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Tight Multiple Twins in Permutations

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Abstract. Two permutations are similar if they have the same length and the same relative order. A collection of $r \geqslant 2$ disjoint, similar subsequences of a permutation π forms r-twins in π. We study the longest guaranteed length of r-twins which are tight in the sense that either each twin alone forms a block or their union does. We address the same question with respect to a random permutation.

1. Introduction

By a *permutation* we mean any finite sequence of distinct integers. We say that two permutations $(x_1, \dots, x_k)$ and $(y_1, \dots, y_k)$ are *similar* if their entries preserve the same relative order, that is, $x_i < x_j$ if and only if $y_i < y_j$ for all pairs $\{i, j\}$ with $1 \leqslant i < j \leqslant k$. For instance, $(2, 1, 3)$ is similar to $(5, 4, 8)$.

Large pairs of similar sub-permutations (called *twins*) in a given, or random, permutation have recently attracted some attention (cf. [3,4,8]). Here we are exclusively devoted to twins which appear in blocks.

A *block* in a permutation π is any subsequence of π occupying a non-empty segment of consecutive positions. For instance, the permutation below contains 8 blocks of length 6. Some of them, like the highlighted one, enjoy a property which is of special interest to us: they consist of two similar blocks of length 3:

$$(12, 6, 7, \boxed{2,1,3}, \boxed{5,4,8}, 13, 10, 9, 11).$$

We call such blocks *order repetitions*.

How large are order repetitions guaranteed in every permutation of a given length? Clearly, every permutation of length at least 2 contains trivial

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order repetitions of length 1. Surprisingly, this is all what you get: as proved by Avgustinowich, Kitaev, Pyatkin, and Valyuzhenich in [2], there exist arbitrarily long permutations without order repetitions of any length greater than 1. This result is a permutation analog of the famous theorem of Thue [12] from 1906, establishing the existence of arbitrarily long words over a 3-letter alphabet avoiding word repetitions of any length, even 1 (see [9]). Both results are constructive and provide simple recursive procedures for generating these objects.

In [4] we introduced a stronger avoidance property of permutations, defined as follows. A block in a permutation forms *tight twins* of *length k* if it consists of two similar disjoint *subsequences* of length k each. For instance, in a permutation below there are tight twins of length 3, namely $(2,1,3)$ and $(5,4,8)$, that do not form a repetition:

$$(12, 6, 7, \boxed{2}, \boxed{5}, \boxed{4}, \boxed{1}, \boxed{3}, \boxed{8}, 13, 10, 9, 11).$$

Note that the containment of tight twins, and, in particular, order repetitions, is not monotone in the sense that the absence of tight twins of length k does not exclude the presence of tight twins longer than k. By using the probabilistic method, we proved in [4, Thm. 3.6], that there exist arbitrarily long permutations without tight twins longer than 12. Most likely this constant is not optimal, but it cannot go all the way down to 1, as every permutation of length 6 contains tight twins of length 2 (see [4, Prop. 3.7]).

In this paper we study generalizations of such problems to multiple twins. Let $r \geqslant 2$ be a fixed integer and π be a permutation. We say that a block in π consisting of r similar, pairwise disjoint subsequences, each of length k, forms *tight r-twins* of *length k*. For example, the following permutation contains tight 4-twins of length 3, namely $(2,1,3)$, $(5,4,8)$, $(15,7,17)$, $(12,9,16)$:

$$(14, 18, \boxed{2}, \boxed{5}, \boxed{4}, \boxed{1}, \boxed{15}, \boxed{12}, \boxed{7}, \boxed{17}, \boxed{8}, \boxed{9}, \boxed{16}, \boxed{3}, 6, 10, 11, 13).$$

How large are tight r-twins guaranteed in every permutation of length n? How large are tight r-twins contained, with high probability, in a *random* permutation of the set $[n] = \{1, 2, \ldots, n\}$?

Let $tt^{(r)}(n)$ denote the largest integer k such that every permutation of length n contains tight r-twins of length k. Our result from [4], mentioned above, states that $tt^{(2)}(n) \leqslant 12$. Here we prove (Theorem 4.2) that for every $r \geqslant 3$ we have

$$tt^{(r)}(n) \leqslant 15r.$$

Note that this bound is independent of n. In contrast, we show (Theorem 5.2) that a random permutation of $[n]$ with high probability contains tight r-twins of length $\sim \frac{\log n}{(r-1)\log\log n}$.

We also consider a related function $f(r,k)$ defined as the least n such that every permutation of length n contains tight r-twins of length *exactly* k. It is not hard to see that $f(2,2) = 6$. We also determined, with a little help of computer, that $f(3,2) = 12$. However, for every $r \geqslant 2$ and $k \geqslant 3$, we provide a construction of arbitrarily long permutations avoiding tight r-twins of length

k. In other words, $f(r,k) = \infty$ for all pairs (r,k) with $r \geqslant 2$ and $k \geqslant 3$ (Propositions 4.3 and 4.4). For the remaining cases we only have a quadratic lower bound $f(r,2) \geqslant r(r+5) - 12$, $r \geqslant 3$ (Proposition 4.5).

Another, more relaxed variant of multiple twins can be defined as follows. A family of r pairwise disjoint and similar blocks in a permutation π, each of length k, is called *block r-twins* of *length* k. For example, the following permutation contains block 4-twins of length 3 (highlighted):

$$(14, \boxed{2,1,3}, 18, 6, 10, \boxed{5,4,8}, \boxed{15,7,17}, 11, 13, \boxed{12,9,16}).$$

How large are block r-twins guaranteed in every permutation of length n? How large are block r-twins contained, with high probability, in a *random* permutation of $[n]$? We answer both these questions with asymptotic precision.

Let $bt^{(r)}(n)$ denote the largest integer k such that every permutation of length n contains block r-twins of length k. We prove (Theorem 3.2) that

$$bt^{(r)}(n) = (1 + o(1))\frac{\log n}{\log\log n},$$

where the term $o(1)$ hides a dependence on r. We also demonstrate (Theorem 5.1) that a random permutation of length n with high probability contains block r-twins of length $\sim \frac{r\log n}{(r-1)\log\log n}$. Both these results were first proved for $r = 2$ in [4].

One may also ask a reverse question: given an integer k and a permutation π, for how large r are there block or tight r-twins of length k in π? We denote the largest such r by $r_{bt}^{(k)}(\pi)$ and $r_{tt}^{(k)}(\pi)$, resp. We show, in particular, that for n even, with high probability, $r_{tt}^{(2)}(\Pi_n) = n/2$, that is, a random permutation Π_n contains tight $n/2$-twins of length 2 (Theorem 6.1).

Finally, let us return to the starting point and define *block-tight r-twins* as block r-twins which are at the same time tight r-twins. This means that r similar blocks occur in a permutation consecutively, with no gaps in-between, as in the following example:

$$(14, 18, 6, \boxed{2,1,3}, \boxed{5,4,8}, \boxed{15,7,17}, \boxed{12,9,16}, 10, 11, 13).$$

For $r = 2$ this notion coincides with the order repetitions discussed at the beginning.

In fact, the result of Avgustinovich et al. [2] mentioned above implies that there exist permutations with no block-tight r-twins of length greater than 1, for any $r \geqslant 2$. Curiously, a random permutation, with high probability, contains block-tight r-twins of length $\sim \frac{\log n}{(r-1)\log\log n}$, which asymptotically agrees with the case of tight r-twins (Theorem 5.2).

In the forthcoming sections we give proofs of the above stated results: about $bt^{(r)}(n)$ in Sect. 3, and about $tt^{(r)}(n)$ and $f(r,k)$ in Sect. 4. Section 6 is devoted to functions $r_{bt}^{(k)}(\pi)$ and $r_{tt}^{(k)}(\pi)$, while all results about the length of block, tight, and block-tight r-twins in random permutations are proved in Sect. 5. The next section contains a technical probabilistic lemma, while the last one presents some open problems.

2. Independence of Occurrences of Twins

In this section we prove a technical result which will be used in several places of the paper. It is about the conditional probabilities of occurrences of r-twins in a random permutation.

