A small non-4-choosable planar graph

MARYAM MIRZAKHANI

Department of Mathematical Sciences
Sharif University of Technology
P.O. Box 11365–9415
Tehran, Iran
Member of Iranian
Mathematics Olympiad Team
m_khani@irearn.bitnet

Abstract

We give a smaller example of a graph, with 63 vertices, which shows that, planar graphs are not necessarily 4-choosable. This was conjectured in 1979 by P. Erdös, A. L. Rubin, and H. Taylor. The earlier example by Margit Voigt given in 1993 has 238 vertices. Our example also shows that there exists 3-colorable planar graph which is not 4-choosable, a counterexample to a claim made by T. Jensen.

A *list coloring* of a graph G is an assignment of colors to the vertices such that adjacent vertices receive distinct colors and such that each vertex v receives a color from a prescribed list $L(v)$ of colors. A graph G is *k-choosable* if such a coloring always exists provided that each $L(v)$ has k colors.

In 1979 Erdös, *et al.* [2] conjectured that every planar graph is 5-choosable, but not necessarily 4-choosable. Recently, Voigt [6] gave an example of a planar graph with 238 vertices which is not 4-choosable and Thomassen [4] proved that every planar graph is 5-choosable. In [6] the question was raised whether that graph is the smallest example or not. In the following first, we give an example of a graph with 69 vertices, which can easily be checked that is not 4-choosable.

Proposition. *Let H be the following graph with the list of colors given in Figure 1. Then H, does not have a list coloring with the given list of colors.*

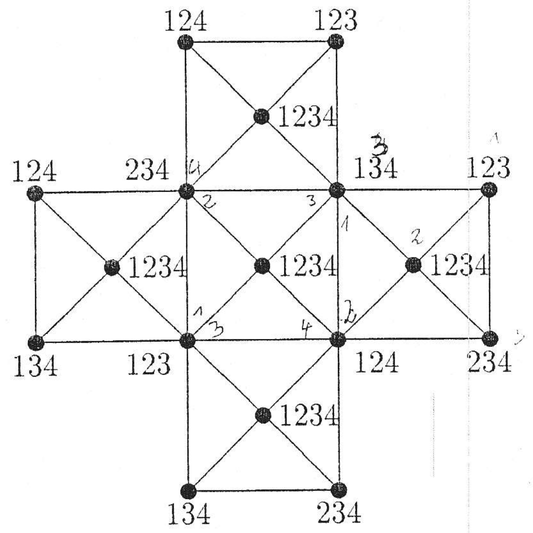
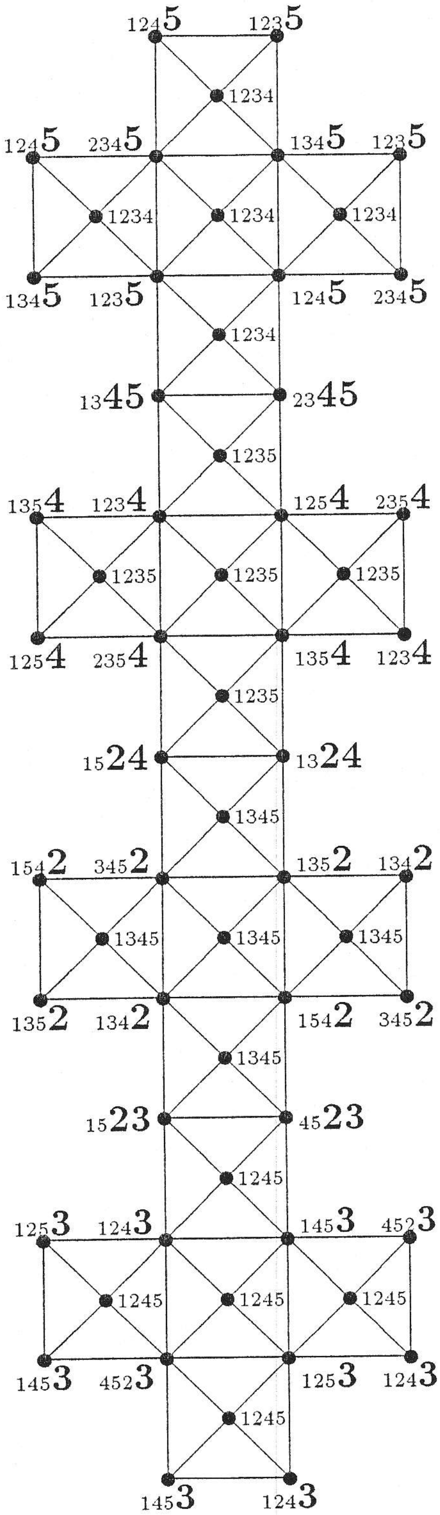


