, d)\, m^2 p^{d-1}\right\} = 1 - o(1)$$

the equation (17) holds for all $v \in V$ and all $U \subset G(v)$, $m = |U| \geq pn/3$.

(b)**:** For distinct $u, v \in V$, expose the random common neighborhood $U = G(u) \cap G(v) \subset V$. Since **a.a.s.** $|U| = (1 + o(1))p^2 n$, we condition on $m = |U| > 0.99p^2 n$. As $p \geq m^{-2/(d-1)}$, we apply (22) to the random variable $\mathbb{X} = \binom{U}{K_d}$. It follows by the union bound that for all choices of distinct u, v, the set $U = G(u) \cap G(v)$ satisfies (17).

(c)**:** This can be established by the union bound over all large subsets $U \subset V$ using the exponential bound given by (22).

(V): By **(I)**(a), **a.a.s.** every $v \in V$ is such that $|G(v) \cap V_0| = (1 + o(1))p\,|V_0|$. Hence, applying **(IV)**(a) to $U = G(v) \cap V_0$ with $d-1$ in place of d yields

$$\begin{aligned}\binom{G(v) \cap V_0}{K_{d-1}} &= (1 + o(1)) p^{\binom{d-1}{2}} \binom{(1 + o(1))p\,|V_0|}{d-1} \\ &= (1 + o(1)) \frac{d}{|V_0|} \binom{V_0}{K_d},\end{aligned}$$

where for the last equality we used **(IV)**(c) applied to $U = V_0$. $\square$

5.2 Proof of Lemma 3

Since X_i is 2-independent, the neighborhoods $H(x)$ are disjoint for all $x \in X_i$. Let $k_x = |f_{i-1}(H(x))|$. To unify our approach, for each x with $k_x < d$ we find a set $D_x \subset \mathrm{im}(f_{i-1})$ such that $f_{i-1}(H(x)) \subset D_x$ and all D_x are pairwise disjoint. Define a subgraph $A_i^* \subset A_i$ by replacing $H(x)$ with D_x in (8), that is

$$A_i^* = \big\{(x,v) \;:\; f_{i-1}\big(D_x\big) \subset G(v)\big\}. \tag{23}$$

Clearly, for every $Q \subset X_i$ we have $|A_i(Q)| \ge |A_i^*(Q)|$, and so, it suffices to prove (18) for A_i^*. For the ease of notation we will write A_i instead of A_i^*.

The proof is split into two cases according to whether Q is small ($|Q| \le \omega p^{-d}$) or large ($|Q| > \omega p^{-d}$). First consider the case when Q is small, and let $Q' \subset Q$ be an arbitrary subset with

$$|Q'| = \min\big\{\delta p^{-d}, |Q|\big\} \ge \frac{\delta |Q|}{\omega}. \tag{24}$$

Notice that

$$|A_i(Q')| \ge |A_i(Q') \cap R_i| = \Big|\big\{w \in R_i \;:\; G(w) \supset f_{i-1}(D_x) \text{ for some } x \in Q'\big\}\Big|. \tag{25}$$

Applying Property (**III**)(a) to $\mathcal{X} = \{f_{i-1}(D_x) \;:\; x \in Q'\}$ and $U = R_i$ yields that the cardinality of the last set in the above inequality is at least $(1-2\delta)p^d|R_i|\,|Q'|$. In particular, for C large enough, we have

$$|A_i(Q)| \ge |A_i(Q')| \overset{(4)}{\ge} (1-2\delta)\varepsilon p^d n\,|Q'| \ge \delta^{-1}\omega\,|Q'| \ge |Q|.$$

Consequently, (18) holds when Q is small.

