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SMALL SUBGRAPHS OF RANDOM GRAPHS - A SURVEY

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1. Introduction

This survey paper is thought to update the stage of knowledge about the limit distribution of the number of small subgraphs of a random graph $K_{n,p}$. It can be viewed as a supplement to Chapter IV of *Random Graphs* by Béla Bollobás [Bo 85].

The model we deal with is a random graph $K_{n,p}$, the finite probabilistic space whose elements are all graphs on vertex set $V_n=\{1,...,n\}$ and the probability assigned to any particular graph with l edges is

$$p^l(1-p)^{\binom{n}{2}-l}\ , \quad 0<p=p(n)<1 .$$

Originally, another random graph model, $K_{n,N}$, was investigated the most. However, the two models happen to be asymptotically (as $n\to\infty$) equivalent and all results presented below can be translated into $K_{n,N}$ under equation $N=\binom{n}{2}p$. (For the best equivalence theorem see [Łu 90]).

We consider here only subgraphs of fixed size, i.e. not depending on n. This restriction could easily be dropped to gain an order o(log$n$) but this would decrease the clarity of the presentation. Another restriction we impose in the paper is that we count subgraphs isomorphic to *one* fixed graph and not to a member of a family of graphs. This allows us to present our results in the purest form.

The history of the subject is a little bit peculiar. It was originated by a result of Erdös and Rényi [ER 60] (Theorem 0 below), which gave rise to the notion of balanced graphs. It took then twenty one years until Bollobás [Bo 81] generalized that result to arbitrary graphs (Threshold Theorem below). Recently, three simpler proofs were found and we present them in Section 3. Two of them as well as Bollobás' proof are based on Theorem 0. Most surprisingly, the third proof (Proof 1 in Section 3) uses exactly the same method of second moment applied by Erdös and Rényi for proving Theorem 0 but makes no reference to balanced graphs.

In Summer 1987 an exponential bound for the probability of nonexistence of a copy of G in $K_{n,p}$ was proven. A trivial consequence of that powerful result is again Bollobás' Threshold Theorem. Here we treat that result briefly since another paper [JŁR 90] in this volume is entirely devoted to it.

Meanwhile, the distributional questions have been posed, attacked, and partially solved on the threshold. This is the content of Section 4. Despite the existence problem, balanced graphs play a crucial role here. We gather their properties in the preliminary Section 2.

Beyond the threshold the normality of the number of copies of an arbitrary graph G in $K_{n,p}$ has been suspected. In Section 5 we fully confirm this prediction.

In the course of the paper we use the asymptotic notation $a_n \asymp b_n$ which means that both $a_n = O(b_n)$ and $b_n = O(a_n)$ hold. For the graph theory notation see Section 2.

2. Density and balance of graphs

The aim of this section is to present a number of structural results on graphs, which will play an important role in our further investigations of the problem of subgraphs of random graphs. They involve only the number of vertices, number of edges and the notion of a subgraph. Given a graph G, $|G|$ and $e(G)$ stand for its number of vertices and edges, respectively. We write $H \subset G$ if H is a subgraph of G. The density of G is defined as $d(G) = e(G)/|G|$, and the maximum subgraph density as $m(G) = \max\{d(H):\ H \subset G\}$. A graph G is balanced if $m(G) = d(G)$. Any subgraph of G with the density equal to $m(G)$ is called extreme. Every extreme subgraph is both balanced and induced. Every component of a balanced graph is extreme. Also, all regular graphs are balanced. A graph G is said to be strictly balanced if G itself is the only extreme subgraph of G. For instance, all connected regular graphs are such ones. Another class of strictly balanced graphs are k-trees and, in particular, trees. For the definition of k-trees see, for example, [Mo 60].

Given two graphs $G_i = (V_i, E_i)$, $i = 1, 2$, we define $G_1 * G_2 = (V_1 * V_2, E_1 * E_2)$, $* = \cup, \cap$. Our first result deals with unions of balanced graphs having the same density. It happens that when taking such unions one never decreases the density. Moreover, it remains unchanged if and only if all pairwise intersections are extreme or empty. In the proof below as well as in the proof of our next result we prefer to deal with differences rather than ratios. Having this in mind, note that every graph function of the form $f(G) = a|G| + be(G)$ is modular, i.e. $f(G_1 \cup G_2) = f(G_1) + f(G_2) - f(G_1 \cap G_2)$. Parts of Theorem 1 below can be found in [Bo 81, KR 83, Ka 84].

Theorem 1. *Let $G_1, \ldots, G_r$ be balanced graphs with density d and let $F_r = \bigcup_{i=1}^{r} G_i$. Then*

(i) $d(F_r) \geqslant d$,

(ii) *$d(F_r) = d$ if and only if, for all $1 \leqslant i < j \leqslant r$, $d(G_i \cap G_j) = d$ or $G_i \cap G_j = \emptyset$.*

Proof. Let $f(H) = d|H| - e(H)$. The thesis is equivalent to

(i) $f(F_r) \leqslant 0$,

(ii) $f(F_r) = 0$ if and only if, for all $1 \leqslant i < j \leqslant r$, $f(G_i \cap G_j) = 0$.

We shall depend heavily on the modularity of f and on the fact that for all $i = 1, \ldots, r$ and for all $H \subset G_i$ $f(H) \geqslant 0$ with equality for extreme subgraphs only. We shall use the induction on r. Since $f(F_2) = -f(G_1 \cap G_2)$, the thesis is true in the case $r = 2$. Assume it is true for $r-1$. To prove (i) note that

$$f(F_r) = f(F_{r-1}) - f(F_{r-1} \cap G_r) \leqslant 0. \tag{1}$$

Now assume that $f(F_r) = 0$ and suppose, to the contrary, that, say, $f(G_1 \cap G_{r-1}) > 0$. By (1), $f(F_{r-1}) = 0$ and we arrive at a contradiction with the induction assumption. Finally, assume that, for all $1 \leqslant i < j \leqslant r$, $f(G_i \cap G_j) = 0$. Then $F_{r-1} \cap G_r$ is the union of $r-1$ graphs of density d, as $F_{r-1} \cap G_r = \bigcup_{i=1}^{r-1} (G_i \cap G_r)$. By the induction assumption $f(F_{r-1} \cap G_r) = 0$ and $f(F_{r-1}) = 0$, so, by (1), $f(F_r) = 0$. □

Theorem 1 has the following consequence. Let $G_1, G_2, \ldots$ be all extreme subgraphs of G. Then $\bigcup_i G_i$ is also extreme and so the maximal (in respect to containment) extreme subgraph of G is well defined. We call it the core of

G and denote by $\bar{G}$. At this point it is worth mentioning that any minimal extreme subgraph of G is strictly balanced.