Let Π_n be a random permutation chosen uniformly from the set of all $n!$ permutations of $[n]$. For integers $r, k \geqslant 2$ and a family of r pairwise disjoint subsets $A_1, \ldots, A_r$ of $[n]$, each of size k, let $\mathcal{E}(A_1, \ldots, A_r)$ be the event that there are r-twins in Π_n on positions determined by the subsets $A_1, \ldots, A_r$. Then,

$$\mathbb{P}(\mathcal{E}(A_1, \ldots, A_r) = 1) = \frac{\binom{n}{k}\binom{n-k}{k}\cdots\binom{n-(r-2)k}{k}\cdot(n-(r-1)k)!\cdot 1}{n!} = \frac{1}{k!^{r-1}}. \tag{2.1}$$

Lemma 2.1. *For integers $r, t, k \geqslant 2$ and $s \in [r]$, let $A_j^{(i)}$, $j = 1, \ldots, r$, $i = 1, \ldots, t$, be k-element subsets of $[n]$ such that for each $i = 1, \ldots, t$ all sets $A_1^{(i)}, \ldots, A_r^{(i)}$ are pairwise disjoint and*

$$\bigcup_{j=1}^{s} A_j^{(1)} \cap \bigcup_{i=2}^{t}\bigcup_{j=1}^{r} A_j^{(i)} = \varnothing.$$

Then, setting $\mathcal{E}^{(i)} := \mathcal{E}(A_1^{(i)}, \ldots, A_r^{(i)})$,

$$\text{if} \quad s \leqslant r-2\,, \quad \text{then} \quad \mathbb{P}(\mathcal{E}^{(1)} \cap \cdots \cap \mathcal{E}^{(t)}) \leqslant \frac{1}{k!^s}\mathbb{P}(\mathcal{E}^{(2)} \cap \cdots \cap \mathcal{E}^{(t)}),$$

while if $s \geqslant r-1$, then event $\mathcal{E}^{(1)}$ is mutually independent of the family of events $\{\mathcal{E}^{(2)}, \ldots, \mathcal{E}^{(t)}\}$, that is,

$$\mathbb{P}(\mathcal{E}^{(1)} \cap \cdots \cap \mathcal{E}^{(t)}) = \frac{1}{k!^{r-1}}\mathbb{P}(\mathcal{E}^{(2)} \cap \cdots \cap \mathcal{E}^{(t)}) = \mathbb{P}(\mathcal{E}^{(1)})\mathbb{P}(\mathcal{E}^{(2)} \cap \cdots \cap \mathcal{E}^{(t)}).$$

Proof. Let $N := N(A_j^{(i)} : 1 \leqslant i \leqslant t, 1 \leqslant j \leqslant r)$ be the number of permutations of an $(n-sk)$-element set D on positions in $[n] \setminus \bigcup_{j=1}^{s} A_j^{(1)}$, that is bijections $f : D \to [n] \setminus \bigcup_{j=1}^{s} A_j^{(1)}$, such that there are $(r-s)$-twins on position sets $A_{s+1}^{(1)}, \ldots, A_r^{(1)}$, as well as, r-twins on position sets $A_1^{(i)}, \ldots, A_r^{(i)}$ for all $i = 2, \ldots, t$. Observe that

$$|\mathcal{E}^{(1)} \cap \cdots \cap \mathcal{E}^{(t)}| = \frac{n!N}{k!^s(n-sk)!} \quad \text{and} \quad |\mathcal{E}^{(2)} \cap \cdots \cap \mathcal{E}^{(t)}| \geqslant (n)_{sk}N,$$

where the equality follows from the fact that once the values of $\Pi_n(i)$ are fixed on $[n] \setminus \bigcup_{j=1}^{s} A_j^{(1)}$, the rest of Π_n is determined by assigning k-element subsets to each position set $A_i^{(1)}$, $j = 1, \ldots, s$, while the inequality is a result of dropping the part of definition of N requesting that there are $(r-s)$-twins on position sets $A_{s+1}^{(1)}, \ldots, A_r^{(1)}$. For $s \geqslant r-1$, there is nothing to drop, so we have equality there. Hence,

$$\mathbb{P}(\mathcal{E}^{(1)} \cap \cdots \cap \mathcal{E}^{(t)}) = \frac{n!N}{k!^s(n-sk)!n!} \leqslant \frac{|\mathcal{E}^{(2)} \cap \cdots \cap \mathcal{E}^{(t)}|}{k!^s(n)_{sk}(n-sk)!} = \frac{1}{k!^s}\mathbb{P}(\mathcal{E}^{(2)} \cap \cdots \cap \mathcal{E}^{(t)}),$$

where, again, for $s = r-1$, we have equality. □

The first statement of Lemma 2.1 will only be used in the proof of Theorem 5.1. The second one will be applied in several proofs, whenever independence of occurrences of r-twins is sought, for example, in both applications of the Local Lemma.

3. Block Twins

We say that a collection of disjoint, similar blocks $\{\sigma_1, \ldots, \sigma_r\}$ in a permutation π forms *block r-twins* in π. Let $bt^{(r)}(\pi)$ denote the longest length of block r-twins in π, that is,

$$bt^{(r)}(\pi) = \max\{|\sigma_1| : (\sigma_1, \ldots, \sigma_r) \text{ form block } r\text{-twins in } \pi\}$$

and let

$$bt^{(r)}(n) = \min\{bt^{(r)}(\pi) : \pi \text{ is a permutation of } [n]\}.$$

Note that containment of block r-twins of length k is monotone, that is, their absence in a permutation excludes both, block r-twins of length $k+1$ and block $(r+1)$-twins of length k.

The goal of this section is to pin-point $bt^{(r)}(n)$ asymptotically. To this end will need the standard Local Lemma.

For events $\mathcal{E}_1, \ldots, \mathcal{E}_n$ in any probability space, *a dependency graph* $D = ([n], E)$ is any graph on vertex set $[n]$ such that for every vertex i the event $\mathcal{E}_i$ is jointly independent of all events $\mathcal{E}_j$ with $ij \notin E$.

Lemma 3.1 (The Local Lemma; Symmetric Version [6] (see [1])). *Let $\mathcal{E}_1, \ldots, \mathcal{E}_n$ be events in any probability space. Suppose that the maximum degree of a dependency graph of these events is at most Δ, and $\mathbb{P}(A_i) \leqslant p$, for all $i = 1, 2, \ldots, n$. If $ep(\Delta+1) \leqslant 1$, then $\mathbb{P}\left(\bigcap_{i=1}^n \overline{\mathcal{E}_i}\right) > 0$.*

The following result gives an asymptotic formula for the function $bt^{(r)}(n)$. The term $o(1)$ depends on r.

Theorem 3.2. *We have*

$$bt^{(r)}(n) = (1+o(1))\frac{\log n}{\log\log n}.$$

Proof. First we show the lower bound. Let $n = k((r-1)k!+1)$ and let π be any permutation of $[n]$. Divide π into $(r-1)k!+1$ blocks, each of length k. By the pigeonhole principle, there are r blocks that induce similar sub-permutations (forming thereby r-twins). The choice of n, together with the Stirling formula, imply that $k = (1+o(1))\frac{\log n}{\log\log n}$.

For the upper bound we use the probabilistic method based upon Lemmas 3.1 and 2.1. Let $n = \lfloor k!(erk)^{-1/(r-1)} \rfloor$ and let $\Pi := \Pi_n$ be a random permutation. An r-tuple of indices $i_1, \dots, i_r$ satisfying

$$1 \leqslant i_1 \leqslant i_2 - k \leqslant i_3 - 2k \cdots \leqslant i_r - (r-1)k \leqslant n - rk,$$

is called *k-spread*. For a k-spread r-tuple let $\mathcal{E}_{i_1,\dots,i_r}$ be the event that segments $(\Pi(i_j), \Pi(i_j+1)), \dots, \Pi(i_j+k-1))$, $j = 1, \dots, r$, form block r-twins in Π. We are going to apply Lemma 3.1 to events $\mathcal{E}_{i_1,\dots,i_r}$ over all choices of k-spread r-tuples $i_1, \dots, i_r$.

