List-edge colorings

LEC1

~~Clearly~~ Define $ch'(G)$ as the smallest k: there is an edge-k-coloring of G from any set of lists of size k.

Clearly $ch'(G) \geq \chi'(G)$, but unlike the vertex-case, <u>no</u> graph is known with $ch'(G) > \chi'(G)$.

<u>List coloring conj.</u> $\forall G: ch'(G) = \chi'(G)$

We will prove it for bipartite graphs

<u>Thm 1 (Galvin 1995)</u> $\forall$ bip. G: $ch'(G) = \chi'(G)$ $(= \Delta(G))$

Will use <u>orientations</u> D obtained from graphs G by orienting their edges, i.e. replacing edge $\{u,v\}$ with directed edges (u,v) or (v,u) (but not both)

<u>Def</u> An indep set $U \subseteq V(D)$ is a <u>kernel</u> if $\forall v \in V(D) - U$ $\exists u \in U: (v,u) \in E(D)$

<u>Lemma 2</u> Given a graph H and a set of lists of colors $(S_v)_{v \in V(H)}$, if H has an orientation D with $d^+(v) < |S_v|$ $\forall v$ ~~then~~ and such that every $D' \overset{i}{\subseteq} D$ has a kernel, then H can be colored from lists S_v.

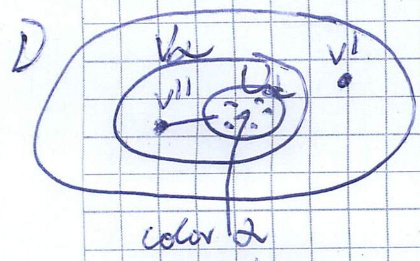
<u>Proof</u> Induction on $|H|$. For $|H| = 1$, $d^+(v) < |S_v| \Rightarrow S_v \neq \emptyset$ and one can color v. Let $|H| > 1$ and assume truth for $< |H|$.

Let $\alpha \in \bigcup S_v$ & D be as stated.

Let $V_\alpha = \{v : \alpha \in S_v\}$, $D_\alpha = D[V_\alpha]$, D_α has a kernel $(\neq \emptyset)$ U_α

Let's color U_α by color α and set $S'_v = S_v - \{\alpha\}$

For $D' := D - U_\alpha$ we have $d^+_{D'}(v) < |S'_v|$ $\forall v \in V(D) - U_\alpha$.



Check: $S'_{v'} = S_{v'}$, $d^+_{D'}(v') \le d^+_D(v')$

$S_{v''} = S_{v'} - \{\alpha\}$, $d^+_{D'}(v'') \le d^+_D(v'') - 1$

So, D' sat. ass. & by ind. can be colored from S'_v.

As they don't contain color α, we get a col. of D from S_v. $\square$

Proof of Thm 1.1 Let $G = (X \cup Y, E)$, $\chi'(G) = k$,

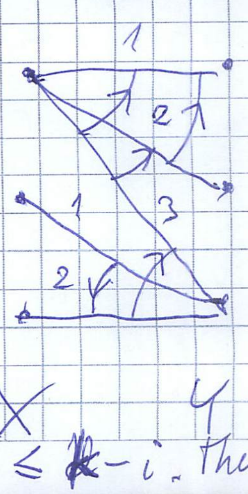
$c: E(G) \to [k]$. Clearly, $ch'(G) \ge k$. We prove $ch'(G) \le k$.

We will use the equality $ch'(G) = ch(L(G))$.

Let $H = L(G)$. Need to find an orient. D sat. ass. of Lemma 2.

Consider adjacent $e, e' \in E$ with, say, $c(e) < c(e')$

If $e \cap e' \in X$ ($e \wedge e'$), then orient $e'e \in D$; if $e \cap e' \in Y$ ($e \vee e'$) then $ee' \in D$



We have $d^+_D(e) < k$ $\forall e \in D$

Indeed, if $c(e) = i$, let $N^+_X(e)$ be the set of e': $ee' \in D$, $e \cap e' \in X$.

$\forall e' \in N^+_X(e)$: $c(e') \in [i-1]$ & all $c(e')$ differ.