Our next result, conjectured by Karoński and Ruciński in [KR 82], goes in the opposite direction. Instead of looking for a balanced subgraph of G, we establish the existence of a balanced supergraph of G with the same density as $\bar{G}$. This leads to a "sandwich-type" conclusion that for every graph G there exist graphs H and F, both balanced, such that

$$H \subset G \subset F \quad \text{and} \quad d(H)=m(G)=d(F).$$

Theorem 2 [GRR 85]. *For every graph G there exists a balanced graph F such that $G \subset F$ and $m(G)=d(F)$.*

Proof (*outline*). We outline here the proof from [RV 86]. If $G=\bar{G}$ then there is nothing to prove. Let G^* minimizes $f(H)=m(G)|H|-e(H)$ over all $\bar{G} \subsetneq H \subset G$, and let $x \in V(G^*)-V(\bar{G})$. Furthermore, let B be an auxiliary graph satisfying, with $v=|B|$ and $e=e(B)$,

$$\text{for all} \quad H \subset B, \quad \frac{e(H)}{|H|-1} \leqslant \frac{e}{v-1}, \tag{2}$$

and

$$f(G^*)+m(G)v-e-m=0. \tag{3}$$

The existence of B can be shown by first constructing for all v and e, $1 \leqslant v-1 \leqslant e \leqslant \binom{v}{2}$, a graph satisfying (2) alone, and then by proving that (3) has a solution (v,e) for which the above system of inequalities holds.

Let $G_1=G \cup B$, $V(G \cap B)=\{x\}$. Using (2), it can be easily shown that $m(G_1)=m(G)$. Condition (3) is equivalent to $f(G^* \cup B)=0$. Summarizing, $G \subset G_1$, $m(G)=m(G_1)$ and $|G_1|-|\bar{G}_1|<|G|-|\bar{G}|$, since $\bar{G}_1 \supset G^* \cup B \supsetneq \bar{G}$. Repeating this construction at most $|G|-|\bar{G}|$ times we arrive at the required graph F. □

We conclude this section with some elementary graph enumeration facts. Let $\operatorname{aut}(G)$ denote the number of automorphisms of a graph G. Set $c(G)$ for the number of graphs on vertex set $\{1,...,|G|\}$ which are isomorphic to G. For $H \subset G$, $V(H)=\{1,...,|H|\}$, set $c(H,G)$ for the number of graphs on vertex set $\{1,...,|G|\}$ which contain H and are isomorphic to G.

Observation. a) $c(G)=|G|!/\operatorname{aut}(G)$.

b) *Let $f(H,G)$ be the number of copies of H in G. Then*

$$c(H,G)=f(H,G)(|G|-|H|)!\,\operatorname{aut}(H)/\operatorname{aut}(G).$$

Proof. Part a) follows easily from Burnside's Lemma (see [HP 73]). Part b) is equivalent to the identity

$$f(H,G)\,c(G)=\binom{|G|}{|H|}c(H)\,c(H,G). \qquad \square$$

3. Four roads to the threshold

Let G be a graph and let $X_n(G)$ count the subgraphs of a random graph $K_{n,p}$ isomorphic to G.

Theorem 0 (Erdős and Rényi, 1960). *If G is balanced then*

$$\lim_{n\to\infty} P(K_{n,p} \supset G)=\begin{cases} 0 & \text{if} \quad np^{d(G)}=o(1), \\ 1 & \text{if} \quad np^{d(G)} \to \infty. \end{cases}$$

Threshold Theorem (Bollobás, 1981). *For arbitrary graph G,*

$$\lim_{n\to\infty} P(K_{n,p} \supset G)=\begin{cases} 0 & \text{if} \quad np^{m(G)}=o(1), \\ 1 & \text{if} \quad np^{m(G)} \to \infty. \end{cases}$$

Proof 1. For any extreme subgraph H of G,

$$P(X_n(G)>0) \leqslant P(X_n(H)>0) \leqslant EX_n(H)$$

$$=\binom{n}{|H|}c(H)p^{e(H)} \asymp n^{|H|}p^{e(H)}=(np^{m(G)})^{|H|},$$

which proves the first statement.

Let $G_1, G_2, ..., G_l$, $l=\binom{n}{|G|}c(G)$, be all copies of G in the complete graph on vertex set $\{1,...,n\}$. Define the indicator random variables $I_n^{(i)}$ by

$$I_n^{(i)} = \begin{cases} 1 & \text{if } G_i \subset K_{n,p}, \\ 0 & \text{otherwise} \end{cases}$$

$i=1,2,\dots,l$. Clearly, $I_n^{(i)}$ and $I_n^{(j)}$ are independent if and only if G_i and G_j are edge-disjoint and so

$$\operatorname{var} X_n(G) = \sum_{1\leq i,\, j\leq l}\sum \operatorname{cov}(I_n^{(i)}, I_n^{(j)}) \asymp \sum_{\substack{H\subset G\\ e(H)>0}} n^{2|G|-|H|} p^{2e(G)-e(H)}$$

provided $p \not\to 1$.

