Ilustr. dowodu: $\pi_1 = 2\,8\,9\,1\,\underline{4}\,\overline{6}\,7\,5\,3$

$\pi_2 = \underline{4}\,\overline{6}\,7\,9\,2\,3\,1\,8\,5$

$\pi_3 = 9\,1\,8\,2\,7\,3\,\overline{6}\,\underline{4}\,5$

$i=4,\ j=6$, i popr. j w π_1, π_2

$l_4^{12} = 4$ (np. 4675), $l_6^{12} = 3$ (np. 675)

Uwaga Tw2 jest optymalne (ćw)

→ 11

SKOJARZENIA (UPORZĄDKOWANE)

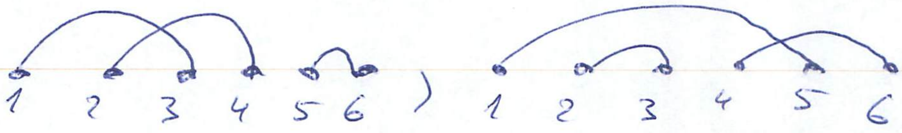
V - zb. lin. uporz., $|V| = 2n$

→ 7a

Def Skojarzenie to podział V na n par.

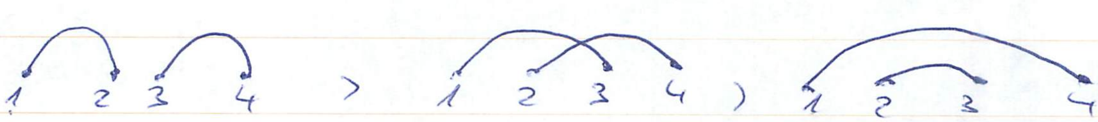
Np. $n=3$, $V=[6]$: 13, 24, 56 lub 15, 23, 46

graficznie



symbolicznie: A B A B C C A B B C A C

Ile? $\dfrac{(2n)!}{2^n n!}$, $n=3: 15$, $n=2: 3$



AA BB — linia

ABAB — fala

ABBA — stos

Ogólniej, linia, fala, stos mocy k to:

A A B B C C ... , A B C A B C ... , A B C C B A

Analogia z relacjami $<$, $>$ dla permutacji

<u>Cel:</u> analog Tw E-Sz. dla skojarzeń

Skojarzenie <u>bez stosu</u> naz. <u>krajobrazem</u>.

W krajobrazie „prawe" końce występują w tej samej kolejności co lewe końce

Np. $\underline{1}\ \underline{2}\ \ \underline{3}\ \ \overline{4}\ \overline{5}\ \ \underline{6}\ \ \overline{7}\ \overline{8}$

A B C A B D C D - przetasowanie (shuffle)

<u>Tw 3.</u> (Dudek, Grytczuk, R. ~~22+~~ '24)

Niech $l, s, w \geq 1$, $n = lsw+1$. Każde skojarzenie mocy n ($2n$ elem.) zawiera linię mocy $l+1$, ~~falę~~ stos mocy $s+1$ <u>lub</u> falę mocy $w+1$.

<u>1 dowód tw 3:</u> Niech $M = \{a_i b_i\}$, $i=1,\dots,n$ będzie skojarzeniem zb. $[2n]$, $n = lsw+1$, $a_1=1 < a_2 < \dots < a_n$; $a_i < b_i$.

Niech s_i = moc najw. stosu zacz. się w a_i,

k_i = moc najw. krajobrazu – " – – " – .

Przyp., że $\forall i: s_i \le s$, $k_i \le lw$. Wtedy

f. $f(i) = (s_i, k_i)$, $f: [n] \to [s] \times [lw]$ ma

$\le slw < n$ różnych wart., tzn. nie jest różnow.

Z drugiej strony, $\forall i<j$, rozważmy pary $a_i b_i$; $a_j b_j$.

Jeśli tworzą stos $a_i\, a_j\, b_j\, b_i$, to $s_i > s_j$. Jeśli

tworzą linię $a_i\, b_i\, a_j\, b_j$ lub falę $a_i\, a_j\, b_i\, b_j$, to

$k_i > k_j$. Zatem f jest różnowart. ↯. Tzn.