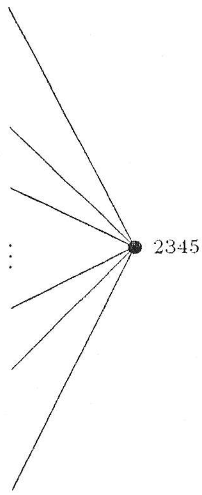
Figure 1

Proof. First we note that $|V(H)| = 17$, and the "inner" vertices have a list of 4 colors each, and the list of "outer" vertices have 3 colors each. The graph H does not have a list coloring with the given list of colors. There are 5 squares in H with horizontal and vertical sides. We denote them by T, R, B, L, and C, for top, right, bottom, left, and center squares respectively. Since the color of inner vertex in each of these squares is either 1, 2, 3, or 4, therefore in any list coloring with the given list of colors, each of these squares has the property that at least two opposite vertices in the corners receive the same color. We call this property as *Property P*. We claim that in any list coloring of H, in square C, either the color of the upper right vertex (URVC) is 1 or the color of the upper left vertex (ULVC) is 2. For, suppose in contrary, neither the color of (URVC) is 1 nor the color of (ULVC) is 2; then we have two cases:
Case (i). The color of (URVC) is 3, then using Property P in T, the color of (ULVC) must also be 3, which is impossible. ✓
Case (ii). The color of (URVC) is 4, then using Property P in C, the color of (ULVC) must be 4, which is also impossible. ✓
If in C the color of (URVC) is 1, then using Property P in square R, the color of the lower right vertex in C must be 2; thus by using Property P in square B the color of the lower left vertex in C must be 3. This by using Property P in L, results that the color of the upper left vertex in C (ULVC) is 4, which is inconsistent with the Property P of C. If in C the color of (ULVC) is 2, similar to the above argument, it leads to a contradiction. ✓ □

Now let H_1, H_2, H_3, and H_4 be 4 disjoint copies of H, such that the color $4+i$, $i = 1, 2, 3, 4$ is added to the list of each outer vertex in H_i; i.e. the vertices whose lists have 3 colors so far. Now, we add a new vertex v with list of colors $\{5, 6, 7, 8\}$ and join it to all of the outer vertices in each H_i. The resulting graph is obviously planar and not 4-choosable.

One can make this example even smaller, to a graph with 63 vertices, by making 6 vertices of H_i's overlap each other. See the following figure:





The following is an open problem stated in [3], (open problem 2.13 page 46), also see [5].

■ *Is every 3-colorable planar graph 4-choosable?*

Our example shows that the answer is negative.

This graph is also a counterexample for Research Problem number 172 in [1], which is to prove or disprove,

$$\#(G) \leq h(G),$$

for every graph G; where $\#(G)$ is the *choice number* of G, defined as

$$\#(G) = \min\{k \mid G \text{ is } k\text{-choosable}\},$$

and $h(G)$, the *Hadwiger number* of G, is the maximum number of vertices of a complete graph to which G can be contracted.

I wish to thank S. Akbari, who introduced the problem and he is the one who mentioned that it is a counterexample to the problem in [3]. I also thank Roya Beheshti, who made my earlier example of 81 vertices smaller to the above example with 69 vertices. And my thank goes to Roya Beheshti, M. Mahdian, and M. R. Salavatipour, all from the Iranian Mathematics and Informatics Olympiad teams, who independently, reduced the earlier examples to the one with 63 vertices.

References

[1] M. BOROWIECKI, *Research problem* 172, Discrete Math., **121** (1993), pp. 235–237.

[2] P. ERDÖS, A. L. RUBIN, AND H. TAYLOR, *Choosability in graphs*, in Proceedings, West Coast Conference on Combinatorics, Graph Theory and Computing, Arcata, CA, Sept. 5–7, 1979, no. **26** in Congr. Numer.

[3] T. R. JENSEN AND B. TOFT, *Graph Coloring Problem*, Wiley, New York, 1995.

[4] C. THOMASSEN, *Every planar graph is 5-choosable*, J. Comb. Theory B, **62** (1994), 180–181.

[5] ——, 3-*list-coloring planar graphs of girth 5*, J. Comb. Theory B, **64** (1995), 101–107.

[6] M. VOIGT, *List colourings of planar graphs*, Discrete Math., **120** (1993), 215–219.