By (2.1), we may set $p := \mathbb{P}(\mathcal{E}_{i_1,\dots,i_r}) = 1/k!^{r-1}$. Notice that by Lemma 2.1, case $s = r-1$, a fixed event $\mathcal{E}_{i_1,\dots,i_r}$ is jointly independent of all events $\mathcal{E}_{i'_1,\dots,i'_r}$ for which

$$\bigcup_{j=1}^{r-1} \{i_j, i_j+1, \dots, i_j+k-1\} \cap \bigcup_{j=1}^{r} \{i'_j, i'_j+1, \dots, i'_j+k-1\} = \varnothing.$$

Thus, there is a dependency graph D for these events with maximum degree at most

$$\Delta = (r-1)kn^{r-1} \leqslant rkn^{r-1} - 1.$$

This and the choice of n yields that

$$e(\Delta+1)p \leqslant e \cdot rkn^{r-1} \cdot \frac{1}{k!^{r-1}} \leqslant 1.$$

Consequently, Lemma 3.1 implies that there exists a permutation π of $[n]$ with no block r-twins of length k, that is with $bt^{(r)}(\pi) < k$. In turn, $bt^{(r)}(n) \leqslant bt^{(r)}(\pi) < k$. Again, the Stirling formula yields that $k = (1+o(1))\frac{\log n}{\log\log n}$. □

4. Tight Twins

In this section we consider r-twins whose union occupies a block of consecutive positions in a permutation π. We call them *tight r-twins*. Note that, unlike block twins, tight twins are not 'monotone', that is, the absence of tight r-twins of length k in a permutation does not exclude the presence of longer tight r-twins. Likewise, it does not exclude the presence of tight $(r+1)$-twins of length k.

4.1. Upper Bound

Let $tt^{(r)}(\pi)$ denote the maximum length of tight r-twins in π, that is,

$$tt^{(r)}(\pi) = \max\{|\sigma_1| : (\sigma_1, \dots, \sigma_r) \text{ form } r\text{-tight twins in } \pi\},$$

and let

$$tt^{(r)}(n) = \min\{tt^{(r)}(\pi) : \pi \text{ is a permutation of } [n]\}.$$

We will prove that for every fixed r there is a constant $c = c(r)$ such that $tt^{(r)}(n) \leqslant c$ for all n. We intend to apply again the probabilistic method. However, due to the lack of monotonicity, in order to show that $tt^{(r)}(n) < k$, we need to find a permutation π without r-twins of any length $m \geqslant k$. To this end, the most suitable tool seems to be the following version of the Local

Lemma, which is equivalent to the standard asymmetric version (see [1]). The dependency graph was defined in Sect. 3.

Lemma 4.1. (The Local Lemma; Multiple Version (see [1])) *Let* $\mathcal{E}_1, \dots, \mathcal{E}_n$ *be events in any probability space with a dependency graph* $D = (V, E)$*. Let* $V = V_1 \cup \dots \cup V_t$ *be a partition such that all members of each part* V_k *have the same probability* p_k*. Suppose that the maximum number of vertices from* V_m *adjacent to a vertex from* V_k *is at most* Δ_{km}*. If there exist real numbers* $0 \le x_1, \dots, x_t < 1$ *such that* $p_k \le x_k \prod_{m=1}^{t}(1 - x_m)^{\Delta_{km}}$*, then* $\Pr\left(\bigcap_{i=1}^{n} \overline{\mathcal{E}_i}\right) > 0$.

Equipped with this tool, we may now prove the main result of this section.

Theorem 4.2. *For every* $n \geqslant 1$ *and* $r \geqslant 3$ *we have* $tt^{(r)}(n) \leqslant 15r$*.*

Proof. Let Π be a random permutation of $[n]$. We will apply Lemma 4.1 in the following setting. For a fixed block K of length rk, $k \geqslant cr$ (where $c = c(r)$ will be specified later), let $\mathcal{A}_K$ denote the event that a sub-permutation of Π occupying K consists of tight r-twins. Let V_k denote the collection of all such events $\mathcal{A}_K$ for all possible blocks K of length rk. Note that $\mathcal{A}_K = \bigcup_{K_1,\dots,K_r} \mathcal{E}(K_1, \dots, K_r)$, where the union extends over all partitions of K into r disjoint subsets of size k and $\mathcal{E}(K_1, \dots, K_r)$ is the event defined prior to Lemma 2.1. Thus, by (2.1) and the union bound, for every $\mathcal{A}_K \in V_k$,

$$\mathbb{P}(\mathcal{A}_K) \leqslant \sum_{K_1,\dots,K_r} \mathbb{P}(\mathcal{E}(K_1, \dots, K_r)) = \frac{1}{r!}\binom{rk}{k}\binom{(r-1)}{k}\cdots\binom{2k}{k}\cdot\frac{1}{(k!)^{r-1}}$$
$$= \frac{1}{r!}\cdot\frac{(rk)!}{(k!)^{2r-1}}.$$

Hence, we may take $p_k = \frac{(rk)!}{r!(k!)^{2r-1}}$.

By Lemma 2.1, case $s = r$, any event $\mathcal{E}(K_1, \dots, K_r)$ is mutually independent of all events $\mathcal{E}(M_1, \dots, M_r)$ such that $\bigcup_{i=1}^{r} K_i \cap \bigcup_{i=1}^{r} M_i = \varnothing$. In turn, any event $\mathcal{A}_K$ depends only on those events $\mathcal{A}_M$ for which $M \cap K \neq \varnothing$. Hence, if M is any block of length rm, with $m \geqslant c$ and $M \neq K$, then we may take $\Delta_{km} = rk + rm - 1$. Furthermore, set $q = 1/2$ for convenience and define $x_m = q^m$.

We are going to prove that for every $k \geqslant c$,

$$p_k \le x_k \prod_{m=c}^{n/r}(1 - x_m)^{\Delta_{km}}.$$

Since $x_m \leqslant 1/2$ for all $m \geqslant 1$, we may use the inequality $1 - x_m \geqslant e^{-2x_m}$ and obtain the bound

$$\prod_{m=c}^{n/r}(1 - x_m)^{\Delta_{km}} \geqslant \prod_{m=c}^{n/r}(1 - x_m)^{r(k+m)} \geqslant \exp\left(-2r\sum_{m=c}^{\infty} x_m(k+m)\right)$$
$$= \exp\left(-2rk\sum_{m=c}^{\infty} q^m\right)\cdot\exp\left(-2r\sum_{m=c}^{\infty} mq^m\right).$$

Since $\sum_{m=c}^{\infty} q^m = \frac{q^c}{1-q} =: A$ and $\sum_{m=c}^{\infty} mq^m = \frac{q^c(-qc+q+c)}{(1-q)^2} = \frac{q^c}{1-q} \cdot \left(c + \frac{q}{1-q}\right)$ $=: B$, we will be done by showing that for all $k \geqslant c$

$$\frac{(rk)!}{r!(k!)^{2r-1}} \leqslant \frac{q^k}{e^{2rkA} \cdot e^{2rB}}.$$

It is not hard to see that when r is fixed, the inequality holds for sufficiently large k. Let us make more precise calculations to derive the dependence of c on r.

First we bound the left-hand side by using the well known consequences of Stirling's formula, $n^n e^{-n}\sqrt{2\pi n} \leqslant n! \leqslant n^n e^{-n+1}\sqrt{n}$, which are valid for all positive integers n. Thus, we obtain

$$\frac{(rk)!}{r!(k!)^{2r-1}} \leqslant \frac{(rk)^{rk} \cdot e \cdot \sqrt{rk}}{e^{rk}} \cdot \frac{e^{k(2r-1)}\sqrt{2\pi k}}{r! \cdot k^{k(2r-1)} \cdot (2\pi)^r \cdot k^r},$$

which simplifies to

$$\frac{(rk)!}{r!(k!)^{2r-1}} \leqslant \frac{e \cdot \sqrt{2\pi r}}{r! \cdot (2\pi)^r} \cdot \frac{(r^r)^k \cdot e^{k(r-1)}}{k^{(k+1)(r-1)}} < (r^r)^k \cdot \left(\frac{e^k}{k^{(k+1)}}\right)^{r-1},$$

for $r \geqslant 2$. On the other hand, recalling that $q = 1/2$, we have $A = \frac{2}{2^c}$ and $B = \frac{2}{2^c}(c+1)$. So, we will be done by showing that

$$(r^r)^k \cdot (e^{r-1})^k \cdot (e^{4r/2^c})^k \cdot e^{4r(c+1)/2^c} \cdot 2^k < k^{(k+1)(r-1)} \tag{4.1}$$

for all $k \geqslant c$. Set $c = 15r$.

