Discrete Mathematics 2

Problem set #4

Due: Wednesday, December 12

1. For n even, $n \geq 2$, find an intersecting family of subsets of X, $|X| = n$, of size 2^{n-1} which is not $\mathcal{F}_x$ for any $x \in X$.

2. Prove that every maximal intersecting family of subsets of X, $|X| = n$, is increasing and has size 2^{n-1}. Here *maximal* means that for every $A \notin \mathcal{F}$, $\mathcal{F} \cup \{A\}$ is not intersecting, while *increasing* means that $\mathcal{F}$ is closed under taking supersets.

3. Show that for every $x \in X$, the family $\mathcal{F}_x$ is maximal.

4. Let $\mathcal{F}$ be a maximal family of subsets of X, $|X| = n$, such that whenever $A, B \in \mathcal{F}$ then $A \cup B \neq X$. Prove that $\mathcal{F}$ is decreasing (i.e., closed under taking subsets) and has size 2^{n-1}.

5. If $\mathcal{F} \subset \binom{X}{\leq r}$ is an intersecting Sperner System, $r \leq n/2$, then $|\mathcal{F}| \leq \binom{n-1}{r-1}$.

6. Fix $1 \leq k \leq n \leq 2k$ and show that if $\mathcal{F}$ is an SS consisting of proper subsets of X, each of size at least k, and not containing two sets whose union is X then $|\mathcal{F}| \leq \binom{n-1}{k}$.

7. Complete the 1st proof of Erdős-Ko-Rado Theorem by showing that for every cyclic order σ at most r sets in $\mathcal{F}$ form intervals of σ.

8. Show that for all $\mathcal{A} \in \binom{X}{r}$
$$|\partial_{r-1}\mathcal{A}| \geq \frac{\binom{n}{r-1}}{\binom{n}{r}}|\mathcal{A}|.$$

9. With notation from the 2nd Proof of the E-K-R Theorem, prove Observation 1, that is, that no set of $\mathcal{G}_1$ is contained in any set of $\mathcal{G}_0$.

10. If $1 \leq s < r < n$ and $\mathcal{F} \subset \binom{X}{r}$ is such that for every $A, B \in \mathcal{F}$, $A \neq B$, we have $|A \cap B| \leq s$ then
$$|\mathcal{F}| \leq \frac{\binom{n}{s+1}}{\binom{r}{s+1}}.$$