$\exists i: s_i \ge s+1$ co kończy dowód <u>lub</u> $\exists i: k_i \ge lw+1$.

W drugim przyp. wiemy tylko że $M \supset K$, $|K| \ge lw+1$.

~~Niech L – najdł. linia w K, $L = \{e_1 < e_2 < \ldots < e_q\}$ (lewe końce)~~ → 9a

~~$W_i = \{f \notin L : e_i\, f\} \cup \{e_i\}$, $i = 1, \ldots, q$.~~ Przyp., że $q \le l$.

$$\Rightarrow \frac{\sum_{i=1}^{q} |W_i|}{q} = \frac{|K|}{q} = \frac{lw+1}{q} \ge w + \frac{1}{q} \Rightarrow \exists i: |W_i| \ge w+1 \qquad \square$$

$\boxed{W_i \text{ są falami!}}$

<u>Grafy uporządkowane</u> – to grafy, których zbiory wierzch. są uporządkowane liniowo.

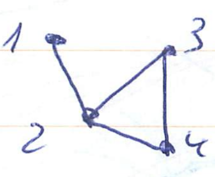
G, H, gdzie $V(G)=\{u_1<\ldots<u_k\}$, $V(H)=\{w_1<\ldots<w_k\}$ są <u>porządkowo-izomorficzne</u>, gdy $\forall\, 1\le i<j\le k$

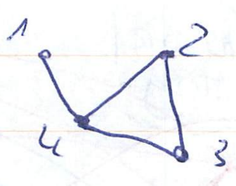
$$u_iu_j\in G \Leftrightarrow v_iv_j\in H.$$

Przypominamy, że G, H są <u>izomorficzne</u>, gdy $\exists f: V(G)\to V(H)$ (<u>izomorfizm</u>): $\forall\, 1\le i<j\le k$

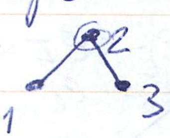
$$u_iu_j\in G \Leftrightarrow f(u_i)f(u_j)\in H.$$

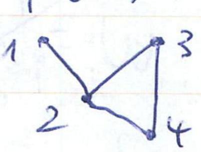
Zatem, gdy $f(u_i)=v_i$, to mamy <u>porz.-izom.</u>

<u>Pr</u> 

i 

są izom., ale nie są porz. izom., bo wsz. (oba) izomorfizmy muszą spełniać $f(2)=4$.

Liczba kopii 

w 

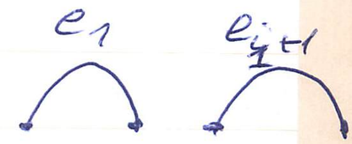
to 5, ale kopii zachow. porządek, tylko 3 (123, 124, 234; <u>nie</u> 234, 243)

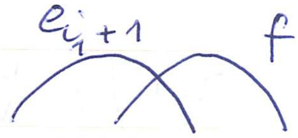
Niech $p = l\omega + 1$ oraz

$$K = \{e_1 < e_2 < \dots < e_p\}.$$

Rozkładamy K na rozłączne fale $W_1, \dots, W_q$:

$W_1 = \{f \in K:$  $\} \cup \{e_1\}$

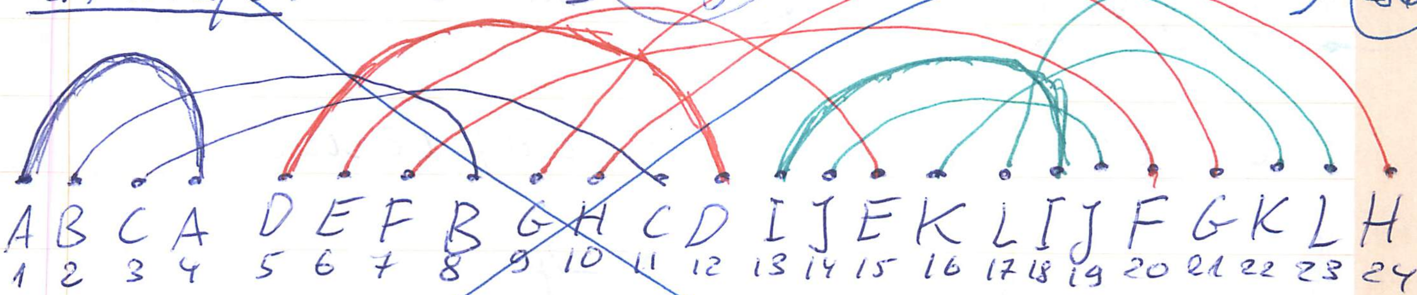
- W_1 jest falą
- niech $W_1 = \{e_1 < e_2 < \dots < e_{i_1}\} \Rightarrow$ 

