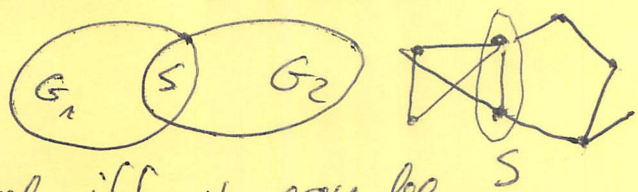
PERFECT GRAPHS

Let $\omega(G)=\max\{r : G\supseteq K_r\}$ and $\alpha(G)=\omega(\bar{G})$.
G is <u>perfect</u> if $\forall\, G' \overset{i}{\subseteq} G$: $\chi(G')=\omega(G')$.
Several classes of graphs are perfect, e.g.
bipartite graphs (trivial), their complements (HW), ...
We examine more closely the <u>chordal graphs</u>, i.e. where every cycle of length ≥ 4 has a chord
(is not induced)
Let's first characterize their structure.

<u>Def.</u> If $G \overset{i}{\supseteq} G_1, G_2, S$ such that $G=G_1\cup G_2$, $S=G_1\cap G_2$, we say that G arises from G_1 & G_2 by <u>pasting</u> them together <u>along S</u>.



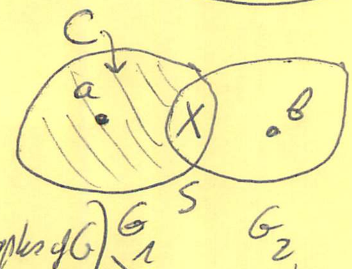
<u>Prop. 5</u> A graph is chordal iff it can be constructed recursively by pasting along complete subgraphs, starting from complete graphs.

<u>Proof</u> $\Leftarrow$ easy: any induced cycle of G must lie entirely within G_1 or within G_2 (b/c S is complete). So, it has to be a triangle (b/c G_1, G_2 - perfect).
$\Rightarrow$ Let G be chordal. We'll show by induction on $|G|$ that it can be constructed as described in Prop 5. Trivial when G is complete ($G_1=G_2=S=G$). Assume not, in particular, $|G|>1$. Let $a,b\in V(G)$, $ab\notin E(G)$.
Let $X\subset V(G)\setminus\{a,b\}$ be a <u>minimal $a-b$ separator.</u>
Let C be a component of $G-X$ containing a,

$G_1 = G[V(C) \cup X]$, $G_2 = G - V(C)$, $S = G[X]$

(here it could be that $X = \emptyset$ and $S = \emptyset$)

Note: $b \notin V(G_1)$, $a \notin V(G_2)$, so we may apply ind.!



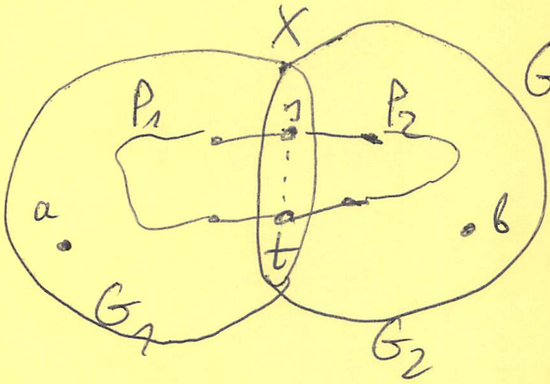
Both G_1, G_2 are chordal (as induced subgraphs of G), and $|G_1|, |G_2| < |G|$!

It remains to show that S is complete.

Suppose $s, t \in S$ are non-adjacent. Since X is a minimal a-b separator, both s & t have neighbors in both C and $V(G_2) \setminus X$ — b/c in $G - (X - \{s\})$ there is an a-b path going through s; similarly in $G - (X - \{t\})$ ....

(In particular, the neighbors of s & t in $V(G_2) \setminus X$ are in the same comp.)

Let P_1, P_2 be the shortest s-t paths in G_1, G_2.



$G \supset P_1 \cup P_2$ – cycle without chords – ↯

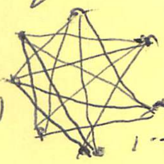
explain: b/c shortest!

G is chordal!

□

Prop 6 Every chordal graph is perfect.

Proof Since complete graphs are perfect, it suffices to show that any ~~perfect~~ graph G obtained from perfect graphs G_1, G_2 by pasting along a complete sub. S is again perfect. Let $H \overset{i}{\subseteq} G$. We'll show that $\chi(H) \leq \omega(H)$! Let $H_i := H \cap G_i$, $i = 1, 2$, $T := H \cap S$, so T is again complete and H arises from H_1, H_2 by pasting along T. As ~~a~~ an induced sub. of G_i, $\chi(H_i) \leq \omega(H_i)$, $i = 1, 2$. Since T is complete, the colorings of H_1 & of H_2 can be combined (adjusted) into a col. of H with $\max\{\omega(H_1), \omega(H_2)\} \leq \omega(H)$ colors. □

By def., every induced sub. of a perfect graph is perfect. Thus, the property of perfection can be characterized by the absence of induced sub. isomorphic to members of some family of graphs $\mathcal{H}$.
(Trivially, one can take $\mathcal{H} = \{$all imperfect graphs$\}$.)
Berge (1963) conjectured (the strong perfect graph conj.) that as $\mathcal{H}$ one can take only all odd cycles of length ≥ 5 and their complements (holes & antiholes, e.g. , , ...

It was proved very recently (200 page proof!)

Thm 12 (Chudnovsky, Robertson, Seymour, Thomas, 2006)
G is perfect iff neither G nor $\overline{G}$ contains an odd cycle C_n, $n \geq 5$, as an induced subgraph. $\square$

Berge also formulated a weak perfect graph conj: proved a few years later.

