Coloring vertices

Def A vertex-coloring of $G=(V,E)$ is a map $c: V \to S$: $\forall\, uv \in E$, $c(u) \neq c(v)$ (el. of S – colors)

The smallest k : G has a coloring $c: V \to [k]$ is called the chromatic number of G.

Notation $\chi(G)$

A graph G is k-colorable if $\chi(G) \leq k$

Obs A coloring is a partition of $V(G)$ into independent sets called color-classes

E.g. bipartite = 2-colorable (incl. K_n^c).

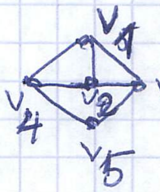
Prop 1. If $|E(G)|=m$, then $\chi(G) \leq \frac{1}{2} + \sqrt{2m+\frac{1}{4}}$.

Proof If $k=\chi(G)$, then $m \geq \frac{1}{2}k(k-1)$ (at least one edge between any pair of color classes – otherwise, we could combine them into one color class) □

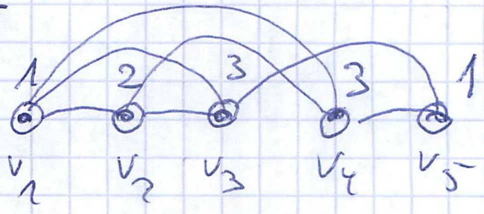
Greedy (on-line) coloring: order vertices $v_1, \dots, v_n$ and color them one by one (from left to right) by the 1st available color.

This shows that $\chi(G) \leq \Delta(G)+1$

Can we do better? Ex.



$\Delta = 3$



$\chi = 3$

Def. The coloring # $col(G) = \min\{k : \exists (v_1, \dots, v_n) : d_{G[v_1,\dots,v_i]}(v_i) < k\}$

Prop 2 $\chi(G) \leq col(G) = \max\{\delta(H) : H \subseteq G\} + 1$

(HW)

It follows from Prop. 2 that if $\chi(G)=k$, then $G \supset H: \delta(H) \geq k-1$. In fact, even more is true. CV-2

<u>Lemma 1</u> $\forall G$ with $\chi(G)=k$ $\exists H \subset G: \chi(H)=k$ & $\delta(H) \geq k-1$.

<u>Proof</u> Let $H \subseteq G$, H - <u>minimal</u> w/r to $\subset$ with $\chi(H)=k$. If $\exists v \in H$: $\deg_H(v) \leq k-2$, we could extend a $(k-1)$-col. of $H-v$ to H. ↯ □

<u>Thm 6 (Brooks, 1941)</u> $\forall$ connected G, $G \neq K_n$, $G \neq C_{2n+1}$,

$$\chi(G) \leq \Delta(G).$$

I will give a new, elegant proof by Mariusz Zając '18?

<u>Thm 6' (Brooks, 1941)</u> Let $k \geq 3$ & G have $\Delta(G) \leq k$. If $G \not\supset K_{k+1}$, then $\chi(G) \leq k$.

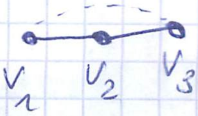
Before the proof, let's define procedure PATHCOLOR$(v_1, \dots, v_{j-1}; v_j)$

<u>Input</u>: partially ~~...~~ colored G, $\Delta(G) \leq k$ with $\leq k$ colors and a path $P = v_1 v_2 \dots v_j \subset G$ totally uncolored.

<u>Output</u>: extension of the coloring to $v_1, \dots, v_{j-1}$.

<u>Explanation</u>: $\forall i=1,\dots,j-1$, after coloring $v_1, \dots, v_{i-1}$, vertex v_i has at least one uncolored neighbor (v_{i+1}) so the number of forbidden colors is $\leq k-1$. $(\Delta(G)-1)$

<u>Proof of Thm 6'</u> Induction on $|G|=n$. For $n \leq k$ - trivial. Let $n \geq k+1$. We may assume that G is k-regular. Otherwise, delete a vertex of degree $< k$ and apply ind. Fix $v \in V(G)$. Since $G \not\supset K_{k+1}$, $\exists x, y$: $vx, vy \in E$ but $xy \notin E$.