$W_2 = \{f \in K:$  $\} \cup \{e_{i_1+1}\}$

itd, aż wyczerpiemy wszystkie krawędzie K.

$K = \bigcup_{i=1}^{q} W_i$; $e_1, e_{i_1+1}, \dots, e_{i_{q-1}+1}$ – linia dług. q

Ilustracja: BŁĄD → tu są! stary. → 9b



$W_1 = \{\underset{e_1}{e_A} < \underset{e_2}{e_B} < \underset{e_3}{e_C}\}$, $W_2 = \{\underset{e_4}{e_D} < \underset{e_5}{e_E} < \underset{e_6}{e_F} < \underset{e_7}{e_G} < \underset{e_8}{e_H}\}$

$W_3 = \{\underset{e_9}{e_I} < \underset{e_{10}}{e_J} < \underset{e_{11}}{e_K} < \underset{e_{12}}{e_L}\}$; $\underset{e_1}{e_A}, \underset{e_4}{e_D}, \underset{e_8}{e_I}$ – linia (najdłuższa!)

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§Appendix A. Proof of Theorem 1.1

Since the proof of Theorem 1.3 is inductive with the base step $r = 2$, we provide here, for completeness, a proof of Theorem 1.1 which differs slightly from that given in [11].

Proof of Theorem 1.1. Let M be an ordered matching consisting of edges $\{a_i, b_i\}$, $i = 1, 2, \ldots, n$, with the left ends satisfying $a_1 < \cdots < a_n$. Notice that the right ends of the edges define a permutation $\pi = (j_1, j_2, \ldots, j_n)$ accordingly to the order $b_{j_1} < b_{j_2} < \cdots < b_{j_n}$.

By the original Erdős-Szekeres theorem this permutation contains either a decreasing subsequence of length $s + 1$ or an increasing subsequence of length $p = w\ell + 1$. In the former case we are done, since any such decreasing subsequence corresponds to a stack. In the latter case we get a sub-matching L with p edges whose right ends come in the same order as the left ends. We call L *a landscape* (see Fig. A.1). Notice that no pair of edges in a landscape may form a nesting.

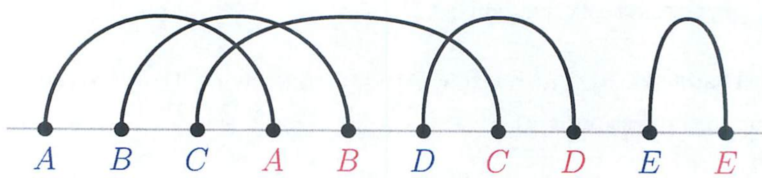

Figure A.1. A landscape with five edges.

Let us order the edges of L as $e_1 < e_2 < \cdots < e_p$, accordingly to the linear order of their left ends. Decompose L into edge-disjoint waves, $W_1, W_2, \ldots, W_k$, in the following greedy way. For the first wave W_1, pick e_1 and all edges whose left ends are between the two ends of e_1, say, $W_1 = \{e_1 < e_2 < \ldots < e_{i_1}\}$, for some $i_1 \geqslant 1$. Clearly, W_1 is a genuine wave since there are no lines (and no nestings) in W_1. Also notice that the edges e_1 and e_{i_1+1} form an alignment, since otherwise the latter edge would be included in W_1.

96

Now, we may remove the wave W_1 from L and repeat this step for $L - W_1$ to get the next wave $W_2 = \{e_{i_1+1} < e_{i_1+2} < \ldots < e_{i_2}\}$, for some $i_2 \geqslant i_1 + 1$. We iterate this procedure until there are no edges of L left. Let the last wave be $W_k = \{e_{i_{k-1}+1} < e_{i_{k-1}+2} < \ldots < e_{i_k}\}$, with $i_k \geqslant i_{k-1} + 1$. Clearly, the sequence $e_1 < e_{i_1+1} < \ldots < e_{i_{k-1}+1}$ of the leftmost edges of the waves W_i, $i = 1, \ldots, k$, forms a line (see Figure A.2).

