Discrete Mathematics 2

Problem set #3

Due: Wednesday, November 15

1. Show that every family $\mathcal{F}$ satisfying the property in the definition of $\alpha_r(x_1, \ldots, x_n)$ is $r - SS$.

2. Prove the Dual Dilworth Theorem: In every finite poset X, the size of a largest chain equals the smallest number of antichains which cover X.

   Hint: use the notion of the rank of an element:

   $$r(x) = \max\{|L| - 1 : \quad L \in \mathcal{L}_x\},$$

   where $\mathcal{L}_x$ is the set of all chains with the largest element x.

3. Prove the Dilworth Lemma: For positive integers a, b, every poset X of size $|X| \geq ab + 1$ contains either a chain of size $a + 1$ or an antichain of size $b + 1$.

4. Prove the Erdős-Szekeres Theorem (1935): every sequence of $rs + 1$ real numbers contains a non-decreasing subsequence of length $r + 1$ or a non-increasing sequence of length $s + 1$.

5. Deduce Corollary 1 from Theorem 2, that is, show how the Aharoni-Haxell (2000) hypergraph extension of Hall's Theorem implies its defect (deficiency) form.

6. Deduce Hall's Theorem from Theorem 2.

7. Let $f(r)$ be the smallest number of edges in an (r, r)-graph with $\nu(H) = 1$ and $\tau(H) \geq r - 1$. Prove that (a) $f(r) \geq 2r - 3$, (b) $f(4) = 6$.