$$\mathbb{E}X_k = \frac{1}{2}\binom{n}{2k}\binom{2k}{k}\frac{1}{k!} = \frac{n!}{2k!\,2(n-2k)!\,k!} =$$

$$= \frac{n(n-1)\dots(n-2k+1)}{2k!^3} < \frac{n^{2k}}{2k!^3} < \left(\frac{e^3 n^2}{k^3}\right)^k \to 0,$$

gdy $k \geq cn^{2/3}$, $c > e$, $k_0 = \lceil cn^{2/3} \rceil$

$P(X_{k_0} \geq 1) \leq \mathbb{E}X_{k_0} \to 0 \Rightarrow$ pnp. $t(\Pi_n) < k_0$ □

Zmiana przestrzeni: $\mathcal{M}_n = (\Omega_n, P_n)$,

Ω_n - zb. wsz. $\frac{(2n)!}{n!\,2^n} := d_n$ skojarzeń zb. $[2n]$

Tw ~~6~~ 5. Pnp. $\mathcal{M}_n$ zawiera linie, stosy i fale wielkości $\Theta(\sqrt{n})$

~~25.10~~ ~~8.11~~

dowód. Najpierw z góry.

~~Niech $(M_k)_1^\infty$ - ciąg dow. skojarzeń rzędu k~~

$X_k = \#$ kopii linii mocy k w $\mathcal{M}_n$

$$\forall S \in \binom{[2n]}{2k}: I_S = \begin{cases} 1 & \mathcal{M}_n \text{ ma kopię } M_k \text{ na } S \\ 0 & \text{w przeciwnym wypadku} \end{cases}$$

$$\mathbb{E}X_k = \binom{2n}{2k} \cdot 1 \cdot \frac{\alpha_{n-k}}{\alpha_n} = \frac{2^k}{(2k)!}\,\frac{n!}{(n-k)!} \leq \left(\frac{e^2 n}{2k^2}\right)^k \to 0$$

gdy $k \geq c\sqrt{n}$, $c > e/\sqrt{2}$; $k_0 = \lceil c\sqrt{n}\,\rceil$.

Ale $X_k = 0 \not\Rightarrow X_{k+1} = 0$ (w ogólności; dla linii, stosów, fal mamy $\Rightarrow$)

$$P(\exists k \geq k_0 : X_k > 0) \leq \sum_{k=k_0}^{n} P(X_k \geq 1) \leq$$

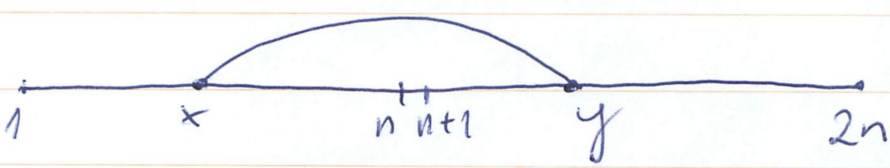
$$\leq \sum_{k=k_0}^{n} \mathbb{E}X_k \leq \sum_{k=k_0}^{n} \left(\frac{e^2 n}{2k^2}\right)^k \leq \sum_{k=k_0}^{n} \left(\frac{e^2 n}{2k_0^2}\right)^k$$

$$\leq \sum_{k=k_0}^{n} \underbrace{\left(\frac{e^2}{2c^2}\right)}_{<1}^k \leq n \left(\frac{e^2}{2c^2}\right)^{c\sqrt{n}} \to 0$$

To samo dla stosów i fal.

Z dołu (tylko fale; stosy na ćw; linie opuszczamy)

Niech $\mathcal{M}_n' = \{xy \in \mathcal{M}_n : x \in [n],\ y \in [2n] \setminus [n]\}$



$$X = |\mathcal{M}_n'|, \quad X = \sum_{\substack{1 \leq x \leq n \\ n+1 \leq y \leq 2n}} I_{xy}, \quad I_{xy} = \begin{cases} 1 & xy \in \mathcal{M}_n \\ 0 & \end{cases}$$

$$P(I_{xy} = 1) = \frac{1}{2n-1} \Rightarrow \mathbb{E}X = \frac{n^2}{2n-1} \underset{(>)}{\sim} \frac{n}{2}$$

Nier. Czebyszewa $P(|X-EX|\ge \varepsilon EX) \le \frac{Var\, X}{\varepsilon^2 (EX)^2}$

$$= \frac{1}{\varepsilon^2}\left(\frac{E(X(X-1))+EX-(EX)^2}{(EX)^2}\right) = \frac{1}{\varepsilon^2}\left(\frac{E(X(X-1))}{(EX)^2}+\frac{1}{EX}-1\right)$$

$$E(X(X-1)) = \sum_{\substack{1\le x\le n\\ n+1\le y\le 2n}} \sum_{\substack{1\le x'\le n\\ n+1\le y'\le 2n\\ x'\ne x,\ y'\ne y}} E(I_{xy} I_{x'y'}) = \frac{n^2(n-1)^2}{(2n-1)(2n-3)}$$

$$\frac{E(X(X-1))}{(EX)^2} - 1 = \frac{\cancel{n^2}(n-1)^2}{\cancel{(2n-1)}(2n-3)}\cdot\frac{(2n-1)^{\cancel{2}}}{n^{\cancel{4}\,2}} - 1 = \dots = O\left(\frac{1}{n}\right) \to 0$$

$$\Rightarrow P(|X-EX|\ge\varepsilon EX)\to 0 \quad \forall\, \varepsilon>0.$$

$\Rightarrow$ pnp $X \ge (1-\varepsilon)EX \ge (1-\varepsilon)\frac{n}{2}$

~~Z Tw 3 ⇒~~ Z 1. części dowodu (z góry):

pnp. M_n (a więc tym bardziej M_n') nie ma stosu rozmiaru $k_0 = \lceil c\sqrt{n}\rceil$, $c > \frac{e}{\sqrt{2}}$

Stos. Tw 4 z $l=1$, $s=k_0-1$, $w=\left\lfloor\frac{X-1}{k_0-1}\right\rfloor$,

do M_n': $X \ge w(k_0-1)+1 = lsw+1$, skąd

w M_n' (a więc i w M_n) $\exists$ fala mocy

$w+1 > \frac{X-1}{k_0-1} = \Omega(\sqrt{n})$ [bo w M_n' nie ma linii ≥ 2] $\square$