When Q is large, that is, $|Q| > \omega p^{-d}$ set $U = W_i \setminus A_i(Q)$ and suppose that $|U| \ge \omega p^{-d}$. Then, by Property (**III**)(b), there is an edge in A_i between Q and U, a contradiction. Thus $|U| < \omega p^{-d}$ which establishes (18). □

5.3 Proof of Lemma 4

Our goal is to prove that **a.a.s.** the random embedding f_0 satisfies (19) for all $Y \subset V$ with $|Y| \le \delta(4p)^{-d}$. Recall that the images $f_0(S_i)$ are created by randomly selecting from V_0 pairwise disjoint d-sets $\kappa_1, \dots, \kappa_t$, each inducing a clique in G, and then define f_0 in any way that satisfies

$f_0(S_i) = \kappa_i$ for all i. Let Ω be the space of all possible sequences $\boldsymbol{\kappa} = (\kappa_1, \ldots, \kappa_t)$. A sequence $\boldsymbol{\kappa}$ is sampled from Ω by first selecting a d-set κ_1 uniformly from $\binom{V_0}{K_d}$, and selecting each subsequent κ_i, $i = 2, \ldots, t$, uniformly from

$$\binom{V_0 \setminus \bigcup_{j=1}^{i-1} \kappa_i}{K_d}.$$

Fix

$$m \le \delta(4p)^{-d} \tag{26}$$

and set

$$\alpha = p^{\binom{d}{2}} \quad \text{and} \quad M = m\alpha \binom{pn/4}{d}. \tag{27}$$

From now on we will focus on a fixed set

$$Y \subset V \text{ with } |Y| = m, \tag{28}$$

and define a random variable $\mathbb{C}_Y = |A_1(Y) \cap X_{d^2+2}|$. Observe that

$$\begin{aligned} \mathbb{C}_Y &\overset{(8)}{=} \left|\left\{x_i \,:\, f_0(H(x_i)) \subset G(y) \text{ for some } y \in Y\right\}\right| \\ &= \left|\{i \in [t] \,:\, \kappa_i \subset G(y) \text{ for some } y \in Y\}\right|. \end{aligned} \tag{29}$$

We will ultimately show that in the random model described above, the inequality

$$\mathbb{C}_Y \ge \frac{1}{2}\left(\frac{p}{5}\right)^d tm \tag{30}$$

fails with such a small probability that the union bound can be applied over all possible choices for Y still yielding a failure probability $o(1)$. Consequently, **a.a.s.** (19) will hold for all choices of Y and thus Lemma 4 will follow.

In view of (30), we are interested in estimating how many d-sets κ_i are contained in the neighborhood $G(y)$ of some $y \in Y$. To this end, we introduce two families of d-cliques. For each $i = 1, \ldots, t$, given disjoint d-cliques $\kappa_1, \ldots, \kappa_{i-1}$, define

$$\mathcal{F}_Y(\kappa_1, \ldots, \kappa_{i-1}) = \bigcup_{y \in Y} \binom{(G(y) \cap V_0) \setminus \bigcup_{j=1}^{i-1} \kappa_j}{K_d}. \tag{31}$$

Note that

$$\mathbb{C}_Y = \sum_{i=1}^{t} \mathbf{1}[\kappa_i \in \mathcal{F}_Y(\kappa_1, \ldots, \kappa_{i-1})].$$

Now, define a family $\mathcal{A}_Y(\kappa_1, \ldots, \kappa_{i-1})$ as follows. Let

$$N = |\mathcal{F}_Y(\kappa_1, \ldots, \kappa_{i-1})|.$$

If $N \geq M$ then set $\mathcal{A}_Y(\kappa_1, \ldots, \kappa_{i-1}) = \mathcal{F}_Y(\kappa_1, \ldots, \kappa_{i-1})$, otherwise let $\mathcal{A}_Y(\kappa_1, \ldots, \kappa_{i-1})$ be an enlargement of the family $\mathcal{F}_Y(\kappa_1, \ldots, \kappa_{i-1})$ with exactly M elements, each being a vertex set of a copy of K_d—for concreteness we can select the $M - N$ lexicographically smallest[4] elements from

$$\binom{V_0 \setminus \bigcup_{j=1}^{i-1} \kappa_i}{K_d} \setminus \mathcal{F}_Y(\kappa_1, \ldots, \kappa_{i-1}).$$

Observe that because of Property (**IV**)(c), our choice of t in (5), and by (26) and (27),

$$\begin{aligned} \left|\binom{V_0 \setminus \bigcup_{j=1}^{i-1} \kappa_i}{K_d}\right| &\geq (1-\delta)\alpha\binom{\left|V_0 \setminus \bigcup_{j=1}^{i-1} \kappa_i\right|}{d} \;>\; \alpha\binom{n/2}{d} > \alpha\frac{(n/4)^d}{d!} \\ &= \alpha p^{-d}\frac{(pn/4)^d}{d!} \overset{(26)}{>} \delta^{-1}\alpha 4^d m\binom{pn/4}{d} \overset{(27)}{>} M. \end{aligned} \tag{32}$$

Consequently, we can always construct a family $\mathcal{A}_Y(\kappa_1, \ldots, \kappa_{i-1})$.