Since $4r/2^c \leqslant 4r(c+1)/2^c \leqslant 1$, we get

$$(e^{4r/2^c})^k \leqslant e^k \quad \text{and} \quad e^{4r(c+1)/2^c} \leqslant e.$$

Thus, (4.1) will follow from

$$(er)^{rk} e^{k+1} < k^{(k+1)(r-1)}.$$

Finally, since $k \geqslant c = 15r \geqslant 2e^2 r$, we get

$$k^{(k+1)(r-1)} \geqslant c^{(k+1)(r-1)} = (2r)^{(k+1)(r-1)} e^{2(k+1)(r-1)}$$

and so it remains to show that $(2r)^{(k+1)(r-1)} \geqslant r^{rk}$ and $e^{2(k+1)(r-1)} \geqslant e^{rk+k+1}$. Observe that the first inequality is equivalent to $2^{(k+1)(r-1)} \geqslant r^{k-r+1}$, which holds, since for $r \geqslant 3$

$$2^{(k+1)(r-1)} \geqslant 2^{kr/2} = \left(2^{r/2}\right)^k \geqslant r^k \geqslant r^{k-r+1}.$$

The second inequality is equivalent to $k(r-3) + 2r - 3 \geqslant 0$, which clearly holds, too. □

Note that the above defined quantity p_k is greater than $(r/ek)^{rk} r^{-r} \geqslant 1$ for $2 \leqslant k \leqslant r/(2e)$, r large, so the bound in Theorem 4.2 cannot be improved to $o(r)$ by this method (see Problems 7.1 and 7.2 in Sect. 7).

4.2. Function $f(r, k)$, or the Problem Turned Around

Here we put the cart before the horse and consider the following extremal problem. Given integers $r, k \geqslant 2$, determine the function

$$f(r, k) = \min\{n : \text{ every permutation of } [n] \text{ contains tight } r\text{-twins of length } k\}.$$

If no such n exists, then we set $f(r, k) = \infty$. Note that f is not monotone in either variable, that is, in general, it is not true that $f(r, k) \leqslant f(r+1, k)$ or $f(r, k) \leqslant f(r, k+1)$. For example, permutation

$$\pi_{12} = (4, 5, 6, 9, 8, 7, 1, 2, 3, 12, 11, 10)$$

contains no tight 2-twins of length 3 (see the proof of Prop. 4.3 for explanation) but it does contain tight 3-twins of length 3, namely $(4, 9, 1)$, $(5, 8, 2)$, and $(6, 7, 3)$, all three similar to $(2, 3, 1)$. Also, π_{12} does contain tight 2-twins of length 6, namely $(4, 5, 6, 9, 8, 7)$ and $(1, 2, 3, 12, 11, 10)$, both similar to $(1, 2, 3, 6, 5, 4)$. Due to this inconvenience, there is no obvious relation between functions $f(r, k)$ and $tt^{(r)}(n)$.

As we show below in Propositions 4.3 and 4.4, quite surprisingly, $f(r, k) = \infty$ for any $r \geqslant 2$ and $k \geqslant 3$. For clarity of presentation we chose to first present the proof in the case $r = 2$, so that the general case will be easier to comprehend. For a sequence of integers $A = (a_1, \ldots, a_m)$, set $-A = (-a_1, \ldots, -a_m)$ and $\overleftarrow{A} = (a_m, \ldots, a_1)$.

Proposition 4.3. *For all $k \geqslant 3$, we have $f(2, k) = \infty$.*

Proof. For each $k \geqslant 2$, we construct an infinite sequence of distinct integers which is free of tight 2-twins of length $2k - 1$ and $2k$. Consider a partition of all natural numbers into consecutive blocks of length $2k - 1$,

$$\mathbb{N} = \bigcup_{m \geqslant 1} A_m,$$

where, for $m \geqslant 1$, $A_m = ((m-1)(2k-1)+1, \ldots, m(2k-1))$ is viewed as a sequence. Then we define

$$\pi^{(k)} = (-\overleftarrow{A_1})\overleftarrow{A_2}(-\overleftarrow{A_3})\overleftarrow{A_4}\cdots$$

For example,

$$\pi^{(2)} = (-3, -2, -1,\ 6, 5, 4,\ -9, -8, -7,\ 12, 11, 10, \cdots)$$

(see Fig. 1a). Of course, for any fixed n divisible by $2k - 1$, we may extract a permutation of length n as the initial segment of $\pi^{(k)}$ and, to get rid of negative integers, rewrite it in the reduced form, that is, as a permutation of $[n]$ similar to it. For instance, for $k = 2$ and $n = 12$, we then recover the permutation π_{12} presented above.

We will now show that $\pi := \pi^{(k)}$ contains no tight 2-twins of length $2k-1$ or $2k$. By symmetry, it suffices to consider only blocks ('windows') of length $4k - 2$ and, respectively, $4k$, which begin at one of the first $2k - 1$ elements of π. Let such a window begin at the s-th element of $(-\overleftarrow{A_1})$, that is at $\pi(s)$, $1 \leqslant s \leqslant 2k - 1$. It then stretches over the entire block $\overleftarrow{A_2}$ and some initial

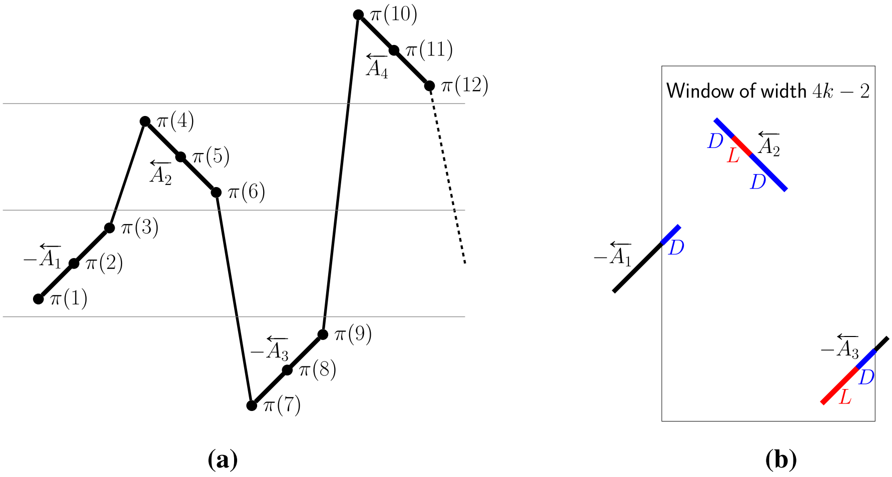

FIGURE 1. **a** The shape of $\pi = \pi^{(2)}$ from the proof of Proposition 4.3; **b** defining D and L in the proof of Proposition 4.3

segment of $(-\overleftarrow{A_3})$. When the window has length $2k$ and $s = 2k-1$, it even reaches the first element of $\overleftarrow{A_4}$.

Suppose there are tight 2-twins of length $2k-1$ in π beginning at $\pi(s)$. They yield a partition of the set $\{\pi(s), \pi(s+1), \ldots, \pi(4k-2+s-1)\} = D \cup L$ of length $4k-2$, where $|D| = |L| = 2k-1$ (see Fig. 1b). Clearly, none of the twins can coincide with $\overleftarrow{A_2}$. For s odd, since then also $2k-1-(s-1) = 2k-s$ is odd, w.l.o.g., the twin on D (call it *Daphne*, cf. [11]) begins with an increasing segment longer than the initial increasing segment of the L-twin (call it *Laurel*), a contradiction with Daphne and Laurel being twins. For s even, we look at the other end, that is, at the first $s-1$ elements of $(-\overleftarrow{A_3})$, where again, due to the oddity of $s-1$, Daphne, say, captures more elements than Laurel. This means, however, that Daphne ends with a longer increasing segment than Laurel, a contradiction, again.