From Chebyshev's inequality it follows that for arbitrary random variable X, $P(X=0)\leq \operatorname{var} X/(EX)^2$, provided $EX\neq 0$. Thus

$$P(X_n(G)=0) = O\Big(\sum_{\substack{H\subset G\\ e(H)>0}} n^{-|H|}p^{-e(H)}\Big) = o(1) \text{ provided } np^{m(G)}\to\infty.$$

□

The simplicity of the above proof is striking in comparison with the original proof in [Bo 81]. Another, trivially looking proof, goes as follows.

Proof 2. Let $H\subset G\subset F$, H and F balanced, $d(H)=m(G)=d(F)$. By Theorem 0,

$$P(K_{n,p}\supset G)\leq P(K_{n,p}\supset H)=o(1) \quad \text{if} \quad np^{m(G)}=np^{d(H)}=o(1)$$

and

$$P(K_{n,p}\supset G)\geq P(K_{n,p}\supset F)=1-o(1) \quad \text{if} \quad np^{m(G)}=np^{d(F)}\to\infty.$$

□

The nontrivial part of this proof, however, is contained in the proof of Theorem 2.

Our third approach is closest to Bollobás' proof. It is also based on Theorem 0.

Proof 3. As a matter of fact, we shall prove the following more general statement.

Theorem 3. *For all sequences* $p=p(n)$, $0\leq p(n)\leq 1$,

$$P(K_{n,p}\supset G)-P(K_{n,p}\supset \bar G)=o(1),$$

where $\bar G$ *is the core of* G.

Proof of Theorem 3. Assume $\bar G\neq G$ and let

$$\alpha = \max_{\bar G\subsetneq H\subset G} \frac{e(H)-e(\bar G)}{|H|-|\bar G|}. \quad \text{Since } d(H)<d(\bar G),\ \alpha<m(G).$$

Pick any β, $\alpha<\beta<m(G)$. Let I and J be sets of integers defined by

$$I=\{n: p\leq n^{-1/\beta}\},\quad J=\{n: p\geq (\log\log n/n)^{1/m(G)}\}.$$

Set $\{a_n\}_I$ for the subsequence of a_n determined by I. By Theorem 0 $\{P(K_{n,p}\supset\bar G)\}_I=o(1)$ and so $\{P(K_{n,p}\supset G)\}_I=o(1)$. Set $\bar p=(\log\log n/n)^{1/m(G)}$. Then

$$\{P(K_{n,p}\supset \bar G)\}_J\geq\{P(K_{n,p}\supset G)\}_J\geq\{P(K_{n,\bar p}\supset G)\}_J=1-o(1)$$

by Theorem 0, provided that $P(K_{n,\bar p}\supset G)\sim P(K_{n,\bar p}\supset \bar G)$.

Consequently, without loss of generality one can assume that

$$n^{-1/\beta}\leq p(n)\leq (\log\log n/n)^{1/m(G)}.$$

We split $\{1,\dots,n\}=V_1^{(n)}\cup V_2^{(n)}$, $V_1^{(n)}\cap V_2^{(n)}=\emptyset$, $|V_2^{(n)}|\sim \frac{n}{\log n}$. We have

$$P(K_{n,p}\supset\bar G,\ V(\bar G)\cap V_2^{(n)}\neq\emptyset)=O(n^{|\bar G|}p^{e(\bar G)}/\log n)=o(1)$$

and so,

$$P(K_{n,p}\supset \bar G)-P(K_{n,p}[V_1^{(n)}]\supset\bar G)=o(1).$$

Let $\bar G_1,\dots,\bar G_l$ be the copies of $\bar G$ in the complete graph on vertex set $V_1^{(n)}$. A graph G_0 is called a $V_2^{(n)}$-extension of $\bar G_i$ if G_0 is isomorphic to G, $G_0\supset \bar G_i$ and $V(G_0)-V(\bar G_i)\subset V_2^{(n)}$, $i=1,\dots,l$. Let $\mathscr A_i$ be the event that $\bar G_1,\dots,\bar G_{i-1}\not\subset K_{n,p}$ and $\bar G_i\subset K_{n,p}$, $i=1,\dots,l$. Moreover, let $\mathscr B_i$ be the event that there is at least one $V_2^{(n)}$-extension of $\bar G_i$ in $K_{n,p}$, $i=1,\dots,l$. We have

$$P(K_{n,p}\supset G)=\sum_{i=1}^{l}P(K_{n,p}\supset G/\mathscr A_i)P(\mathscr A_i)\geq\sum_{i=1}^{l}P(\mathscr B_i/\mathscr A_i)P(\mathscr A_i)$$

$$=P(\mathscr B_1/\mathscr A_1)P(K_{n,p}[V_1^{(n)}]\supset\bar G)=P(Y_n>0)P(K_{n,p}[V_1^{(n)}]\supset\bar G),$$

where Y_n is the number of $V_2^{(n)}$-extensions of $\bar{G}_1$ in a random graph $K^*_{n,p}=K_{n,p}[V_2^{(n)}]\cup\bar{G}_1$. Easily,

$$EY_n\asymp(n/\log n)^{|G|-|\bar{G}|}\,p^{e(G)-e(\bar{G})}$$

and

$$\operatorname{var} Y_n\asymp\sum_{\bar{G}\subsetneq H\subset G}(n/\log n)^{2|G|-|\bar{G}|)-(|H|-|\bar{G}|)}\,p^{2(e(G)-e(\bar{G}))-e(H)-e(\bar{G}))},$$

so

$$P(Y_n=0)\leqslant\frac{\operatorname{var} Y_n}{(EY_n)^2}=O\Big(\sum_{\bar{G}\subsetneq H\subset G}(n^{-1}\log n)^{|H|-|\bar{G}|}\,p^{e(\bar{G})-e(H)}\Big)$$

$$=O((\log n)^{|G|}/np^{\alpha})=O((\log n)^{|G|}\,n^{\alpha/\beta-1})=o(1).$$

This completes the proof of Theorem 3. □

For $np^{m(G)}\to c$ the above result was also proved in [BW**].

The last road to the threshold is the most recent one and appeared as a corollary of the following result.

