Discrete Mathematics 2

Problem set #2

Due: Wednesday, November 7

The old stuff

1.7 Let G be a bipartite graph with bipartition (V_1, V_2) and let A be the set of vertices of maximum degree.

(a) Show that there is a matching saturating $A \cap V_1$.

(b) Deduce from part (a) and Problem 4 that G contains a matching saturating A.

1.8 *An $r \times s$ Latin rectangle based on* $[n]$ is an $r \times s$ matrix A such that each entry belongs to $[n]$ and each integer from $[n]$ occurs in each row and column at most once.

(a) Prove that every $r \times n$ Latin rectangle can be extended to an $n \times n$ Latin square.

(b) Show that an $r \times s$ Latin rectangle can be extended to an $n \times n$ Latin square iff for each $i = 1, \ldots, n$ occurs in A at least $r + s - n$ times.

The new stuff

1. Prove the following reformulation of Cor. 4: for all $r \le \lceil n/2 \rceil$ there is a surjection $f_r : \binom{X}{r} \to \binom{X}{r-1}$ such that $A \supset f_r(A)$, while for every $r \ge \lfloor n/2 \rfloor$ there is a surjection $g_r : \binom{X}{r} \to \binom{X}{r+1}$ such that $A \subset g_r(A)$.
2. Show that Theorem 2 (LYM Inequality) implies Theorem 1 (Sperner).
3. Prove that if an SS $\mathcal{F}$ consists of sets of size at most k only, $k \le n/2$, then $|\mathcal{F}| \le \binom{n}{k}$.
4. What is the largest size of an SS with at least one set of size at most 2, at least one set of size at least $n-2$, and no sets of size i, for any $3 \le i \le n-3$.
5. For an integer $s \ge 1$, let $\mathcal{F}$ be an s-Sperner System (s-SS), that is, $\mathcal{F}$ does not contain any chain of length $s+1$. Set $a_k = \left|\mathcal{F} \cap \binom{X}{k}\right|$. Prove that

$$\sum_{k=0}^{n} a_k \binom{n}{k}^{-1} \le s.$$

6. Show that the LYM inequality becomes equality iff $\mathcal{F} = \binom{X}{k}$ for some $k \in [n]$.
7. For a positive integer r and a sequence of real numbers $(x_1, \ldots, x_n)$, satisfying $|x_i| \ge 1$, $i = 1, \ldots, n$, let

$$\alpha_r(x_1, \ldots, x_n) = \max_{\mathcal{F} \subset \mathcal{P}(n)} \{|\mathcal{F}| : \forall A, B \in \mathcal{F}, A \ne B, \quad |x_A - x_B| < r\}.$$

Show that $\alpha_r(x_1, \ldots, x_i, \ldots, x_n) = \alpha_r(x_1, \ldots, -x_i, \ldots, x_n)$

8. Let $y_A = \sum_A x_i - \sum_{A^c} x_i$. Show that

$$\alpha_r(x_1, \ldots, x_n) = \max_{\mathcal{F} \subset \mathcal{P}(n)} \{|\mathcal{F}| : \exists x \quad \forall A \in \mathcal{F} \quad y_A \in (x - r, x + r)\}.$$