It remains to exclude tight 2-twins of length $2k$ in π. Suppose that there are such twins, D and L, or Daphne and Laurel. We are now looking at a window on positions $\{s, s+1, \ldots, 4k+s-1\} = D \cup L$, $1 \leqslant s \leqslant 2k-1$, of length $4k$, where $|D| = |L| = 2k$. The argument is similar to that for twins of length $2k-1$. Clearly, neither Daphne or Laurel can contain an entire block $\overleftarrow{A_2}$ or $(-\overleftarrow{A_3})$. As before, for s odd, Daphne begins with an increasing segment longer than the initial increasing segment of Laurel, a contradiction. For s even, the ending segment consisting of the first $s+1$ elements of $(-\overleftarrow{A_3})$ has an odd length at least 3 and, again, Daphne, say, ends with a longer increasing segment than Laurel, which is a contradiction. □

Now we show how the above construction can be generalized to yield a similar result for any $r \geqslant 3$.

Proposition 4.4. *For all $r, k \geqslant 3$, we have $f(r,k) = \infty$.*

Proof. We use the previous construction with blocks of length $rk-1$, $k \geqslant 2$, obtaining a permutation $\pi_r^{(k)}$. Let us denote its consecutive blocks of length $rk-1$ by $B_1, B_2, \ldots$, that is, $B_1 = -\overleftarrow{A_1}$, $B_2 = \overleftarrow{A_2}$, etc. We will show that $\pi_r^{(k)}$ does not contain tight r-twins of length $\ell \in \{2k-1, 2k\}$. Suppose the opposite and let $T_1, \ldots, T_r$ be tight r-twins of length ℓ in $\pi_r^{(k)}$.

W.l.o.g., consider a window $W = T_1 \cup \cdots \cup T_r$ of length $r\ell$ beginning at the s-th element of B_1. Then, since $\ell \geqslant 2k-1$, $s \geqslant 1$ and $r \geqslant 2$,

$$|W \cap B_2| \geqslant \min\{2\ell - (rk-s), rk-1\} \geqslant \min\{2rk - r - (rk-s), rk-1\} \geqslant rk - r + 1.$$

Also, $W \cap B_4 = \varnothing$, except for $\ell = 2k$ and $s = rk-1$, when $|W \cap B_4| = 1$. By taking the average, there is T_{i_0} with $|T_{i_0} \cap B_2| = \min_i |T_i \cap B_2| \leqslant k-1$ and T_{i_1} with $|T_{i_1} \cap B_2| = \max_i |T_i \cap B_2| \geqslant k$. Moreover, if for no i, $|T_i \cap B_2| \geqslant k+1$, then for all i, $|T_i \cap B_2| \geqslant k-1 \geqslant 1$.

We claim that for all i, $|T_i \cap B_2| \geqslant 1$. Suppose that $T_{i_0} \cap B_2 = \varnothing$. Then, in view of the above, $|T_{i_1} \cap B_2| \geqslant k+1 \geqslant 3$. This means that T_{i_1} has a decreasing segment of length at least 3, while T_{i_0} does not, as $T_{i_0} \subset B_1 \cup B_3 \cup \{f\}$, where f is the first element of B_4.

Also, for all i, $|T_i \cap B_1| \geqslant 1$. Indeed, otherwise there would be a twin which begins with a decreasing segment and a twin which begins with an increasing segment, a contradiction. Finally, compare T_{i_0} with T_{i_1}. There is in T_{i_1} a decreasing segment of length k with all elements larger than the first element of T_{i_1}. On the other hand, there is not such a segment in T_{i_0} (there might be a decreasing segment of length k which, however, ends in B_3, thus, below the first element of T_{i_0} which must belong to B_1). This is a contradiction, again, and the proof is completed. ☐

Proposition 3.7 in [4] together with permutation $\pi_2 = (1,4,3,2,5)$ show that $f(2,2) = 6$. Thus, in view of Proposition 4.3, $f(2,k)$ is determined for all $k \geqslant 2$. Let us now focus on $f(r,2)$, $r \geqslant 3$, the only remaining case. For simplicity, we will call tight r-twins of length 2 just *r-twins*. Besides $f(2,2) = 6$, we also know that $f(3,2) = 12$. Indeed, permutation

$$\pi_3 = (11,2,3,8,7,6,5,4,9,10,1)$$

of length 11 is free of 3-twins, while a computer verification of all permutations of length 12 reveals that each one of them contains 3-twins. For $r \geqslant 4$, however, we only have a lower bound on $f(r,2)$, which is quadratic in r.

Proposition 4.5. *For every $r \geqslant 3$ we have $f(r,2) \geqslant r(r+5) - 12$.*

Proof. We begin by constructing a suitable permutation of length $r(r+5)-13$.

For $r = 3$, permutation π_3 presented above is of the required length $11 = 3(3+5) - 13$. Fix $r \geqslant 4$ and consider the following sequence of sequences of consecutive integers of (mostly) diminishing lengths: $A_0 = (1, \ldots, r-1)$, $A_1 = (r, \ldots, 2r-2)$, $A_2 = (2r-1, \ldots, 3r-4)$, $A_3 = (3r-3, \ldots, 4r-7)$, $\ldots$, $A_{r-1} = (\binom{r}{2} + r - 1)$, $A_r = (\binom{r}{2} + r)$, $\ldots$, $A_{3r-7} = (\binom{r}{2} + 3r - 7)$. Note that the first two sequences have the same length $r-1$, then each next one is shorter by one, and, finally $|A_{r-1}| = |A_r| = \cdots = |A_{3r-7}| = 1$. In total, their concatenation makes up the sequence $(1, \ldots, \binom{r}{2} + 3r - 7)$. Define permutation

$$\pi'_r = ((-1)^{r-1}\overleftarrow{A_{3r-7}}) \cdots \overleftarrow{A_2}\ (-\overleftarrow{A_1})\ \overleftarrow{A_0}\ 0\ (-A_0)\ A_1\ (-A_2) \cdots (-1)^r A_{3r-7}$$

of length

$$2 \times \left(\binom{r}{2} + 3r - 7\right) + 1 = r(r+5) - 13.$$

(Of course, for singleton classes the overhead arrow is redundant.) Further, let π_r be the reduced form of π'_r, obtained, simply, by adding $\binom{r}{2} + 3r - 6$ to all elements of π'_r. For instance,

$$\pi'_4 = (-11, 10, -9, 8, 7, -6, -5, -4, 3, 2, 1, 0, -1, -2, -3, 4, 5, 6, -7, -8, 9, -10, 11)$$

becomes

$$\pi_4 = (1, 22, 3, 20, 19, 6, 7, 8, 15, 14, 13, 12, 11, 10, 9, 16, 17, 18, 5, 4, 21, 2, 23)$$

(see Fig. 2). One more example:

$$\pi_5 = (37, 2, 35, 4, 33, 6, 7, 30, 29, 28, 11, 12, 13, 14, 23, 22, 21, 20,$$
$$19, 18, 17, 16, 15, 24, 25, 26, 27, 10, 9, 8, 31, 32, 5, 34, 3, 36, 1).$$

In general, π_r consists of a decreasing sequence of length $2r - 1$ located in the middle which is extended in both directions by alternatingly increasing and decreasing sequences of lengths getting shorter by one in each step, except for the very end where, on each side, we have a zigzag pattern of singletons of length $2r - 5$. Moreover, what is crucial here, each next monotone segment (to the left or to the right) is entirely below or entirely above all previous elements appearing on the same side.

We are going to prove that there are no tight r-twins of length 2 (shortly, r-twins) in π_r. For the proof we need to distinguish two types of r-twins. We call r-twins *increasing* if they are similar to $(1, 2)$ and *decreasing* if they are similar to $(2, 1)$. Suppose that there are r-twins in π_r. As they occupy a block (window) of length $2r$, let us consider all its possible locations. Owing to symmetry, it suffices to consider windows with the right end belonging to the right half of π_r, that is, to be to the right of 0 in π'_r (for the ease of description we will look at π'_r, not π_r, which is, of course, equivalent).