Recall that for $p\not\to 1$,

$$\operatorname{var} X_n(G)\asymp\sum_{\substack{H\subset G\\ e(H)>0}}n^{2|G|-|H|}\,p^{2e(G)-e(H)}.$$

We call H a leading overlap of G if $H\subset G$, $e(H)>0$ and $\operatorname{var} X_n(G)=O(n^{2|G|-|H|}\,p^{2e(G)-e(H)})$.

Theorem 4. ([JŁR 90]) *Let H be a leading overlap of G. Then there exist constants $c_1, c_2>0$ such that*

$$\exp\{-c_1\,EX_n(H)\}\leqslant P(X_n(G)=0)\leqslant\exp\{-c_2\,EX_n(H)\}.$$

Proof. The left-hand inequality follows immediately by the FKG-inequality (see [Bo 86]). Two quite different proofs of the right-hand inequality are given in [JŁR 90]. Here, however, we present yet another proof due to Boppana and Spencer [BS 89].

Let R be a random subset of Ω, $|\Omega|<\infty$, with $P(\omega\in R)=p_\omega$, $\omega\in\Omega$, and the events "$\omega\in R$" mutually independent. Furthermore, let $\Omega_1,\dots,\Omega_l$ be subsets of Ω and let A_i denote the event "$\Omega_i\subset R$", $i=1,\dots,l$.

Obviously $P(\bigcap \bar{A}_i)=\prod_{i=1}^{l}P(\bar{A}_i|\bigcap_{j=1}^{i-1}\bar{A}_j)$. Let us fix i and denote by S the set of all $j<i$ with $\Omega_j\cap\Omega_i\neq\emptyset$. Then

$$P(A_i|\bigcap_{j=1}^{i-1}\bar{A}_j)\geqslant P(A_i)\{1-\sum_{j\in S}P(A_j|A_i\cap\bigcap_{s\notin S}\bar{A}_s)\}$$

$$\geqslant P(A_i)(1-\sum_{j\in S}P(A_j|A_i),$$

the last inequality following from the fact that

$$P(A_i\cap A_j)P(\bigcup_{s\notin S}A_s)\leqslant P(A_i\cap A_j\cap\bigcup_{s\notin S}A_s),$$

an easy consequence of the FKG-inequality. Thus

$$P(\bigcap_{i=1}^{l}\bar{A}_i)\leqslant\exp\{-\sum_{i=1}^{l}P(A_i)+\sum\sum P(A_i\cap A_j)\}$$

or, in terms of indicators $I_i=I(A_i)$ with $X=\sum_{i=1}^{l}I_i$,

$$P(X=0)\leqslant\exp\{-EX+\sum\sum E(I_iI_j)\},\tag{4}$$

where the double summation is taken over all unordered pairs $\{i,j\}$ with $\Omega_i\cap\Omega_j\neq\emptyset$. Let us denote this double sum by M.

The remainder of the proof is not contained in [BS 89] but is based on a personal communication from Spencer (see [Sp**]). Note that (4) is also true for any subset of indices $J\subset[l]$, i.e.

$$\log P(X=0)\leqslant-\sum_J E(I_i)+\sum_J\sum E(I_iI_j).\tag{5}$$

Now let J be a random subset of $[l]$ with $P(i\in J)=\lambda$ and the events "$i\in J$" mutually independent. Taking the expectation of both sides of (5) we get

$$\log P(X=0)\leqslant-\lambda EX+\lambda^2 M$$

which, minimized over λ, gives

$$P(X=0)\leqslant \exp\{-(EX)^2/4M\}$$

provided $EX\leqslant 2M$ (this ensures that $\lambda\leqslant 1$). Otherwise, directly from (4), we have

$$P(X=0)\leqslant \exp\{-0.5EX\}.$$

Applying all the above with $\Omega=[n]^2$, $p_\omega=p=p(n)$, $\Omega_i=$ the edge set of G_i, $I_i=I_n^{(i)}$, $i=1,\dots,l=\binom{n}{v}c(G)$, $v=|G|$, we complete the proof since if $EX\leqslant 2M$,

$$M\asymp \operatorname{var} X_n(G)\asymp (EX_n(G))^2/EX_n(H)$$

and $EX>2M$ means that G is a leading overlap of itself. □

The set of leading overlaps of G keeps changing as $K_{n,p}$ evolves (i.e. as the decay of $p(n)\to 0$ decreases) and there are induced subgraphs of G which never become leading. Whole information can be read out from the structure of G (see [JŁR 90] for details). In particular, if $np^m\to\infty(0)$ arbitrarily slowly then the smallest (largest) extreme subgraph H is leading and, of course, $EX_n(H)\to\infty(0)$. This, once again, implies the Threshold Theorem.

4. On the threshold

Theorem 3 means that the problem of the existence of a copy G in $K_{n,p}$ can be reduced to balanced graphs G only. On the threshold, i.e. when $np^{m(G)}\sim c$, $0<c<\infty$, the same can be concluded even with respect to the limit distribution of $X_n(G)$. Our next result makes this precise. From now on, given $H\subset G$, we define an extension of H_0, a copy of H, as any copy G_0 of G such that $H_0\subset G_0$. Moreover, we denote by $X_n(H_0,G)$ the number of extensions of H_0 in $K_{n,p}\cup H_0$. The following result was suggested by T. Łuczak.

Theorem 5. *For arbitrary graph G and for every $\varepsilon>0$, if $np^{m(G)}\sim c$, $0<c<1$, then* $\lim_{n\to\infty} P(|X_n(G)/EX_n(\bar G_0,G)-X_n(\bar G)|>\varepsilon)=0$.