Unlike the families $\mathcal{F}_Y(\kappa_1, \ldots, \kappa_{i-1})$, the families $\mathcal{A}_Y(\kappa_1, \ldots, \kappa_{i-1})$ have a uniform lower bound of M for their cardinalities. Thus, we are in position to use Proposition 1 in our analysis of $\mathcal{A}_Y(\kappa_1, \ldots, \kappa_{i-1})$. In Claim 5.3 below we will show that **a.a.s.** $\mathcal{A}_Y(\kappa_1, \ldots, \kappa_{i-1}) = \mathcal{F}_Y(\kappa_1, \ldots, \kappa_{i-1})$ for all Y, which will allow us to apply the conclusions of that analysis to $\mathcal{F}$.

Let

$$\mathbb{A}_i = \mathbf{1}[\kappa_i \in \mathcal{A}_Y(\kappa_1, \ldots, \kappa_{i-1})].$$

Notice that, again by Property (**IV**)(c),

$$\mathbf{P}[\mathbb{A}_i = 1 \mid \kappa_1, \ldots, \kappa_{i-1}] = \frac{\mathcal{A}_Y(\kappa_1, \ldots, \kappa_{i-1})}{\binom{V_0 \setminus \bigcup_{j=1}^{i-1} \kappa_i}{K_d}} \geq \frac{M}{(1+\delta)\alpha\binom{n}{d}} > m\left(\frac{p}{5}\right)^d.$$

Since a sequence of d-sets $(\kappa_1, \ldots, \kappa_{i-1})$ determines the values of $\mathbb{A}_1, \ldots, \mathbb{A}_{i-1}$, the above inequality implies that for all $i = 1, \ldots, t$,

$$\mathbf{P}[\mathbb{A}_i = 1 \mid \mathbb{A}_1, \mathbb{A}_2, \ldots, \mathbb{A}_{i-1}] \geq q_1, \quad \text{where} \quad q_1 = q_1(m) := m\left(\frac{p}{5}\right)^d. \tag{33}$$

[4] Recall that the vertex set $V = [n]$ has a natural linear order.

Let $\mathbb{A} = \sum_{i=1}^t \mathbb{A}_i$. Proposition 1(a) from Section 5.4 implies that

$$\mathbf{P}[\mathbb{A} \le tq_1/2] \le \exp\{-c_1 t q_1\} \tag{34}$$

for some constant $c_1 > 0$.

Define $Z = G(Y) \cap V_0$ and let $s = |Z|$. Note that κ_i might intersect Z and not be contained in $\mathcal{A}_Y(\kappa_1, \ldots, \kappa_{i-1})$. Since the κ_i's must be disjoint, this might effectively reduce the number of choices for κ_{i+1} in $\mathcal{A}_Y(\kappa_1, \ldots, \kappa_i)$.

To deal with this potential difficulty we introduce another random variable $\mathbb{B}$ to keep track of how many vertices of Z are "consumed" by the sequence $\boldsymbol{\kappa}$. Let

$$\mathbb{B}_i = \mathbf{1}[\kappa_i \cap Z \ne \emptyset]$$

and

$$\mathbb{B} = \sum_{i=1}^t \mathbb{B}_i.$$

By Property (**V**) it follows that

$$\mathbf{P}[\mathbb{B}_i = 1 \mid \mathbb{B}_1, \ldots, \mathbb{B}_{i-1}] \le (1+\delta)\frac{ds}{|V_0|}\left|\binom{V_0}{K_d}\right| \cdot \left|\binom{V_0 \setminus \bigcup_{j=1}^{i-1} \kappa_i}{K_d}\right|^{-1}. \tag{35}$$