ILUSTRACJA

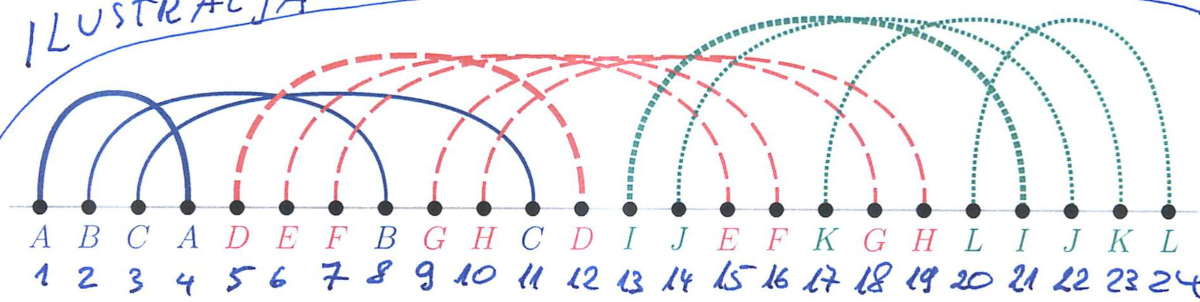

FIGURE A.2. Greedy decomposition of a landscape into waves. The leftmost edges of the waves, forming a line, are bold.

If $k \geqslant \ell + 1$, then we are done. Otherwise, $k \leqslant \ell$, and, since $p = \ell w + 1$, some wave W_i must have at least

$$\frac{p}{k} = \frac{\ell w + 1}{\ell} > w$$

edges. This completes the proof. $\square$

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$W_1 = \{e_A < e_B < e_C\}$, $W_2 = \{e_D < e_E < e_F < e_G < e_H\}$

$W_3 = \{e_I < e_J < e_K < e_L\}$

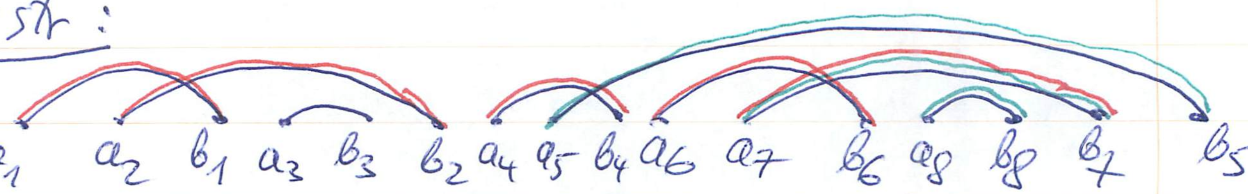
W_1, W_2, W_3 - fale

$\{e_A, e_D, e_I\}$ - najdłuższa linia

2. dowód tw. 3: $M=\{a_i<b_i\}$, $1 = a_1<\ldots<a_n$

Indeksy prawych końców par $(a_i;b_i)$ tworzą perm. π $[n]$.
Na podst. Tw 1, π zawiera podc. rosn. dług. $lw+1$ lub podc. mal. dł. $s+1$. W drugim przyp. ten podciąg gen. w M stos dł. $s+1$. W pierwszym – ~~fala~~ krajobraz dł. $lw+1$. Dalej jak w 1. dow. $\square$

Ilustr:



$$\pi = \bar{1}\ 3\ \bar{2}\ \bar{4}\ \bar{6}\ \underline{8}\ \underline{\bar{7}}\ \underline{5}$$

[np. $s_2=2$, $s_3=1$, $s_5=3$, $k_1=5$, $k_6=2$]

Uwaga 2 Tw 3 jest optymalne, tzn. nie jest prawdą dla $n=lsw$ (ćw)

Wn 3 Każde skojarzenie zb. $[2n]$ zawiera linię, stos lub falę dł. $\lceil n^{1/3}\rceil$. (Ćw)