Graph Theory II

Problem Set - correction (in Problems 4, 5c, 6, 7) #8

due: Tuesday, June 6

1. Show that every pair (X, Y) which is ϵ-regular in G is also ϵ-regular in the complement $\bar{G}$ of graph G.

2. If (A, B) jest ϵ-regular with density d and $Y \subseteq B$, $|Y| \geqslant \epsilon|B|$, then all but at most $\epsilon|A|$ vertices $v \in A$ have each at least $(d-\epsilon)|Y|$ neighbors in Y.

3. Let $G = (X, Y, E)$, $n = |X| = |Y|$, be a bipartite ϵ-regular graph with density $d_G(X, Y) = d > 2\epsilon$. Show that if $\delta(G) \geqslant \epsilon n$, then G has a perfect matching. Can the assumption of ϵ-regularity be weakened?

4. Given are 3 sets X, Y i Z, all of size n. Show that if $G_1 = (X, Y, E_1)$, $G_2 = (X, Z, E_2)$, and $G_3 = (Y, Z, E_3)$ are ϵ-regular bipartite graphs, each of density d, then the number of triangles T in the union of these graphs $G_1 \cup G_2 \cup G_3$ satisfies the inequalities

$$(1-2\epsilon)(d-\epsilon)^3 n^3 < T < \left[2\epsilon d + (d+\epsilon)^3\right] n^3.$$

   In addition, consider the version where only G_2 and G_3 are ϵ-regular (and all 3 have density d).

5. Let $G = (X, Y, E)$ be an ϵ-regular bipartite graph of density $d_G(X, Y) = d$. Show that

   (a) if $d > 2\epsilon$, then there exists a subset $A \subseteq \binom{X}{2}$ of size $|A| \geqslant (1-6\epsilon)\binom{|X|}{2}$ such that for all pairs of vertices $u, v \in A$ we have

$$(d-\epsilon)|Y| \leqslant \deg u, \deg v \leqslant (d+\epsilon)|Y|$$

   and

$$(d-\epsilon)^2|Y| \leqslant \deg(u, v) \leqslant (d+\epsilon)^2|Y| \ ;$$

   (b) if $A \subseteq X$, $B \subseteq Y$, $|A| > \eta|X|$, and $|B| > \eta|Y|$, where $\eta \leqslant 1/2$, then subgraph $G[A \cup B]$ of graph G induced by the sets A i B is ϵ/η-regular;

   (c) if $E' \subseteq E$, $|E'| = \eta|E|$, then subgraph $G - E' = (X, Y, E - E')$ is $(\epsilon + \eta\frac{d}{\epsilon^2})$-regular.

6. Let $G = (X, Y, E)$, $n = |X| = |Y|$ be an ϵ-regular bipartite graph of density $d_G(X, Y) = d$. Let $N(S) = \bigcap_{v \in S} N_G(v)$ be the set of all *common* neighbors of the vertices from a set $S \subseteq X$. We say that S is *good*, if

$$(d-\epsilon)^{|S|} n \leqslant |N(S)| \leqslant (d+\epsilon)^{|S|} n \ .$$

   Fix n, k such that $n \geqslant 3(k-1)$. Show that if $\epsilon \leqslant (d-\epsilon)^k$, then

   (a) every good set S of size k is contained in at most $2\epsilon n$ bad (= not good) sets of size $k+1$;

   (b) all k-element sets $S \subseteq X$, except at most $3\epsilon k\binom{n}{k}$, are good.

   Hint for part (b): induction on k and double counting.

7. Show that if a graph G on n vertices possesses an ϵ-regular partition $(V_0, V_1, ..., V_k)$, where $|V_0| < \epsilon n$ and $\epsilon \leqslant 1/9$, then it also possesses a $3\sqrt{\epsilon}$-regular partition $(V_0', V_1', ..., V_k')$, where $|V_0'| \leqslant k-1$.