Graph Theory II

Problem Set (correction to Problem 5, hints) #9

due: Tuesday, June 6

1. Determine the value of $ex(n, K_{1,r})$ for all n and r. Hint: It's easy to find an upper bound. A bit harder that it can be achieved for all r and n (all parities have to be considered).

2. Show that $t_{r-1}(n) \leqslant \frac{1}{2}n^2\frac{r-2}{r-1}$, with equality whenever $(r-1)|n$. Hint: Express $n = (r-1)k+i$, $0 \leqslant i < r-1$. Then express $t_{r-1}(n)$ in terms of r, n and i.

3. Show that $t_{r-1}(n)/(n^2/2) \to \frac{r-2}{r-1}$ as $n \to \infty$. Hint: Use the bound $t_{r-1}(n) \geqslant t_{r-1}\left((r-1)\lfloor\frac{n}{r-1}\rfloor\right)$.

4. For $0 < s \leqslant t \leqslant n$, let $z(n,s,t)$ be the maximum number of edges in a bipartite graph with $n+n$ vertices not containing a copy $K_{s,t}$. Show that
$$2ex(n, K_{s,t}) \leqslant z(n,s,t) \leqslant ex(2n, K_{s,t}).$$
Hint: For the lower bound double the vertex set to obtain a bipartite graph with twice as many edges and still no copy of $K_{s,t}$

5. Let $1 \leqslant r \leqslant n$ be integers and G be a bipartite graph with bipartition $\{A, B\}$, $|A| = |B| = n$ and not containing $K_{r,r}$. Show that
$$\sum_{v\in A}\binom{deg_G(v)}{r} \leqslant (r-1)\binom{n}{r}$$
and deduce from this inequality that $ex(n, K_{r,r}) \leqslant c_r n^{2-1/r}$. Hint: Count the number of pairs (x, Y) with $x \in A$, $Y \subseteq B$, $|Y| = r$, and x connected to all vertices of Y. For the second part use $(s/t)^t \leqslant \binom{s}{t} \leqslant s^t$ and that the function $z \to z^r$ is convex.

6. Given a tree T, find an upper bound on $ex(n, T)$ that is linear in n and independent of the structure of T, i.e. depends only on $|T|$. Hint: Prove and combine two facts: 1) every graph G contains a subgraph H with $\delta(H) \geqslant ||G||/|G|$; 2) every graph H with $\delta(H) \geqslant t-1$ contain every tree on t vertices.

7. Erdős and Sós (1963) conjectured that for every tree T and all n, $ex(n, T) \leqslant \frac{1}{2}(|T|-2)n$. Show that this conjecture is best possible in the sense that for every k and infinitely many n, there is a graph on n vertices and $\frac{1}{2}(k-2)n$ edges which contains no tree on k vertices. Hint: easy.

8. Prove the Erdős-Sós conjecture in the case when T is a star. Hint: easy.