Proof. Set $Y_n=X_n(\bar G_0,G)$. We have $P(|X_n(G)/EY_n-X_n(\bar G)|>\varepsilon)\leqslant P(X_n(\bar G)>\log n)+\sum_{k=0}^{\log n} P(|X_n(G)/EY_n-X_n(\bar G)|>\varepsilon$ and $X_n(\bar G)=k)\leqslant EX_n(\bar G)/\log n+\sum_{k=0}^{\log n} P$(there is a copy of $\bar G$ in $K_{n,p}$ with the number of extensions Y_n satisfying $|Y_n-EY_n|>\frac{\varepsilon}{k}EY_n)\leqslant o(1)+(\log n)n^{|\bar G|}p^{e|\bar G|}\times P(|Y_n-EY_n|>\frac{\varepsilon}{\log n}EY_n)\leqslant o(1)+O((\log n)^3\operatorname{var} Y_n/(EY_n)^2)=o(1)$.

The second inequality in the above sequel is due to the fact that each copy of G is an extension of exactly one copy of $\bar G$. □

In view of Theorem 5, in what follows, we focus on the case of a balanced graph G. Then all moments of $X_n(G)$ converge to positive constants.

Lemma 1 [RV 85]. *Let $b_r^{(n)}$ be the r-th binomial moment of $X_n(G)$. If G is balanced and $np^{d(G)}\sim c$, $c>0$, then*

$$b_r=\lim_{n\to\infty} b_r^{(n)}=\sum_{t=|G|}^{r|G|}\frac{c^t}{t!}\alpha(G,t),\quad r=1,2,\dots$$

where $\alpha(G,t)$ is the number of unordered r-tuples of distinct copies of G, $\{G_1,\dots,G_r\}$, such that $V(\bigcup_{i=1}^r G_i)=\{1,\dots,t\}$ and $d(\bigcup_{i=1}^r G_i)=d(G)$.

Proof. Let $I_n^{(1)},\dots,I_n^{(l)}$ be the same as in Proof 1 of the Threshold Theorem (see Section 3). Then, by Theorem 1,

$$b_r^{(n)}=\sum_{1\leqslant i_1<\dots<i_r\leqslant l} E(I_n^{(i_1)}\dots I_n^{(i_r)})=\sum_t\sum_s\binom{n}{t}\alpha(G,t,s)p^s$$

$$=(1+o(1))\sum_t\binom{n}{t}\alpha(G,t)p^{td(G)}\sim\sum_t\frac{c^t}{t!}\alpha(G,t),$$

where $\alpha(G,t,s)$ is defined similarly as $\alpha(G,t)$ with an additional assumption that $e(\bigcup_{i=1}^r G_i)=s$. □

It is unfortunate that the sequence b_r grows usually too fast to satisfy the Carleman criterion (see [ChT 78]) and therefore it does not determine uniquely any distribution. However, in two special cases we are successful.

Corollary 1. *Assume* $np^{d(G)} \sim c > 0$.

a) ([Bo 81, KR 83, RV 85]), $X_n(G) \xrightarrow{\mathcal{D}} \mathrm{Po}(c^{|G|}/\mathrm{aut}(G))$ *if and only if G is strictly balanced.*

b) *If a balanced graph G has exactly one extreme subgraph H different from G then for every* $k = 0, 1, \ldots$

$$\lim_{n\to\infty} P(X_n(G) = k) = P\Big(\sum_{i=1}^{Z} Y_i = k\Big), \quad \textit{where}$$

$Z, Y_1, Y_2, \ldots$ *are independent Poisson random variables with*

$$EZ = c^{|H|}/\mathrm{aut}(H) \quad \textit{and} \quad EY_i = c^{|G|-|H|}\,\mathrm{aut}(H)/\mathrm{aut}(G),\ i = 1, 2, \ldots.$$

Proof. a) From Theorem 1, $d(\bigcup G_i) = d(G)$ if and only if $G_i \cap G_j = \emptyset$ for all $i \neq j$. Thus

$$\alpha(G, t) = \begin{cases} \binom{r|G|}{|G|, \ldots, |G|} \frac{1}{r!} (c(G))^r & \text{if } t = r|G|, \\ 0 & \text{otherwise}, \end{cases}$$

and so $b_r = (c^{|G|}/\mathrm{aut}(G))^r / r!$, i.e. b_r is the r-th binomial moment of the distribution $\mathrm{Po}(c^{|G|}/\mathrm{aut}(G))$. On the other hand, if G is not strictly balanced, i.e. G contains a proper extreme subgraph H, then

$$b_2 \geq \frac{1}{2}(c^{|G|}/\mathrm{aut}(G))^2 + \frac{1}{(2|G|-|H|)!} c^{2|G|-|H|},$$

which excludes any Poisson distribution by Corollary 7 from [ChT 78, p. 254].

b) We shall show that $\sum_{r=0}^{\infty} b_r z^r = \exp\{\lambda_0(e^{\lambda_1 z} - 1)\}$, $\lambda_0 = c^{|H|}/\mathrm{aut}(H)$, $\lambda_1 = c^{|G|-|H|}\,\mathrm{aut}(H)/\mathrm{aut}(G)$, which is the generating function of binomial moments of the random variable $\sum_{i=1}^{Z} Y_i$. From Theorem 1, $d(\bigcup_{i=1}^{r} G_i) = d(G)$ if and only if every component of G_i consists of a copy of H with a number of its extensions. If k is the number of components of $\bigcup G_i$ then $V(\bigcup G_i)$ can be split into two disjoint sets, namely $k|H|$ vertices belonging to the copies of H and $r(|G|-|H|)$ vertices in the extensions. Thus

$$b_r = \sum_{k=1}^{r} \frac{c^t}{t!} \binom{t}{k|H|} \binom{k|H|}{|H|, \ldots, |H|} \frac{1}{k!} \binom{r(|G|-|H|)}{|G|-|H|, \ldots, |G|-|H|}$$

$$\times \frac{1}{r!} (c(H))^k S(r, k) k! (c(H, G))^r,$$

where $t = k|H| + r(|G|-|H|)$ and $S(r, k)$ are the Stirling numbers of the second kind. So,

$$b_r = \sum_{k=1}^{r} \lambda_0^k \lambda_1^r S(r, k)/r!, \quad r = 1, 2, \ldots$$

and

$$\sum_{r=0}^{\infty} b_r z^r = \sum_{r=0}^{\infty} \sum_{k=0}^{r} \lambda_0^k (\lambda_1 z)^r S(r, k)/r! = \exp\{\lambda_0(e^{\lambda_1 z} - 1)\}$$

where $b_0 = 1$ and $S(r, 0) = \begin{cases} 1, & r = 0 \\ 0, & r > 0. \end{cases}$