Note that

$$|V_0 \setminus \bigcup_{j=1}^{i-1} \kappa_i| = |V_0| - (i-1)d > |V_0| - td.$$

By our choice of τ, using the Bernoulli inequality, we may ensure that

$$\left(1 - \frac{td}{|V_0|}\right)^d \ge 1 - \frac{8}{3d^2} \ge \frac{19}{27}$$

as $d \ge 3$. Thus, applying Property (**IV**)(c) to both, V_0 and $V_0 \setminus \bigcup_{j=1}^{i-1} \kappa_i$, we conclude that the R-H-S of (35) is at most

$$\frac{(1+\delta)ds|V_0|^d}{|V_0|(|V_0| - td)^d} = (1+\delta)\frac{ds}{|V_0|}\left(1 - \frac{td}{|V_0|}\right)^{-d} \le (1+\delta)\frac{4ds}{3n}\frac{27}{19} < \frac{2ds}{n} := q_2,$$

for δ small enough. Consequently, Proposition 1(b) from Section 5.4 implies that

$$\mathbf{P}[\mathbb{B} > 3dst/n] \le \exp\{-c_2 dst/n\} \tag{36}$$

for a constant $c_2 > 0$. By (34) and (36)

$$\mathbf{P}[\mathbb{A} \leq tq_1/2 \text{ or } \mathbb{B} > 3dst/n] \leq \exp\{-c_1 tq_1\} + \exp\{-c_2 dst/n\}. \qquad (37)$$

As it follows from the next (deterministic) claim, the second term on the R-H-S of (37) is much smaller than the first one.

Claim.

$$q_1 n = o(s)$$

Proof. First consider the case when $m = |Y| \leq \omega p^{-1}$. Let $Y' \subset Y$ be an arbitrary set of size $|Y'| = \min\{m, \delta/p\}$. Observe that $|Y'| > \frac{\delta m}{\omega}$. By Property $(\mathbf{II})(a)$ applied to Y' we have

$$\begin{aligned} s \ &\geq \ |G(Y') \cap V_0| \geq (1-2\delta)p\,|V_0|\,|Y'| \\ &> \frac{pn\,|Y'|}{2} \geq \frac{\delta pnm}{2\omega} \gg \left(\frac{p}{5}\right)^d mn = q_1 n. \end{aligned}$$

Hence, if $m \leq \omega p^{-1}$, it follows that $q_1 n = o(s)$.

Now suppose that $m = |Y| \geq \omega p^{-1}$ and let $U = V_0 \setminus (G(Y) \cup Y)$. As $B(Y, U) = \emptyset$, in order not to contradict Property $(\mathbf{II})(b)$, we must have $|U| < \omega p^{-1} = o(n)$. Since $|U| \geq |V_0| - |Z| - |Y|$, by (26),

$$s = |Z| \geq |V_0| - m - \omega p^{-1} = \Theta(n),$$

for C large enough. On the other hand, $q_1 = m(p/5)^d \leq (5^d\omega)^{-1} = o(1)$. Hence, again, $q_1 n = o(s)$. $\square$

As a consequence of the above claim and (37),

$$\mathbf{P}[\mathbb{A} \leq tq_1/2 \text{ or } \mathbb{B} > 3dst/n] \leq 2\exp\{-c_1 tq_1\}. \qquad (38)$$

Our next deterministic claim shows that when $\mathbb{B}$ is small, the families $\mathcal{A}_Y(\kappa_1, \ldots, \kappa_{i-1})$ and $\mathcal{F}_Y(\kappa_1, \ldots, \kappa_{i-1})$ coincide.

Claim. Suppose that $\boldsymbol{\kappa} = (\kappa_1, \ldots, \kappa_t) \in \Omega$ is such that $\mathbb{B} = \mathbb{B}(\boldsymbol{\kappa}) \leq 3dst/n$. Then

$$\mathcal{A}_Y(\kappa_1, \ldots, \kappa_{i-1}) = \mathcal{F}_Y(\kappa_1, \ldots, \kappa_{i-1}) \text{ for all } i = 0, 1, \ldots, t. \qquad (39)$$

In particular, there are $\mathbb{A} = \mathbb{A}(\boldsymbol{\kappa})$ indices $i_1, \ldots, i_{\mathbb{A}} \in [t]$ such that for each $j = 1, \ldots, \mathbb{A}$, we have $\kappa_{i_j} \subset G(y_j)$ for some $y_j \in Y$.