$\Rightarrow |N^+_X(e)| \le i-1$. Similarly, $|N^+_Y(e)| \le k-i$. Thus, $|N^+_D(e)| < k$.

It remains to show that $\forall D' \overset{i}{\subseteq} D$ has a kernel.

Induction on $|D'|$. For $|D'|=0$, $\emptyset$ is a kernel.

Assume $|D'| \geq 1$. Let $E' = V(D') \subseteq E(G)$.

$\forall x \in X$ incident with ≥ 1 edge from E',

let $e_x \in E'$ be the edge at x with minimum c.

Let $U = \{e_x : x \in X$ as above$\}$. Then

~~We claim that U~~

$\forall e' \in E' - U \ \exists e \in U$: e', e meet at X & $c(e) < c(e') \Rightarrow e'e \in D'$.

If U is indep., then U is a kernel of D' – done!

Assume then that U is not indep.

Let $e, e' \in U$ be adjacent, assume $c(e) < c(e')$.

By def. of U, e, e' meet at Y, so $ee' \in D'$.

By ind. hyp., $D' - e$ has a kernel U'.

If $e' \in U'$, then U' is also a kernel of D' – done!
(b/c $ee' \in D'$)

If not, $\exists e'' \in U'$: ~~D' has $e'e''$~~ $e'e'' \in D'$ (by def. of kernel)

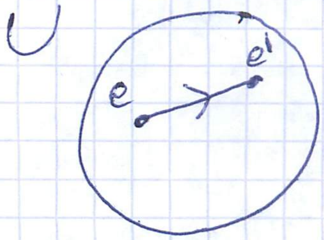
If e', e'' ~~meet~~ at X, then $c(e'') < c(e')$ by def. of D — contradicting $e' \in U$.

Hence e', e'' meet at Y and $c(e') < c(e'')$.

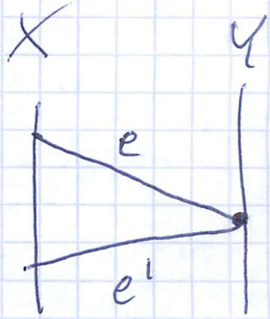
Since e, e' meet at Y too, also e, e'' meet at Y,
so $c(e) < c(e') < c(e'')$. So, $ee'' \in D'$ and again U' is a kernel of D'. $\square$

Illustrations

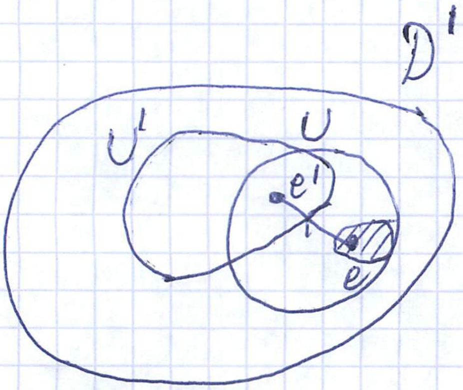
LEC4



, $c(e) < c(e')$ $\overset{\text{def. of } U}{\Longrightarrow}$



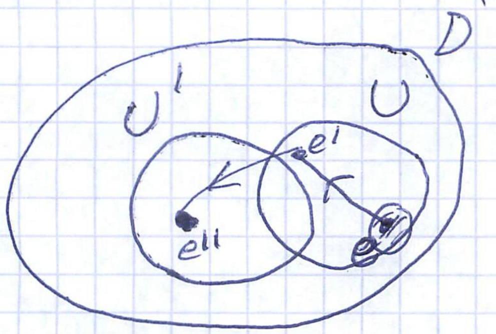
I.

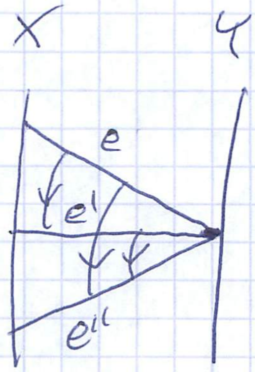


U' - kernel of $D' - e$

$\Rightarrow U'$ - kernel of D' bc $ee' \in D'$, $e' \in U'$

II.





; $c(e) < c(e'')$

$\Downarrow$

$ee'' \in D'$

$\Downarrow$

U' is a kernel of D''