Since $b_r = O(C^r r^r)$ for some $C > 0$, the Carleman criterion is satisfied and the proof is complete. □

Note that part b) of the above result agrees with our intuition. The distribution of Z is the same as the limit distribution of $X_n(H)$ since H is strictly balanced (see part a)). Moreover, the distribution of Y_i coincides with the limit distribution of $X_n(H_0, G)$ where H_0 is any copy of H in the complete graph K_n on vertex set $\{1, \ldots, n\}$. Below we state this observation in a slightly more general form. A special case of it was proved in [RV 86].

Theorem 6. *Let* $H \subset G$ *be such that for all* $H \subsetneq K \subsetneq G$ $[e(K) - e(H)]/[|K| - |H|] < [e(G) - e(H)]/[|G| - |H|]$. *Then*

$$\lim_{n\to\infty} P(X_n(H_0, G) = k) = e^{-\lambda} \lambda^k / k!, \quad k = 0, 1, \ldots,$$

provided $n^{|G|-|H|} p^{e(G)-e(H)} \sim c$, $0 < c < \infty$, *where* $\lambda = c \cdot c(H, G)/(|G| - |H|)!$

Proof. One can easily prove the thesis using the method of moments as we did in Corollary 1a. However, this is the right time to make the reader familiar with a method proposed by Barbour [Ba 82], which requires only checking the first and the second moment. His idea was to estimate the distance between the sequence of random variables in question, X_n, and the sequence of appropriate Poisson distributed random variables, defined as

$$d(X_n)=\sup_{A\subset\{0,1,2,\dots\}}\left|P(X_n\in A)-\sum_{k\in A}e^{-EX_n}(EX_n)^k/k!\right|.$$

Barbour proved that if $\mathscr{A}$ is a family of l-element sets of pairs of elements from $\{1,\dots,n\}$ and the indicators I_α, $\alpha\in\mathscr{A}$, are defined by $I_\alpha=\begin{cases}1 & \text{if } \alpha\subset K_{n,p}\\ 0 & \text{otherwise}\end{cases}$ then, for $X_n=\sum_{\alpha\in\mathscr{A}}I_\alpha$,

$$d(X_n)\leqslant 2p^l+\sum_{\alpha\neq\beta}\ \sum_{\alpha\cap\beta\neq\emptyset}E(I_\alpha I_\beta)/EX_n. \tag{6}$$

Let $G_1, G_2, \dots$ be all extensions of a given copy H_0 of H. Then, with $\mathscr{A}$ being the family of edge-sets $E(G_i)-E(H_0)$, we have

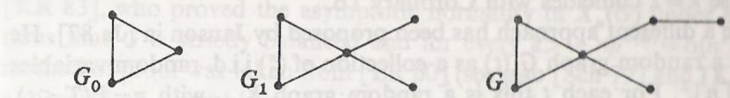
$$d(X_n(H,G))=O\Big(\sum_{H\subsetneq K\subsetneq G}n^{|G|-|K|}p^{e(G)-e(K)}\Big)=o(1),$$

since $[e(G)-e(K)]/(|G|-|K|)>[e(G)-e(H)]/(|G|-|H|)$. □

The above approach could have been used to prove Corollary 1a as well. Note also that if H is an extreme subgraph of a balanced graph G then the assumption of Theorem 6 reduces to the requirement that whenever $H\subset K\subset G$ and K is extreme then either $K=G$ or $K=H$. This is obviously the case in Corollary 1b).

Bollobás and Wierman [BW**] propose a recursive method which allows one, after tedious calculations, to find, in principle, the limit distribution of $X_n(G)$ for arbitrary G. They define a special grading of G, $G_0\subset G_1\subset\dots G_t=G$ which is a refinement of that introduced by Bollobás in [Bo 81]. Here G_0 is the union of all strictly balanced extreme subgraphs of G, G_1 is G_0 plus the union of all minimal subgraphs of $G-E(G_0)$ for which the ratio of edges to vertices not in G_0 is exactly $d(G)$, and so on. A branching conditioning argument gives the asymptotic independence of $X_n(G_0)$ and the numbers of extensions of G_r to G_{r+1}, $r=0,\dots,t-1$ attached at various possible places to the copies of G_r already existing in $K_{n,p}$. From this, it is not too far to the limit distribution of $X_n(G)$.

Let us illustrate the method with a simple example. Let G_0, G_1 and $G_2=G$ be as follows.



Denote by $Y_n(i)$ the number of nontriangular neighbors of the lexicographically i-th triangular vertex of $K_{n,p}$. Similarly, let $Z_n^{(i)}(j)$ be the nontriangular degree of the j-th nontriangular neighbor of the i-th triangular vertex. Then, for each d and l,

$$P(X_n(G)=l)\geqslant\sum_{k=1}^{d}\ \sum_{l_1,\dots,l_{3k}=1}^{d^2}\ \sum^{*}_{t_1^{(i)},\dots,t_{l_i}^{(i)},\,i=1,\dots,3k}P(k,\underline{l},\underline{T}),$$

and

$$P(X_n(G)=l)\leqslant\sum_{k=1}^{d}\sum_{\underline{l}}^{d^2}\sum_{\underline{T}}^{*}P(k,\underline{l},\underline{T})+P(X_n(G_0)>d)+3dP(Y_n(i)>d^2),$$

where $P(k,\underline{l},\underline{T})=P(X_n(G_0)=k, Y_n(i)=l_i, Z_n^{(i)}(j)=t_j^{(i)}, j=1,\dots,l_i, i=1,\dots,3k)$, and the indices under $\sum^*$ satisfy $\sum_{i=1}^{3k}(l_i-1)(t_1^{(i)}+\dots+t_{l_i}^{(i)})=l$. As the authors claim, $P(k,\underline{l},\underline{T})\sim P(X_n(G_0)=k)\prod_i[P(Y_n(i)=l_i)\prod_j P(Z_n^{(i)}(j)=t_j^{(i)}]$ and we are home, since $X_n(G_0)$, $Y_n(i)$, $Z_n^{(i)}(j)$ all converge to appropriate Poisson distributions.

