Graph Theory II

Problem Set #1

due: Wednesday, March 8

1. Show that every regular bipartite graph has a 1-factor and, thus, also a 1-factorization, that is, a decomposition of its edge set into disjoint 1-factors.
2. For each $k \geqslant 2$, give an example of a k-regular graph without a 1-factor. (Begin with small k.)
3. Show that the Petersen graph does not have a 1-factorization.
4. Show that every bipartite graph contains a matching saturating all vertices of maximum degree.
5. Deduce Hall's Theorem from Tutte's Theorem.
6. Find set S from Theorem 3 for the following graphs: n-vertex path P_n, cycle C_n, clique K_n, bipartite complete graph $K_{n,m}$, any tree.
7. Prove Petersen's Theorem from 1891 stating that every bridgeless cubic graph has a 1-factor. Hint: Use Tutte's Theorem and double counting technique.
8. Show that
$$\nu(G) \geqslant \frac{|E(G)|}{2\Delta(G) - 1}.$$
(Recall that $\nu(G)$ denotes the size of the largest matching in G, while $\Delta(G)$ – the maximum vertex degree.)
9. Prove the following defect form of Tutte's Theorem:
For all $1 \leqslant k \leqslant n/2$, every n-vertex graph G has a matching of size k if and only if for all $S \subseteq V(G)$,
$$q(G - S) \leqslant |S| + (n - 2k).$$
Hint: For the nontrivial implication, apply Tutte's Theorem to the graph $G * K_{n-2k}$ obtained by joining every vertex of G to every vertex of a clique K_{n-2k} (the clique is vertex disjoint from G). Alternatively, apply Theorem 3 (see Remark 3).