Discrete Mathematics 2

Problem set #1

Due: Wednesday, October 10

1. Prove the Defect Form of Hall's Theorem (Cor. 1)
2. Prove the Polyandric Form of Hall's Theorem (Cor. 2) and reformulate it in terms of bipartite graphs.
3. Let A be an $n \times n$ matrix. Prove that A has n 1's such that each row and each column contains precisely one of them iff any k rows contain 1's in at least k columns.
4. Let G be a bipartite graph with bipartition (V_1, V_2) and let k be a fixed integer. Suppose that each vertex of V_1 has degree at least k, while each vertex of V_2 has degree at most k. Show that G has a matching saturating V_1. Deduce that every bipartite, regular graph contains a perfect matching.
5. *A 2-factor* of a graph is a 2-regular spanning subgraph, that is, a union of disjoint cycles covering all the vertices. Show that every regular graph of positive even degree has a 2-factor (Petersen, 1891).
6. Let k be a positive integer. Show that any two partitions of a finite set into k-element sets admit a common SDR.
7. Let G be a bipartite graph with bipartition (V_1, V_2) and let A be the set of vertices of maximum degree.

   (a) Show that there is a matching saturating $A \cap V_1$.

   (b) Deduce from part (a) and form Problem 4 that G contains a matching saturating A.
8. *An $r \times s$ Latin rectangle based on* $[n]$ is an $r \times s$ matrix A such that each entry belongs to $[n]$ and each integer from $[n]$ occurs in each row and column at most once.

   (a) Prove that every $r \times n$ Latin rectangle can be extended to an $n \times n$ Latin square.

   (b) Show that an $r \times s$ Latin rectangle can be extended to an $n \times n$ Latin square iff for each $i = 1, \ldots, n$ occurs in A at least $r + s - n$ times.