The above procedure can be repeated, provided enough time, for any particular balanced graph G, giving the limit distribution of $X_n(G)$, which strongly depends on the structure of G. There is no hope, however, for any compact general formula. In the simplest case when G consists of m_i components isomorphic to a strictly balanced graph H_i, $i=1,\dots,k$,

$$X_n(G)\xrightarrow{\mathscr{D}}\prod_{i=1}^{k}\binom{Y_i}{m_i},$$

where Y_i are independent Poisson random variables satisfying $X_n(H_i)\xrightarrow{\mathscr{D}}Y_i$. Another special case, opposite to the above, is when the extreme subgraphs of G form an ascending sequence $H_1\subset H_2\subset\dots\subset H_k=G$. Then the method of Bollobás and Wierman might be helpful to confirm our prediction that

$$X_n(H_l) \xrightarrow{\mathcal{D}} W(l) = \sum_{j=1}^{W(l-1)} Y_j^{(l)}, \quad l=2,\dots,k,$$

where $W(1)$, $Y_j^{(l)}$ are appropriate independent Poisson random variables. The case $k=2$ coincides with Corollary 1b.

Quite a different approach has been proposed by Janson in [Ja 87]. He defines a random graph $G_n(t)$ as a collection of $\binom{n}{2}$ i.i.d. random variables T_e, $e\in[n]^2$. For each t this is a random graph $K_{n,p}$ with $p=P(T_e\leqslant t)$. Such a general setting turns each graph characteristics into a random process and thus the advanced theory of Poisson processes can be applied. That enables one, for instance, to get

$$X_n(G) \xrightarrow{\mathcal{D}} \sum_{i=1}^{Z}\sum_{j=1}^{k}\binom{Y_{ij}}{2},$$

where G is a k-cycle with two pendant edges sharing the root, whereas Z, Y_{ij} are appropriate independent Poisson random variables. Janson's approach seems to work for all graphs G consisting of a strictly balanced graph H and an independent set of vertices, each joined to exactly $d(H)$ vertices of H.

As we have seen earlier, our basic Lemma 1 is of not much use for the problem of limit distribution of $X_n(G)$. It brings, however, some information about the limit of $P(X_n(G)>0)$. We conclude this section with a short proof of the fact that on the threshold it can be neither 0 nor 1.

Corollary 2 [RV 85]. *If $np^{m(G)}\sim c>0$ then*

$$0<\liminf_{n\to\infty} P(K_{n,p}\supset G)\leqslant \limsup_{n\to\infty} P(K_{n,p}\supset G)<1.$$

Proof. Either Theorem 2 or 3 allows us to restrict ourselves to balanced graphs G. Let H be a strictly balanced and extreme subgraph of G. Then $P(K_{n,p}\supset G)\leqslant P(K_{n,p}\supset H)\to 1-e^{-\lambda}$, $\lambda=c^{|H|}/\operatorname{aut}(H)$, and the right-hand side is proved. Now assume, to the contrary, that $\lim_{n\to\infty} P(K_{n_m,p}\supset G)=0$ for some n_m. This means that $X_{n_m}(G)\xrightarrow{\mathcal{D}}0$ but this is a contradiction, since $EX^2_{n_m}=O(1)$ and $EX_{n_m}\not\to 0$ (see [ChT 78, p. 254, Cor. 7]). □

5. Beyond the threshold

It is natural to expect that, as $np^{m(G)}\to\infty$,

$$\tilde{X}_n(G)=\frac{X_n(G)-EX_n(G)}{\sqrt{\operatorname{var} V_n(G)}} \xrightarrow{\mathcal{D}} N(0,1) \quad \text{holds.}$$

First general result in this direction is due to Karoński and Ruciński [KR 83], who proved the asymptotic normality of $\tilde{X}_n(G)$ under restrictions that G is strictly balanced and for every $\varepsilon>0$, $np^{d(G)}=o(n^{\varepsilon})$. The method of proof was taken from [ER 60] (see also [Sch 79] and [Ka 82]) and relied on a combinatorial identity relating the moments of $X_n(G)$ to those of the Poisson distribution $\operatorname{Po}(EX_n(G))$. The idea behind this approach is that since the distribution of $X_n(G)$ is close to $\operatorname{Po}(EX_n(G))$ on the threshold, it should be so near the threshold. But then $EX_n(G)\to\infty$ and, in turn, $\operatorname{Po}(EX_n(G))$, after standardization is close to $N(0,1)$. Basically, the same idea was utilized in Barbour's approach [Ba 82]. Recall that a sequence of random variables X_n is Poisson convergent if

$$d(X_n)=\sup_{A\subset\{0,1,\dots\}}\Big|P(X_n\in A)-\sum_{k\in A}e^{-\lambda_n}\lambda_n^k/k!\Big|=o(1), \quad \lambda_n=EX_n.$$

Easily, $d(X_n)=o(1)$ and $\lambda_n\to\infty$ imply that $(X_n-\lambda_n)/\sqrt{\lambda_n}\xrightarrow{\mathcal{D}}N(0,1)$.

The question when $X_n(G)$ is Poisson convergent is answered in our next theorem. It improves some earlier results from [Ba 82] and [Ka 84].