~~Case IV~~ Set $v_1 = x$, $v_2 = v$, $v_3 = y$  (CV-3)

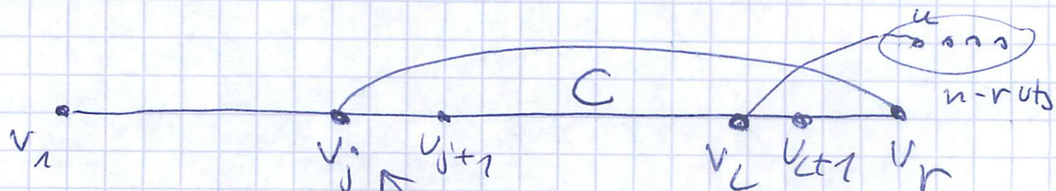
Let $P = v_1 v_2 v_3 \cdots v_r$ be a longest path in G beg. with $v_1 v_2 v_3$.

Case 1 $r = n$: let v_j be a third neighbor of v_2 ($\neq v_1, v_3$)

$k \geq 3$

1) Give v_1 & v_3 the same color
2) Apply PATHCOLOR $(v_4, \dots, v_{j-1}; v_j)$
3) — " — — " — $(v_n, v_{n-1}, \dots, v_j; v_2)$
4) Color v_2 – possible b/c 2 of its nghbs have the same color.

Case 2 $r < n$



the „smallest" nghb of v_r

$G' = G - V(C)$

Color G' by ind. hyp. with $\leq k$ colors

If no edge between G' & C, apply ind. hyp. to $G[V(C)]$ ✓

Let v_l be the „largest" vertex on C with a nghb in G', say u.

Note $l < r$ b/c all nghbs of v_r are on C.

v_{l+1} is not adjacent to G', so let $c(v_{l+1}) = c(u)$

Apply PATHCOLOR $(v_{l+2}, \dots, v_r, v_j, \dots, v_{l-1}; v_l)$!!!

maybe $\boxed{l+2=r}$

Finally, color v_l as it has 2 nghbs (u, v_{l+1}) with the same color.

$\square$

Future remark: The proof works basically the same for list colorings.

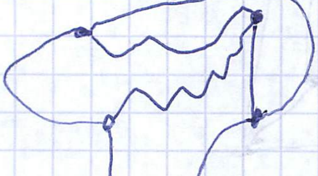
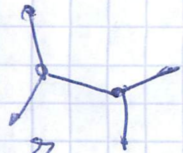
Planar graphs & Colorings from lists (P1)

Def A *plane graph* is a pair (V, E):

(i) $V \subseteq \mathbb{R}^2$ (ii) $\forall e \in E$: e is an arc (curve) between 2 vts (the endpts of e)

(iii) diff. edges have diff. pairs of endpts.

(iv) the interior of an edge contain no pt. of any other edge

 $\cong K_4$ ← 4 faces 

1 face (true for $\forall$ forest)

$\mathbb{R}^2 - G \equiv$ union of regions called *faces*

In a 2-connected plane graph every face is bounded by a cycle.

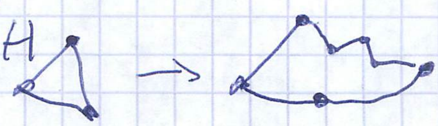
VERTE ← $\boxed{\text{TRIANGULATIONS}}$

Thm 7 (Euler's formula) $\underset{\#\text{vts}}{n} - \underset{\#\text{edges}}{m} + \underset{\#\text{faces}}{l} = 2$

(HW) **Cor1** (a) Every plane graph with $n \geq 3$ vts has $\leq 3n-6$ edges

(b) Every plane graph with no K_3 - - - - $\leq 2n-4$ edges □

(~~HW~~) **Cor2** A plane graph contains neither K_5 nor $K_{3,3}$. □

Def. A *subdivision* of a graph H is obtained by replacing every edge of H by a path (and these paths are internally disjoint) H  TH (many) ← a family of graphs

Notation TH

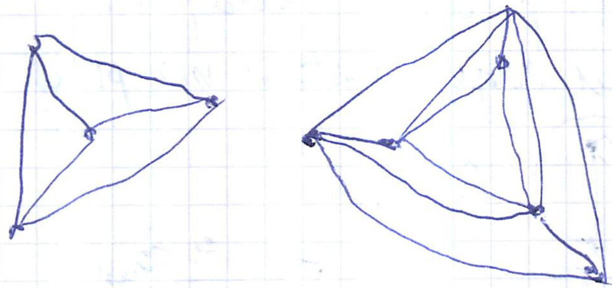
If $G = TH$ is a subgraph of F, then H is a *topological minor* of F, ex F , H , $G = TH$ , H is top min. of F

Cor3 A plane graph contains neither K^5 nor $K_{3,3}$ as a topological minor □ (HW) find a subd. of K_5 or $K_{3,3}$ in Petersen

Def. A plane G is maximal if no $G' \supsetneq G$ with $V(G') = V(G)$ is plane.