Let w be the rightmost element of the window and assume first that it belongs to A_m, m odd. Then, since the last element of the window is the largest one, the r-twins have to be increasing. Thus, the elements of the previous block, $(-A_{m-1})$ have to be all paired with the elements of A_m. If $m \leqslant r - 1$, then, $|A_{m-1}| < |A_m|$, and we get a contradiction. If $w \in A_r \cup \cdots \cup A_{3r-7}$, i.e., it belongs to the zigzag of singletons at the end, the window still contains $A_{r-3} \cup A_{r-2}$. Then each singleton has to be paired with the next one, and the three elements of A_{r-3} have to be paired with the two elements of A_{r-2}, a contradiction again. For m even, the situation is symmetrical: the last element is the smallest, so the r-twins have to be decreasing, which again leads to a contradiction.

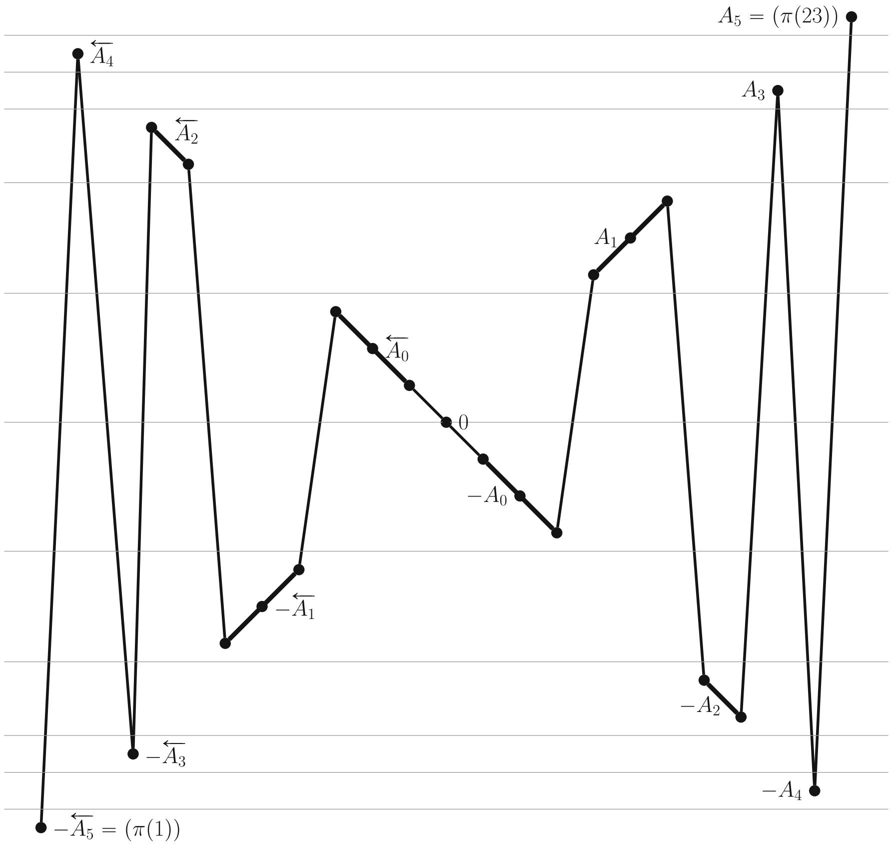

FIGURE 2. The shape of $\pi = \pi_4$ (and as well as π'_4) from the proof of Proposition 4.5

It remains to consider the case when w belongs to $(-A_0)$. Then the window contains a decreasing sub-sequence of length at least $r+1$ (all its elements but those sitting in $(-A_1)$). But, clearly, no two elements of a decreasing subsequence can be paired with each other in increasing twins. We, again, arrive at a contradiction which completes the proof. □

5. Block, Tight, and Block-Tight Twins in Random Permutations

In the previous sections we used a random permutation Π_n as a tool of the probabilistic method to estimate $bt^{(r)}(n)$ and $tt^{(r)}(n)$. Now we are interested in the longest length of block, tight, and block-tight r-twins *in* a random permutation. We say that an event $\mathcal{E}_n$ in the uniform probability space of all $n!$ permutations of $[n]$ holds *asymptotically almost surely*, or *a.a.s.*, for short, if $\mathbb{P}(\mathcal{E}_n) \to 1$, as $n \to \infty$.

5.1. Block Twins

The next result shows that the maximum length of block r-twins in Π_n is a.a.s. just a little bit greater than in the worst case and the difference diminishes with r increasing (cf. Theorem 3.2).

Theorem 5.1. *For a random permutation Π_n, a.a.s. we have*

$$bt^{(r)}(\Pi_n) = \left(\frac{r}{r-1} + o(1)\right)\frac{\log n}{\log\log n}.$$

Proof. For an integer $k \geqslant 2$, recall from the proof of Theorem 3.2 that for a given k-spread r-tuple of indices $i_1, \ldots, i_r$, $\mathcal{E}_{i_1,\ldots,i_r}$ denotes the event which holds if segments $(\Pi(i_j), \Pi(i_j+1)), \ldots, \Pi(i_j+k-1))$, $j = 1, \ldots, r$, form block r-twins. Let $X_{i_1,\ldots,i_r}$ be the indicator random variable of the event $\mathcal{E}_{i_1,\ldots,i_r}$, that is, $X_{i_1,\ldots,i_r} = 1$ if $\mathcal{E}_{i_1,\ldots,i_r}$ holds; otherwise $X_{i_1,\ldots,i_r} = 0$. Set $X(k) = \sum X_{i_1,\ldots,i_r}$, where the sum extends over all k-spread r-tuples $i_1, \ldots, i_r$.

By (2.1), we have $\mathbb{P}(X_{i_1,\ldots,i_r} = 1) = \mathbb{P}(\mathcal{E}_{i_1,\ldots,i_r}) = k!^{-(r-1)}$ and thus $\mathbb{E}(X(k)) = \Theta(n^r k!^{-(r-1)})$, where the hidden constant is less than one. Let

$$k^+ = \left\lceil \frac{(1+\varepsilon_n) r \log n}{(r-1)\log\log n} \right\rceil, \quad \text{where} \quad \frac{\log\log\log n}{\log\log n} \ll \varepsilon_n = o(1).$$

Then, with $c_r = \frac{e(r-1)}{r}$,

$$\mathbb{E}X(k^+) \leqslant n^r (k^+)!^{-(r-1)} \leqslant \left(\frac{e}{k^+}\right)^{k^+(r-1)} n^r \leqslant \left(\frac{c_r \log\log n}{\log n}\right)^{\frac{(1+\varepsilon_n) r \log n}{\log\log n}} n^r \to 0,$$

because, after taking the logarithm,

$$\frac{(1+\varepsilon_n) r \log n}{\log\log n}(\log c_r + \log\log\log n - \log\log n) + r\log n \to -\infty.$$

Hence, by Markov's inequality, a.a.s., $X(k^+) = 0$, that is, $bt^{(r)}(\Pi_n) < k^+$.

We will establish a matching lower bound on $bt^{(r)}(\Pi_n)$ by the second moment method. Set $k^- = \left\lfloor \frac{r\log n}{(r-1)\log\log n} \right\rfloor$. Then, $\mathbb{E}X(k^-) \to \infty$, since

$$\frac{n^r}{(k^-)!^{r-1}} \geqslant \left(\frac{1}{k^-}\right)^{(r-1)k^-} n^r \geqslant \left(\frac{(r-1)\log\log n}{r\log n}\right)^{\frac{r\log n}{\log\log n}} n^r \to \infty,$$

because

$$\frac{r\log n}{\log\log n}\big(\log((r-1)/r) + \log\log\log n - \log\log n\big) + r\log n \to \infty.$$

By the same kind of calculations, we also have,

$$\frac{n^r}{(\log n)^r (k^-)!^{r-1}} \to \infty \quad \text{and} \quad n/(k^-)! = o(1). \tag{5.1}$$

Now, we turn to estimating $Var(X(k^-))$. For two k^--spread r-tuples of indices, $i_1, \ldots, i_r$ and $i'_1, \ldots, i'_r$, let us analyze the covariance $Cov(X_{i_1,\ldots,i_r}, X_{i'_1,\ldots,i'_r})$. Set $A_j = \{i_j, \ldots, i_j + k - 1\}$ and $A'_j = \{i'_j, \ldots, i'_j + k - 1\}$, $j =$