Theorem 7 [Ru 88]. *$X_n(G)$ is Poisson convergent if and only if $np^{d(G)}\to 0$ or $np^{\alpha}\to 0$, where $\alpha=\min\{(e(G)-e(H))/(|G|-|H|)\colon H\subsetneq G\}$ ($\alpha>d(G)$ if and only if G is strictly balanced).*

Proof. (*a sketch*) For every $A\subset\{0,1,\dots\}$,

$$|P(X_n\in A)-P(Y_n\in A)|=O(\lambda_n)=o(1) \quad\text{if}\quad np^{d(G)}\to 0.$$

Also, Barbour's bound (6) from Section 4 is of the same order of magnitude as

$$\sum_{H\subsetneq G} n^{|G|-|H|}p^{e(G)-e(H)}=o(1) \quad\text{if}\quad np^{\alpha}\to 0,$$

and the sufficiency follows. To prove the necessity note that if $n^2q\to c\in[0,\infty)$ then $P(X_n(G)=l)\sim e^{-c/2}\sim P(K(n,p)$ is complete). On the other hand, for all $k=1,2,\dots$ and all $\lambda>0$, $e^{-\lambda}\lambda^k/k!<k^{-1/2}$ and so $\lim_{n\to\infty}P(Y_n=l)=0$. Here $q=1-p$ and $l=\binom{n}{|G|}c(G)$.

The case $n^2q \to \infty$ is more complicated and we restrict ourselves to the case when both $np^{\alpha} \to \infty$ and $np^m \to \infty$. Then $\lambda_n = o(\operatorname{var} X_n(G))$ and, by Theorem 8, $\tilde{X}_n \xrightarrow{\mathscr{D}} N(0,1)$. Hence, it follows from Slutsky's theorem [ChT 78, p. 249] that $(X_n - \lambda_n)/\sqrt{\lambda_n} \xrightarrow{\mathscr{D}} 0$, which contradicts the Poisson convergence. □

Comment. If G is strictly balanced, $X_n(G)$ is Poisson convergent in the same range of $p(n)$ in which G is the only leading overlap of itself, i.e. $\operatorname{var} X_n(G) \sim \lambda_n$. □

At the other end of the range of p, for p constant, the asymptotic normality of $X_n(G)$ was established by Nowicki [No 89]. Then Nowicki and Wierman [NW 88] made an attempt to close the gap but only showed that $\tilde{X}_n(G) \xrightarrow{\mathscr{D}} N(0,1)$ if $np^{e(G)-1} \to \infty$ and $n^2(1-p) \to \infty$, G arbitrary.

In both papers the approach through incomplete U-statistics was applied. Moreover, Nowicki [No 89] and independently Maehara [Ma 87] proved asymptotic normality of $Y_n(G)$, the number of induced copies of G in $K_{n,p}$ for constant $p \neq e(G)/\binom{|G|}{2}$. (As long as $p \to 0$ there is no difference between the asymptotic behavior of $X_n(G)$ and $Y_n(G)$.) The case $p = e(G)/\binom{|G|}{2}$ was settled in [BKR 89] by using the orthogonal projection method. In particular, for some G the limit distribution is then non-normal. Despite Nowicki, Maehara used the old method of moments in a manner he had done earlier in [Ma 80]. That proof was an inspiration for our final result, which brings full solution to the problem of asymptotic normality of $X_n(G)$.

Theorem 8 [Ru 88]. *For arbitrary graph G,*

$$\tilde{X}_n(G) \xrightarrow{\mathscr{D}} N(0,1) \quad \text{iff} \quad np^{m(G)} \to \infty \quad \text{and} \quad n^2 q \to \infty .$$

Moreover if

$$n^2 q \to c \quad \text{then} \quad \tilde{X}_n(G) \xrightarrow{\mathscr{D}} (-\tilde{\mathrm{P}}\mathrm{o}(\tfrac{c}{2})).$$ □

For a detailed proof see [Ru 88]. Here we present the proof only for the simplest case $0 < \lim_{n \to \infty} p(n) < 1$ which, nevertheless, will give the reader a glimpse of the underlying idea of a complete proof.

Set μ_k for the k-th factorial moment of $X_n(G)$. We shall prove that

$$\mu_{2k} \sim \frac{(2k)!}{k!\,2^k}, \quad \mu_{2k+1} = o(\mu_2{}^{k+}1/2), \quad k = 1, 2, \dots,$$

which implies the thesis. We express

$$\mu_k = \sum\nolimits^{(*)} E\{(I_n^{(i_1)} - p^{e(G)}) \dots (I_n^{(i_k)} - p^{e(G)})\} = \sum\nolimits^{(*)} a(i_1, \dots, i_k),$$

where $\sum^{(*)}$ is taken over all $(i_1, \dots, i_k)$, $1 \leq i_1, \dots, i_k \leq l$, such that graphs $G_{i_1}, \dots, G_{i_k}$ satisfy for each $h \in \{1, \dots, k\}$, $e(G_{i_h} \cap \bigcup_{j=1, j \neq h}^{k} G_{i_j}) > 0$.

In the case $p \sim c$, $0 < c < 1$, μ_k is a polynomial in n of degree equal to the maximum number of vertices in $|\bigcup_{j=1}^{k} G_{i_j}|$. For even k, the maximum is achieved only when for each $h \in \{1, \dots, k\}$ there is exactly one $j \neq h$ such that $e(G_{i_h} \cap G_{i_j}) = 1$. Thus

$$\mu_{2k} \sim \binom{2k}{2, \dots, 2} \frac{1}{k!} \mu_2^k .$$

If k is odd, such "perfect matching" is impossible. Hence

$$\mu_{2k+1} = O(n^{(2k+1)|G| - 2(k+1)}) = O(\mu_2^{k+1/2} n^{-1}).$$ □

Recently Theorem 8 was supplied by the rate at which $\tilde{X}_n(G)$ converges to $N(0,1)$. This was made possible by an extensive use of Stein's normal approximation (see [BKR 89]). The result says that the distance between the distribution of $\tilde{X}_n(G)$ and of $N(0,1)$ in a special metric is $O(1/n\sqrt{q})$ if $p > \frac{1}{2}$ and $O((EX_n(H))^{-1})$ if $p \leq \frac{1}{2}$ where H is a leading overlap of G.

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