G is a triangulation if every face is bounded by a triangle (incl. the outer face).

Ex.



Prop. 3 A plane graph of order ≥ 3 is maximal iff it is a triangulation. $\square$

HW - Draw a triangulation on 7 vts (and another one, $\not\cong$)

P2

Def. G is planar if it is isomorphic to a plane graph (G can be embedded into $\mathbb{R}^2$)
(drawn)

Cor A planar graph has $\leq 3n-6$ edges and contains neith K_5 nor $K_{3,3}$ as a top. minor.

Thm 8 (Kuratowski 1930) G is planar iff G contain neith K_5 nor $K_{3,3}$ as a top. minor $\square$

Back to colorings

Famous **4 Color Conjecture** (Guthrie '1852) MAPS

Kempe – 1879 – incorrect proof

Heawood – 1880 – corrected Kempe's proof and showed that 5 colors suffice

1st proof Appel, Haken '1977 – heavy use of computer

Simpler version: Robertson, Sanders, Seymour, Thomas 1997

It follows easily from Prop 2 that $\chi(G) \leq 6$ $\forall$ planar G. HW

We will show that $\chi(G) \leq 5$ follows from a more general result

LIST COLORINGS

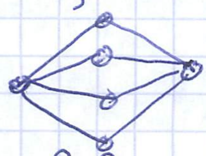
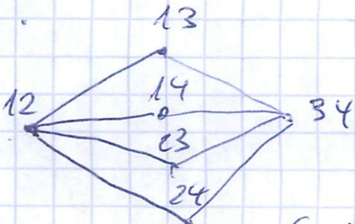
Def Suppose that to each vtx v of G a list S_v of colors is assigned. A coloring of G from the lists $\{S_v\}$ is a coloring c s.t. $\forall v \in V: c(v) \in S_v$

G is k-list-colorable or k-choosable if $\forall (S_v)_{v \in V}$ with $|S_v| = k$ $\forall v$, $\exists$ a coloring from the lists S_v.

The smallest such k is the list-chromatic number or the choice number of G. ch(G)

<u>Obs</u> $ch(G) \geq \chi(G)$ (by taking all $S_v = [k]$, ~~ch~~$\chi(G)=k$)

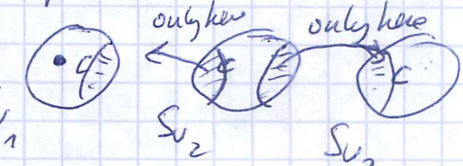
$ch(G)$ may be (much) larger than $\chi(G)$.

<u>Ex.</u> G  $\chi(G)=2$, but $ch(G) > 2$ 

However, Prop 2 & Brooks Thm remains true. Indeed, let $(S_v)_{v \in V}$ be lists of size $|S_v| = col(G)$. Then at each step of the greedy alg., v_i has $\leq col(G) - 1$ colored nghbs, so at least one color from S_{v_i} is free.

<u>As for Brooks Thm</u>, PATHCOLOR works exactly the same (as each list has k colors).

<u>In Case 1</u>, step 1) give v_1, v_3 a pair of colors blocking at most 1 color for S_{v_2}.
Easily done! (HW)



<u>In Case 2</u> similarly, give v_{l+1} & u a pair of colors blocking at most 1 color of S_{v_l}

<u>Thm 9</u> (Thomassen, 1994) Every planar graph is 5-choosable

<u>Proof</u> We shall prove a <u>stronger assertion</u> for all <u>plane</u> graph G on ≥ 3 vts:

<u>SA</u> <u>Suppose</u> all inner faces of G are bounded by triangles, while the outer face by a cycle $C = v_1 \ldots v_k v_1$.