$1, \ldots, r$. Let $0 \leqslant s \leqslant r$ be the largest integer such that there are indices $1 \leqslant j_1 < \cdots < j_s \leqslant r$ with

$$(A'_{j_1} \cup \cdots \cup A'_{j_s}) \cap (A_1 \cup \cdots \cup A_r) = \varnothing. \tag{5.2}$$

Then, by Lemma 2.1 with $t = 2$, for $s \in \{r-1, r\}$, due to independence, $Cov(X_{i_1,\ldots,i_r}, X_{i'_1,\ldots,i'_r}) = 0$, while for $s \leqslant r-2$, using also (2.1),

$$Cov(X_{i_1,\ldots,i_r}, X_{i'_1,\ldots,i'_r}) \leqslant \mathbb{P}(X_{i_1,\ldots,i_r} = X_{i'_1,\ldots,i'_r} = 1) \leqslant \frac{1}{(k^-)!^{s+r-1}}.$$

Moreover, for $s \leqslant r-2$ and a given k^--spread r-tuple $i_1, \ldots, i_r$, the number of k^--spread r-tuples $i'_1, \ldots, i'_r$ satisfying (5.2) is $o\left(n^s (\log n)^{r-s}\right)$. Indeed, for each $j \notin \{j_1, \ldots, j_s\}$ there are no more than $2rk^- = o(\log n)$ choices for placing i_j, while for $j \in \{j_1, \ldots, j_s\}$ "the sky is the limit".

Hence,

$$Var(X(k^-)) = \sum_{i_1,\ldots,i_r} \sum_{i'_1,\ldots,i'_r} Cov(X_{i_1,\ldots,i_r}, X_{i'_1,\ldots,i'_r}) = o\left(n^r \sum_{s=0}^{r-2} \frac{n^s (\log n)^{r-s}}{(k^-)!^{s+r-1}}\right)$$

and, by Chebyshev's inequality and (5.1),

$$\mathbb{P}(X(k^-) = 0) \leqslant \frac{Var(X(k^-))}{(\mathbb{E}X(k^-))^2} = o\left(\frac{(\log n)^r (k^-)!^{r-1}}{n^r} \sum_{s=0}^{r-2} \left(\frac{n}{(k^-)!}\right)^s\right) = o(1).$$

□

5.2. Tight and Block-Tight Twins

It turns out that the longest tight and block-tight r-twins in a random permutation have asymptotically the same length. To see the reason, let $Y(k)$ and $Z(k)$ denote, resp., the number of tight and block-tight r-twins of length k in Π_n. Then

$$\mathbb{E}Y(k) = (n - rk + 1) \times \frac{1}{r!} \binom{rk}{k, \ldots, k} \times \frac{1}{k!^{r-1}} \quad \text{and} \quad \mathbb{E}Z(k) = (n - rk + 1) \times \frac{1}{k!^{r-1}},$$

and the extra factor in $\mathbb{E}Y(k)$, counting the partitions of a block of length rk into r blocks of length k, turns out to be of a negligible order of magnitude.

We put these two results under one theorem, because they have a common proof. Indeed, as every block-tight r-twins are also tight, $Z(k) \leqslant Y(k)$, so it will be sufficient to bound $\mathbb{P}(Y(k) > 0)$ and $\mathbb{P}(Z(k) = 0)$ only. This is quite fortunate, as estimating $\mathbb{P}(Y(k) = 0)$ seems to be much harder. In fact, the estimates needed in the proof of Theorem 5.2 below become very similar to, and even easier than, those in the proof of Theorem 5.1. There is one twist, however. Since the property of possessing tight, as well as block-tight, r-twins of length k is not monotone in k, to prove the upper bound we need to estimate not just $\mathbb{E}Y(k^+)$, but $\sum_{k \geqslant k^+} \mathbb{E}Y(k)$.

Note that, roughly, the longest tight and block-tight r-twins in a random permutation are r times shorter than largest block r-twins. We have already defined function $tt^{(r)}(\pi)$ in Sect. 4. The function $btt^{(r)}(\pi)$ is defined analogously.

Theorem 5.2. *For a random permutation Π_n, a.a.s. we have*

$$tt^{(r)}(\Pi_n) = \left(\frac{1}{r-1} + o(1)\right)\frac{\log n}{\log\log n} = btt^{(r)}(\Pi_n).$$

Proof. For $k \geqslant \log n$, with $c_r = er^{r/(r-1)}$,

$$\mathbb{E}Y(k) \leqslant \frac{nr^{rk}}{k!^{r-1}} \leqslant n\left(\frac{c_r}{k}\right)^{(r-1)k} \leqslant n\left(\frac{c_r}{\log n}\right)^{(r-1)\log n} = o(n^{-1}),$$

and so

$$\sum_{\log n \leqslant k \leqslant n/r} \mathbb{E}Y(k) = o(1).$$

To deal with the lower range of k, let

$$k^+ = \left\lceil \frac{(1+\varepsilon_n)\log n}{(r-1)\log\log n}\right\rceil \quad \text{where} \quad \frac{\log\log\log n}{\log\log n} \ll \varepsilon_n = o(1).$$

Then, for every $k^+ \leqslant k \leqslant \log n$, with $c_r' = (r-1)c_r$,

$$\mathbb{E}Y(k) \leqslant n\left(\frac{c_r}{k}\right)^{(r-1)k} = n\left(\frac{c_r'}{(r-1)k}\right)^{(r-1)k} \leqslant n\left(\frac{c_r'\log\log n}{\log n}\right)^{\frac{(1+\varepsilon_n)\log n}{\log\log n}} \leqslant n^{-\varepsilon_n/2},$$

because, after taking the logarithm,

$$\log n - \frac{(1+\varepsilon_n)\log n}{\log\log n}(\log\log n - \log\log\log n - \log c_r') + \frac{\varepsilon_n}{2}\log n \to -\infty.$$

Hence,

$$\sum_{k^+ \leqslant k \leqslant \log n} \mathbb{E}Y(k) = O\left((\log n)n^{-\varepsilon_n/2}\right) = o(1)$$

and, consequently, by Markov's inequality,

$$\mathbb{P}(\exists k \geqslant k^+ :\ Y(k) > 0) \leqslant \sum_{k^+ \leqslant k \leqslant n/r} \mathbb{E}Y(k) = o(1),$$

that is, a.a.s., $btt^{(r)}(\Pi_n) \leqslant tt^{(r)}(\Pi_n) < k^+$.

We now establish a matching lower bound on $btt^{(r)}(\Pi_n)$ by the second moment method. Set $k^- = \left\lfloor \frac{\log n}{(r-1)\log\log n}\right\rfloor$. Then, $\mathbb{E}Z(k^-) \to \infty$, since

$$\mathbb{E}Z(k^-) \geqslant \frac{n}{2(k^-)!^{r-1}} \geqslant n\left(\frac{1}{k^-}\right)^{(r-1)k^-} \geqslant n\left(\frac{\log\log n}{\log n}\right)^{\frac{\log n}{\log\log n}} \to \infty,$$

because

$$\log n - \frac{\log n}{\log\log n}(\log\log n - \log\log\log n) \to \infty.$$

To bound the variance of $Z(k^-)$, for every block B of length rk in $[n]$ denote by I_B the indicator random variable that Π_n spans on B block-tight r-twins and observe that as a simple consequence of Lemma 2.1 (case $s = r$),

I_B and $I_{B'}$ are independent whenever $B \cap B' = \varnothing$. For $B \cap B' \neq \varnothing$ we will trivially bound $Cov(I_B, I_{B'}) \leqslant \mathbb{P}(I_B = 1) = (k^-)!^{-(r-1)}$. Also observe that for a fixed B the number of choices of B' satisfying $B \cap B' \neq \varnothing$ is $O(1)$. Thus,

$$Var(Z(k^-)) = O\left(\frac{n}{(k^-)!^{r-1}}\right) = \Theta\left(\mathbb{E}Z(k^-)\right)$$

and, consequently,

$$\mathbb{P}(Z(k^-) = 0) \leqslant \frac{Var(Z(k^-))}{\mathbb{E}Z(k^-)^2} = O\left(\frac{1}{\mathbb{E}Z(k^-)}\right) = o(1),$$

that is, a.a.s. $tt^{(r)}(\Pi_n) \geqslant btt^{(r)}(\Pi_n) \geqslant k^-$. □

6. A Third Point of View

So far we have considered two scenarios with respect to the three parameters: n - the length of permutation, r - the multiplicity of twins, and k - the length of twins. For the main objects of interest in this paper, namely functions $bt^{(r)}(n)$, $tt^{(r)}(n)$, etc., we fixed r, let $n \to \infty$, and asked for the largest k. When studying function $f(r,k)$ in Sect. 4.2, we fixed r and k and asked for the smallest n. In this section we consider a third "point of view", where we fix k and n (or let $n \to \infty$) and ask for the largest r.