Proof. Let $W = Z \cap \bigcup_{i=1}^{t} \kappa_i$ and observe that $|W| \le td$ as well as

$$|W| \le \mathbb{B}d \le 3d^2 \frac{st}{n}. \tag{40}$$

Let

$$Y' = \{y \in Y \ : \ |G(y) \cap W| \ge pn/3\}.$$

We will now prove that

$$|Y'| \le \frac{4}{pn}|W|. \tag{41}$$

Let $\tilde{Y} \subset Y'$ be an arbitrary set with

$$|\tilde{Y}| = \min\{|Y'|, \delta/p\}. \tag{42}$$

Set

$$T = \{w \in W \ : \ |G(w) \cap \tilde{Y}| \ge 2\}$$

and observe that by Properties (**II**)(a) and (b), and the definition of T,

$$\begin{aligned}(1-2\delta)p\,|\tilde{Y}|\,|V_0| &\le |G(\tilde{Y}) \cap V_0| \le |T| + e(\tilde{Y}, V_0 \setminus T) \\ &\le e(\tilde{Y}, V_0) - e(\tilde{Y}, T) + |T| \\ &\le (1+2\delta)p\,|\tilde{Y}|\,|V_0| - e(\tilde{Y}, T)/2.\end{aligned} \tag{43}$$

It follows that $e(\tilde{Y}, T) \le 4\delta pn\,|\tilde{Y}|$. Since every vertex $v \in \tilde{Y}$ has at least $pn/3$ neighbors in W,

$$|W| \ge |\tilde{Y}|\,pn\Big(\frac{1}{3} - 4\delta\Big) \ge \frac{pn}{4}|\tilde{Y}| \tag{44}$$

and consequently, due to our choice of τ,

$$|\tilde{Y}| \le \frac{4}{pn}|W| \le \frac{4}{pn}td < \delta/p.$$

From the definition of $\tilde{Y}$ (see (42)) we thus have $\tilde{Y} = Y'$, and (41) follows immediately from (44).

By (41) and (40),

$$|Y'| \overset{(41)}{\le} \frac{4}{pn}|W| \overset{(40)}{\le} \frac{4}{pn} \times d^3 \frac{st}{n} = \frac{s}{2pn} \times (8d^3\tau).$$

On the other hand, by Property (**II**)(a), for every $y \in Y'$

$$|G(y) \cap V_0| \le (1+2\delta)p\,|V_0| < 2pn$$

and thus

$$s = |G(Y) \cap V_0| \leq 2pn\, m.$$

Consequently,

$$|Y'| \leq 8d^3 \tau m. \tag{45}$$

We are now ready to conclude the proof of Claim 5.3. Recall that $m = |Y| \leq \delta(4p)^{-d}$, $|V_0| \geq \frac{3}{4}n$, and $\alpha = p^{\binom{d}{2}}$. By (31) and the Bonferroni inequality, for every i,

$$\begin{aligned} N := |\mathcal{F}_Y(\kappa_1, \ldots, \kappa_{i-1})| &\geq \left| \bigcup_{y \in Y} \binom{(G(y) \cap V_0) \setminus W}{K_d} \right| \\ &\geq \sum_{y \in Y} \left| \binom{(G(y) \cap V_0) \setminus W}{K_d} \right| - \sum_{y \neq y' \in Y} \left| \binom{(G(y) \cap G(y') \cap V_0) \setminus W}{K_d} \right| \\ &\geq \sum_{y \in Y \setminus Y'} \left| \binom{(G(y) \cap V_0) \setminus W}{K_d} \right| - \sum_{y \neq y' \in Y} \left| \binom{G(y) \cap G(y') \cap V_0}{K_d} \right|. \end{aligned}$$