<u>Suppose</u> further that v_1 has been colored by color 1 & v_2 – color 2, equiv., $S_{v_1} = \{1\}$, $S_{v_2} = \{2\}$.

<u>Suppose</u> finally that $|S_v| \geq 3$ for all $v \in V(C) - \{v_1, v_2\}$ & $|S_v| \geq 5$ for all inner v, i.e. $v \in V(G) \setminus V(C)$.

Then G has a coloring from $(S_v)_{v \in V}$.

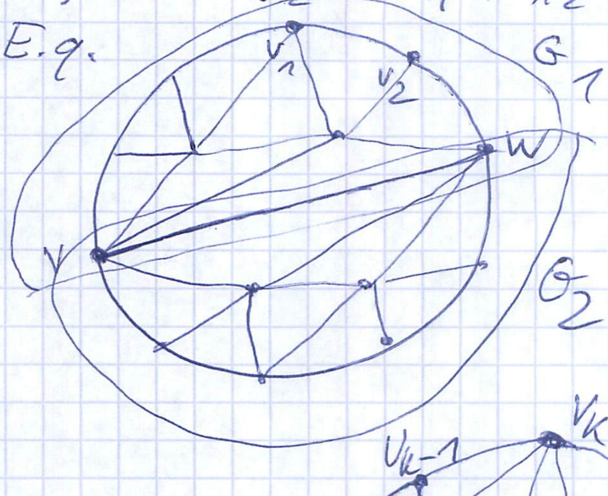
First check that SA $\Rightarrow$ Thm 9.

W.l.o.g. assume G is maximal, so its outer face is 

Assign a list of 5 colors to every vertex. Color v_1, v_2 arbitrarily from their lists and remove these colors (if needed) from S_{v_3}.
Then, the ass. of SA are sat. & one can extend this coloring.

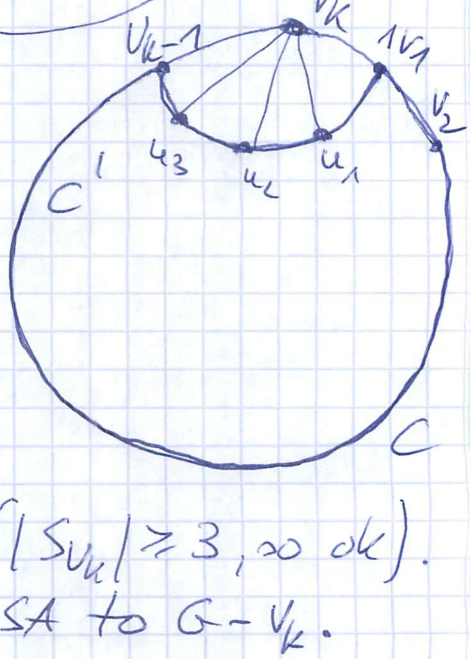
We prove SA by ind. on $|G|$. For $|G|=3$, it is trivial ($G=C$).
Let $|G| \geq 4$ & assume SA for smaller graphs.

Case I C has a chord (diagonal) vw. Then vw lies on 2 unique cycles $C_1, C_2 \subseteq C + vw$, with $v_1 v_2 \in C_1$, $v_1 v_2 \notin C_2$.
For $i=1,2$, let $G_i := G[V(C_i)]$. E.g.



Apply ind. hyp. (SA) to G_1, then to G_2 (with colors of v, w fixed).

Case II. C has no chord.



Let $v_1, u_1, \ldots, u_m, v_{k-1}$ be the neighbors of v_k (in the natural cyclic ordering around v_k)

As all inner faces are triangles,
$P = v_2 u_1 \cdots v_{k-1}$ is a path in G and
$C' = P \cup (C - v_k)$ - a cycle in G.
Choose 2 diff. colors $j, l \neq 1$ from S_{v_k} ($|S_{v_k}| \geq 3$, so ok).
Delete j, l from all S_{u_i}. Apply ind. hyp SA to $G - v_k$.
At least one of j, l is not the color $c(v_{k-1})$, so assign it to v_k. $\square$

Remark $\exists$ planar graph with $ch(G)=5$. (Voigt '93 – 238 vts)
Maryam Mirzakhani – 63 vts! (1996)
(1977 – 2017) Medal Fieldsa

presentation?