Given k and a permutation π, let $r_{bt}^{(k)}(\pi)$ and $r_{tt}^{(k)}(\pi)$ be the largest r such that π contains block, respectively, tight r-twins of length k. (To make this parameter well defined we allow $r = 1$.) Define $r_{bt}^{(k)}(n)$ and $r_{tt}^{(k)}(n)$ as the minimum of $r^{(k)}(\pi)$, respectively, $r_{tt}^{(k)}(\pi)$ over all permutations π of $[n]$.

It follows from the pigeonhole principle (cf. the proof of Theorem 3.2) that $r_{bt}^{(k)}(n) \geqslant \lfloor (n/k - 1)/k! \rfloor + 1$, which for $k = 2$ can be pinpointed to $r_{bt}^{(2)}(n) = \lfloor (n+2)/4 \rfloor$ by considering a permutation π with $\pi(1) < \pi(2) < \pi(3) > \pi(4) > \pi(5) < \pi(6) < \pi(7) > \pi(8) \cdots$. Also, by [5, Thm. 1.2], a.a.s. $r_{bt}^{(2)}(\Pi_n) \geqslant (7/20 + o(1))n$, as one can take every other pair from each alternating sequence in Π_n.

From Propositions 4.3 and 4.4, it follows that $r_{tt}^{(k)}(n) = 1$ for $k \geqslant 3$, while from the lower bound on $f(r,2)$ in Proposition 4.5 we have $r_{tt}^{(2)}(n) = O(\sqrt{n})$.

As far as $r_{tt}^{(k)}(\Pi_n)$ is concerned we have a complete solution for $k = 2$ only. Namely, we show below that a random permutation Π_n, n even, contains a.a.s. $n/2$-twins of length 2, an optimal result comparable with the presence of a perfect matching in a graph. (In fact, we do use Hall's Theorem in the proof.) Although we do not specify it, the proof yields the existence of $n/2$-twins of length 2 similar to $(1,2)$ as well as another set similar to $(2,1)$.

Theorem 6.1. *A.a.s.* $r_{tt}^{(2)}(\Pi_n) = n/2$.

Proof. Set $n = 2r$ and consider an auxiliary bipartite graph B between $U := \{1, \ldots, r\}$ and $W := \{r+1, \ldots, 2r\}$ where $ij \in B$ if $\Pi_n(i) < \Pi_n(j)$. As $\mathbb{P}(ij \in B) = 1/2$, for a fixed $i = 1, \ldots, 2r$, the random variable $X(i) := \deg_B(i)$ has expectation $\mathbb{E}X(i) = r/2$. Similarly, for $\{i,j\} \in \binom{U}{2}$ and $\ell \in W$, or $\{i,j\} \in \binom{W}{2}$

and $\ell \in U$, we have $\mathbb{P}(i\ell \in B,\ j\ell \in B) = 1/3$. Indeed, out of 6 possible relative permutations of $\Pi(i), \Pi(j)$, and $\Pi(\ell)$, exactly 2 are such that $\Pi(i) < \Pi(\ell)$ and $\Pi(j) < \Pi(\ell)$. Thus, the random variable $Y(i,j) := |\{\ell : i\ell \in B,\ j\ell \in B\}|$, or the co-degree of i, j in B, has expectation $\mathbb{E}Y(i,j) = r/3$.

Observe that both, $X(i)$ and $Y(i,j)$, satisfy the Lipschitz condition for permutations with $c = 1$, that is, swapping around two values of a permutation changes the value of the function by at most 1. Thus, one may apply the Azuma–Hoeffding inequality for random permutations (see, e.g., Lemma 11 in [7], or Section 3.2 in [10], or Thm. 2.6 in [4]) and, using also the union bound, conclude that a.a.s. for all i, j and large r, we have

$$|X(i) - r/2| \leqslant r^{2/3} \quad \text{and} \quad |Y(i,j) - r/3| \leqslant r^{2/3}. \tag{6.1}$$

We intend to apply Hall's Marriage Theorem to B. For $S \subset U$ or $S \subset W$, let $N(S)$ be the set of neighbors of S, that is, all vertices with at least one neighbor in S. Observe that if Hall's condition, $|N(S)| \geqslant |S|$, is violated, then it is also violated by some S with $|S| \leqslant \lceil r/2 \rceil$. Indeed, take the smallest set S with $|N(S)| < |S|$ and put $s = |S|$. Say $S \subset U$. Then $|N(S)| = s - 1$ and, crucially, setting $S' = W \setminus N(S)$, we have $N(S') \subset U \setminus S$. But $|S'| = r - s + 1 > r - s = |U \setminus S| \geqslant |N(S')|$, so S' also violates Hall's condition, and by the minimality of S, $s \leqslant r - s + 1$.

Thus, with room to spare, it is enough to check Hall's condition for, say, $|S| \leqslant \frac{7}{12}r$, r large. If $S = \{i\}$, then, by (6.1), a.a.s. $|N(S)| = X(i) \geqslant 1$. If $2 \leqslant |S| \leqslant \frac{7}{12}r$, then, again by (6.1), a.a.s. for any $\{i,j\} \subset S$, we have

$$|N(S)| \geqslant |N(\{i,j\}| = X(i) + X(j) - Y(i,j) \geqslant \frac{2}{3}r - r^{2/3} > |S|.$$

Thus, a.a.s., there is in B a perfect matching which corresponds to tight r-twins of length 2 in Π_{2r}, similar to $(1,2)$. To obtain the other type, $(2,1)$, apply the same proof with the definition of B changed to $ij \in B$ if $\Pi_n(i) > \Pi_n(j)$. □

7. Concluding Remarks

We conclude with some open problems for future considerations. In Theorem 4.2 we proved that $tt^{(r)}(n) \leqslant 15r$.

Problem 7.1. Is it true that $tt^{(r)}(n) \leqslant c$ for some absolute constant c?

As mentioned after the proof of Theorem 4.2, due to the weakness of the bound on $\mathbb{P}(\mathcal{A}_K)$, one will probably need other tools than the Local Lemma. The above probability is, however, of its own interest. To extract the essence of the problem, let $n = kr$ and let $Q^{(r)}(k)$ denote the number of permutations of $[n]$ that are tight r-twins of length k. From the proof of Theorem 4.2 we know already that for large k with respect to r, $Q^{(r)}(k)/(rk)! \leqslant p_k \to 0$ exponentially fast. How about the other way around, that is, when k is fixed and $r \to \infty$?

Problem 7.2. Determine the asymptotic order of $Q^{(r)}(k)$ for every fixed $k \geqslant 2$ and $r \to \infty$.

Recall parameters $r_{bt}^{(k)}(\pi)$ and $r_{tt}^{(k)}(\pi)$ introduced in Sect. 6 and note that $\mathbb{P}(r_{tt}^{(k)}(\Pi_n) = n/k) = Q^{(n/k)}(k)/n!$. Thus, $Q^{(r)}(k) \sim (rk)!$ would mean that a random permutation Π_n a.a.s. contains r-twins (of length $k = n/r$) which cover it entirely, or $r_{tt}^{(k)}(\Pi_n) = n/k$. In Sect. 6 we proved it only for $k = 2$ (cf. Theorem 6.1).

Problem 7.3. Find asymptotic distributions of $r_{bt}^{(k)}(\Pi_n)$ and $r_{bt}^{(k)}(\Pi_n)$ for every fixed k and $n \to \infty$.

In Proposition 4.5 we showed a lower bound on $f(r, 2)$.

Problem 7.4. For $r \geqslant 4$, find an upper bound on $f(r, 2)$, or prove that $f(r, 2) = \infty$.

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