For $y \in Y \setminus Y'$, Property $(\mathbf{I})(a)$ yields that

$$|(G(y) \cap V_0) \setminus W| = |G(y) \cap V_0| - |G(y) \cap W| \geq (1 + o(1))p\,|V_0| - pn/3 > pn/3.$$

while for $y \neq y' \in Y$, Property $(\mathbf{I})(b)$ yields that

$$|G(y) \cap G(y') \cap V_0| = (1 + o(1)p^2 |V_0| < p^2 n.$$

Moreover, by (45) and (5),

$$|Y \setminus Y'| \geq (1 - 8d^3\tau)m \geq \frac{1}{2}m.$$

Consequently, by Properties $(\mathbf{IV})(a)$ and (b), and by (26), we obtain

$$\begin{aligned} N &\geq (1 + o(1)) \left\{ \frac{m}{2} \alpha \binom{pn/3}{d} - \binom{m}{2} \alpha \binom{p^2 n}{d} \right\} \\ &\geq (1 + o(1)) \frac{m\alpha}{2d!} \left\{ (pn/3)^d - (mp^d)(pn)^d \right\} \\ &> m\alpha \binom{pn/4}{d} = M. \end{aligned} \tag{46}$$

It follows from the definition of $\mathcal{A}_Y(\kappa_1, \ldots, \kappa_i)$ and (46) that (39) holds. Thus, by definition, there are $\mathbb{A}$ indices $i_1, \ldots, i_{\mathbb{A}}$ such that $\kappa_{i_j} \in$

$\mathcal{A}_Y(\kappa_1, \ldots, \kappa_{i_j-1})$ for all $j = 1, \ldots, \mathbb{A}$. Because of (39) and (31), for each $j = 1, \ldots, \mathbb{A}$, we have

$$\kappa_{i_j} \in \mathcal{F}_Y(\kappa_1, \ldots, \kappa_{i_j-1}) \subset \bigcup_{y \in Y} \binom{G(y)}{K_d}.$$

Hence, $\kappa_{i_j} \subset G(y_j)$ for some $y_j \in Y$. Therefore the claim is proved. $\square$

In view of (38), with probability at least $1 - 2\exp\{-c_1 t q_1\}$, we have $\mathbb{A} \geq tq/2$ and $\mathbb{B} \leq 3dst/n$. Hence, by our last claim, with such a probability, the number of sets $\kappa_i = f(S_i)$ contained in some neighborhood $G(y)$, $y \in Y$, is

$$\mathbb{A} \geq \frac{tq_1}{2} \overset{(33)}{=} \frac{tm}{2}\left(\frac{p}{5}\right)^d.$$

In other words, with probability at least $1 - 2\exp\{-c_1 t q_1\}$ the random embedding f_0 satisfies (30) for a fixed set Y (see (28)). We will now finish the proof of Lemma 4 by using the union bound.

For a fixed $m \leq \delta(4p)^{-d}$, the probability that there is some $Y \subset V$, $|Y| = m$, which fails to satisfy (19) is at most

$$\binom{n}{m} 2\exp\{-c_3 tmp^d\} \leq 2\exp\{m(\log n - c_3 \varepsilon C^d)\} \leq 2n^{-2}, \tag{47}$$

for C large enough, where $c_3 = c_1/3^d$. Hence, the probability that (30) fails for some Y with $|Y| = m \leq \delta(4p)^{-d} = o(n)$ is $o(n \times n^{-2}) = o(1)$. This completes the proof of Lemma 4.

5.4 A probabilistic lemma

Here we prove an auxiliary probabilistic lemma needed for the proof of Lemma 4.

Proposition 1. *Let $\mathbb{X}_1, \ldots, \mathbb{X}_t$ be a sequence of Bernoulli random variables, $\mathbb{X} = \sum_{i=1}^t \mathbb{X}_i$, $\delta > 0$, and $q_1, q_2 \in [0, 1]$.*

(a) *If for all $i = 1, \ldots, t$*

$$\boldsymbol{P}[\mathbb{X}_i = 1 \mid \mathbb{X}_1, \ldots, \mathbb{X}_{i-1}] \geq q_1,$$

then

$$\boldsymbol{P}[\mathbb{X} \leq (1 - \delta)tq] \leq \exp\{-c_1 t q_1\} \tag{48}$$

for some $c_1 = c_1(\delta) > 0$.