Thm 13 (Lovász, 1972) A graph G is perfect iff $\overline{G}$ is perfect.

Clearly, Thm 12 $\Rightarrow$ Thm 13.

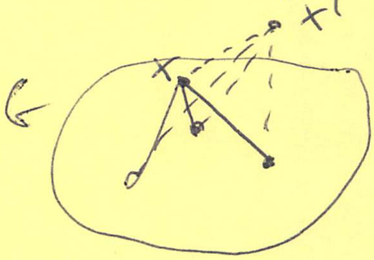
Lovasz also proved a stronger statement.

Thm 14 (Lovász, 1972) G is perfect iff $|H| \leq \alpha(H)\omega(H)$ $\forall H \subseteq G$.

Observe that Thm 12 $\Rightarrow$ Thm 14 $\Rightarrow$ Thm 13. (HW)

We first give the original proof of Thm 13.
For this we need another, preliminary fact.

<u>Def.</u> Given G and $x \in V(G)$, let G' be obtained from G by adding a vertex x' and joining it to x and <u>all</u> nghbs of x. We then say that G' is obtained from G by <u>expanding vx x to an edge xx'.</u>



~~Pr~~ Lemma 3) Any graph obtained from a perfect graph by expanding a vx is perfect.

<u>Proof</u> Induction on $|G|$. True for K_1 (as any expanding is K_2)
Let G be perfect, $|G|>1$, G' obtained from G by expanding x to xx'.
It suffices to show that $\chi(G') \le \omega(G')$. Indeed, every proper ind. sub. H of G' is <u>either</u> isomorphic to an ind. subgraph of G (so $\chi(H) \le \omega(H)$ ~~by~~ G is perfect) <u>or</u> is obtained from a proper ind. sub. of G by expanding x (so $\chi(H) \le \omega(H)$ by ind. hyp. ~~that~~).

Let $\omega := \omega(G)$. Then $\omega(G') \in \{\omega, \omega+1\}$. If $\omega(G') = \omega+1$, then $\chi(G') \le \chi(G)+1 = \omega+1 = \omega(G')$.

Assume that $\omega(G') = \omega$. Then x lies in <u>no</u> $K_\omega \subseteq G$.
Let us color G with ω colors, $X = \{v : c(v) = c(x)\}$.
Every clique K_ω contains a vertex of X (but not x).
Thus, $H := G - (X \setminus \{x\})$ has $\omega(H) < \omega$ (all K_ω's destroyed).
Since G is perfect, $\chi(H) \le \omega - 1$. But $V(G' - H) = X - x + x'$ is indep. (b/c $X - x$ is ind.), so $\chi(G') \le \omega - 1 + 1 = \omega$. $\square$

Proof of Thm 13 Ind. on $|G|$, $|G|=1$ – trivial.

$|G|\geq 2$. Let $\mathcal{K}$ be the family of all cliques (vx sets) in G.
Put $\alpha=\alpha(G)$, let $\mathcal{A}$ – all indep. sets of size α in G.
Suffices to show that $\chi(\overline{G})\leq\omega(\overline{G})=\alpha$ (for proper induced subs it follows by induction).
To this end, we shall find $K\in\mathcal{K}: K\cap A\neq\emptyset\ \forall A\in\mathcal{A}$
b/c then $\omega(\overline{G}-K)=\alpha(G-K)<\alpha=\omega(\overline{G})$, and
by ind. hyp. $\chi(\overline{G})\leq\chi(\overline{G}-K)+1\overset{\text{ind. hyp.}}{=}\omega(\overline{G}-K)+1\leq\omega(\overline{G})$.

Suppose there is no such K, i.e. $\forall K\in\mathcal{K}\ \exists A_K\in\mathcal{A}$:
$K\cap A_K=\emptyset$. We build a new graph G' by replacing every vx x of G by a complete graph G_x of order
$k(x)=|\{K\in\mathcal{K}: x\in A_K\}|$, and joining all vts of G_x to all vts of G_y whenever $xy\in E(G)$, i.e. by $K_{k(x),k(y)}$.
Note that $\alpha(G')\leq\alpha$. [Not „=" b/c not every vx is blown up.]
Thus G' has vertex set $\bigcup_{\substack{x\in V\\ k(x)>0}} V(G_x)$ & $vw\in E(G')\Leftrightarrow x=y$ or $xy\in E(G)$.

Obs. that G' can be obtained from G by repeated vertex expansion of $G[\{x\in V: k(x)>0\}]\subseteq G$.
perfect by ass. on G.

By Lemma 3, G' is thus perfect, in particular, $\chi(G')\leq\omega(G')$.
We will contradict it!
Every clique X' of G' has the form $G'[\bigcup_{x\in X} G_x]$, $X\in\mathcal{K}$.
So $\omega(G')=\sum_{x\in X}k(x)=|\{(x,K): x\in X, K\in\mathcal{K}, x\in A_K\}|=$

$$= \sum_{K \in \mathcal{K}} |X \cap A_K| \leq |\mathcal{K}| - 1 \quad b/c$$

$|X \cap A_K| \leq 1 \ \forall K$ (as A_K is indep., while X is complete)

& $|X \cap A_X| = 0$ (by def. of A_X).

On the other hand, $|G'| = \sum_{x \in V} k(x) = \sum_{K \in \mathcal{K}} |A_K| = |\mathcal{K}| \alpha \geq$

$\geq |\mathcal{K}| \alpha(G')$. Thus

$$\chi(G') \geq \frac{|G'|}{\alpha(G')} \geq |\mathcal{K}| > \omega(G') \ \lightning \quad \square$$