Graph Theory II

Problem Set #2

due: Wednesday, March 15

1. Prove that for any graph G, $\tau(G) \leqslant 2\nu(G)$ and $\tau(G) = n - \alpha(G)$.
2. Show that in every (undirected) graph G there is a path cover of size at most $\alpha(G)$.
3. Let $\mathcal{C} = \{K_1, K_2, C_3, C_4, ...\}$. Prove that in every graph G there is a $\mathcal{C}$-cover of size at most $\alpha(G)$. Hint: Find a subgraph H of G isomorphic to a member of $\mathcal{C}$ and containing a vertex not adjacent to any vertex of $G - V(H)$. Then apply induction on $|G|$.
4. Deduce Hall's Theorem from Gallai-Milgram Theorem. Hint: Direct all the edges from A to B.
5. Deduce from Gallai-Milgram Theorem that every tournament contains a (directed) Hamiltonian path.
6. Show that every graph G with $\delta(G) \geqslant 2$ contains a cycle longer than $\delta(G)$. Hint: Consider a longest path in G and one of its endpoints.
7. Show that if G is connected, then every two longest paths in G have a common vertex.
8. Prove that, for every $k \geqslant 1$, if a connected graph G has at least $2k+1$ vertices and $\delta(G) \geqslant k$, then G contains a path of length at least $2k$. Hint: Consider a longest path P in G and both its endpoints; create a cycle on the vertex set $V(P)$; by connectivity, if the cycle is not Hamiltonian, one can find a path longer than P – a contradiction.
9. Prove that if a graph G has $n \geqslant 5$ vertices and $\alpha(G) < 3$, then G contains a cycle of length at least $n/2$. Hint: Apply induction on n. Check $n = 5$ and $n = 6$ using known facts from Ramsey Theory. For $n \geqslant 7$, remove some two vertices, apply induction hypothesis for $n - 2$ (obtaining a cycle C' of length $\geqslant (n-2)/2$) and construct a cycle C by adding to C' one or both removed vertices.