(b) *If for all* $i = 1, \dots, t$

$$\boldsymbol{P}[\mathbb{X}_i = 1 \mid \mathbb{X}_1, \dots, \mathbb{X}_{i-1}] \le q_2,$$

then

$$\boldsymbol{P}[\mathbb{X} \ge (1+\delta)tq] \le \exp\{-c_2 t q_2\} \tag{49}$$

for some $c_2 = c_2(\delta) > 0$.

Proof. Let $\mathbb{Y}_1, \dots, \mathbb{Y}_t$ be i.i.d. random variables with uniform distribution over the unit interval $[0,1]$. In addition, we assume that $\mathbb{Y}_1, \dots, \mathbb{Y}_t$ are jointly independent of $\mathbb{X}_1, \dots, \mathbb{X}_t$. For any $2 \le k \le t$ and $(b_1, \dots, b_{k-1}) \in \{0,1\}^{k-1}$, denote

$$p(b_1, \dots, b_{k-1}) = \mathbf{P}[\mathbb{X}_k = 1 \mid \mathbb{X}_1 = b_1, \dots, \mathbb{X}_{k-1} = b_{k-1}].$$

(For completeness, define $p(\lambda) = \mathbf{P}[\mathbb{X}_1 = 1]$, where λ is the empty string.) Notice that $p(\cdot) \ge q_1$ by assumption (a) and $p(\cdot) \le q_2$ by assumption (b).

To prove part (a), for each $1 \le k \le t$ set

$$\mathbb{Z}_k = \mathbf{1}[\mathbb{X}_k = 1] \cdot \mathbf{1}[\mathbb{Y}_k \le q_1/p(\mathbb{X}_1, \dots, \mathbb{X}_{k-1})].$$

It is clear that $\mathbb{Z}_k \le \mathbb{X}_k$ for every k. Moreover, we claim that

$$\mathbf{P}[\mathbb{Z}_k = 1 \mid \mathbb{Z}_1, \dots, \mathbb{Z}_{k-1}] = q_1. \tag{50}$$

Indeed, since $\mathbb{Z}_1, \dots, \mathbb{Z}_{k-1}$ are determined by $\{\mathbb{X}_i, \mathbb{Y}_i \ :\ 1 \le i \le k-1\}$ and $\mathbb{Y}_1, \dots, \mathbb{Y}_t$ are independent, it is enough to prove that

$$\mathbf{P}[\mathbb{Z}_k = 1 \mid \mathbb{X}_1, \dots, \mathbb{X}_{k-1}] = q_1.$$

For a fixed $b \in \{0,1\}^{k-1}$, denoting $\mathcal{E} = \{\mathbb{X}_1 = b_1, \dots, \mathbb{X}_{k-1} = b_{k-1}\}$, we have

$$\mathbf{P}[\mathbb{Z}_k = 1 \mid \mathcal{E}] = p(b_1, \dots, b_{k-1}) \cdot \mathbf{P}[\mathbb{Y}_k \le q_1/p(b_1, \dots, b_{k-1})] = q_1, \tag{51}$$

which proves (50). Note that equation (50) implies that $\mathbb{Z}_1, \dots, \mathbb{Z}_t$ are i.i.d. 0-1 random variables, and consequently, $\mathbb{Z} = \sum_{k=1}^t \mathbb{Z}_k$ is a binomial random variable with parameters t and q_1 and such that $\mathbb{Z} \le \mathbb{X}$. Inequality (48) is derived from the Chernoff bound applied to $\mathbb{Z}$.

To establish part (b) one defines the variables $\mathbb{Z}_k$ by

$$\mathbb{Z}_k = \mathbf{1}\left[\mathbb{X}_k = 1 \text{ or } \mathbb{Y}_k \ge \frac{1-q_2}{1-p(\mathbb{X}_1, \dots, \mathbb{X}_{k-1})}\right]$$

and shows an analog of (50). Consequently, $\mathbb{Z} = \sum_{k=1}^t \mathbb{Z}_k$ is a binomial random variable with parameters t and q_2 and such that $\mathbb{Z} \ge \mathbb{X}$. Inequality (49) follows, again, by the Chernoff bound applied to $\mathbb{Z}$.