Coloring edges

An **edge coloring** of $G=(V,E)$ is a map $c: E \to S$ with $c(e) \neq c(f)$ for any adjacent edges e, f.
The smallest k: $\exists$ a **k-edge-coloring**, i.e. when $S=[k]$, is the **edge-chromatic number** or **chromatic index** of G

Notation: $\chi'(G)$

Clearly, $\chi'(G) = \chi(L(G))$, $L(G)$ – line graph of G

Always: $\chi' \in \{\Delta, \Delta+1\}$!!! **Trivially** $\chi' \geq \Delta$

Prop 4 (König, 1916) $\forall$ bip. G: $\chi'(G) = \Delta(G)$.

There is an inductive proof in the **book**, but Prop 4 follows immediately from Problem 1.4: "Every bip. graph contains a matching sat. all vs of max. deg."
Remove such a matching M as color class and apply ind.

Thm 10 (Vizing 1964) $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$

So, all graphs are divided into 2 classes:

class 1: $\chi' = \Delta$ class 2: $\chi' = \Delta+1$

No good characterization. E.g. $\chi'(\text{Petersen}) = \Delta+1 = 4$.

Thm 11 (Csaba, Kühn, Lo, Osthus, Treglown) 2016

$\exists n_0 \, \forall n \geq n_0$, n even, $\forall d \geq \frac{n}{2}$: every d-reg. G ~~has~~ of order n has $\chi'(G) = \Delta(G)$.

?

Proof of Thm 10 (Schrijver) Induction on $|G|$

CE2

It's enough to prove $\forall\, k \geq 0$:

$\boxed{(S_k)}$ If for some $v \in V(G)$: (i) $\deg_G(v) \leq k$

(ii) $\forall\, u \in N_G(v)$: $\deg_G(u) \leq k$,

(iii) $|\{u \in N_G(v) : \deg_G(u) = k\}| \leq 1$

(iv) $\chi'(G - v) \leq k$,

then $\chi'(G) \leq k$

For $|G| = 1$, $\chi'(G) = 0 \leq 1$. Assume $\forall\, |G| < n$ $\chi'(G) \leq \Delta(G) + 1$

Take G with $|G| = n$, $k = \Delta(G) + 1$, v - any vx of G.

The ass. of (S_k) hold: (i)-(iii) trivially, (iv) by ind. ass.

Then $\chi'(G) \leq \Delta(G) + 1$.

It remains to show (S_k) - ind. on k.

$k = 0$ - trivial (v is isolated), so $\chi'(G - v) = \chi'(G)$.

$k \geq 1$ W.l.o.g. assume $\exists\, u_0 \in N_G(v)$: $\deg_G(u_0) = k$

& $\forall\, u \in N_G(v) \setminus \{u_0\}$: $\deg_G(u) = k - 1$

(if not, keep adding pendant vts w

w - new edges can be easily colored)

Let c be a k-coloring of $G - v$.

For $i = 1, \dots, k$, $X_i = \underbrace{\{u \in N_G(v) : \forall\, uw \in G - v : c(uw) \neq i\}}_{\text{"vertices } u \text{ missed by color } i\text{"}}$

Choose c to minimize $\sum_{i=1}^{k} |X_i|^2$

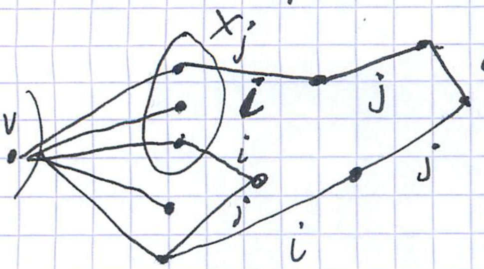
Case I $\forall i$ $|X_i| \neq 1$. Since $\deg_{G-v}(u_0) = k-1$, $\deg_{G-v}(u) = k-2$ $\forall u \neq u_0$

$\sum_{i=1}^{k} |X_i| = 2\deg_G(v) - 1 \leq 2k - 1 \Rightarrow \exists\, i : |X_i| = 0$, $\exists\, j : |X_j| \geq 3$, odd.

Let $H_{ij} = \{e \in G - v : c(e) \in \{i,j\}\}$.

~~this is~~ Let C_{ij} be a comp. of H_{ij} : $V(C_{ij}) \cap X_j \neq \emptyset$

C_{ij} is a path P starting in X_j. Exchange colors i,j on P:



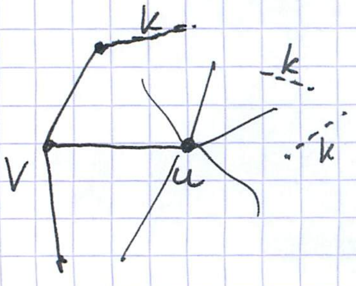
$|X_j'| \le |X_j| - 1$, $|X_i'| = 2$

$$|X_i'|^2 + |X_j'|^2 = 4 + |X_j|^2 - (|X_j|^2 - |X_j'|^2)$$
$$\le 4 + \cancel{|X_j|^2} - \cancel{|X_j|^2} + (|X_j|^2 - 2|X_j| + 1)$$
$$\le 5 + |X_j|^2 - 6 = |X_j|^2 - 1 \quad ↯$$

$(x := |X_j'| \ge 3,\ 0^2 + x^2$ vs $\le 4 + (x-1)^2 = x^2 + 5 - 2x \le x^2 - 1)$

<u>Case II</u> $\exists i : |X_i| = 1$, w.l.o.g. $|X_k| = 1$, $X_k = \{u\}$

$G' = G - vu - \{e \in G : c(e) = k\}$, <u>$G-v$</u> is $(k-1)$-col.



Moreover, $\deg_{G'}(v) \le k-1$,
$\forall u \in N_G(v) : \deg_{G'}(u) \le k-1$
& for at most one $u \in N_G(v) : \deg_G(u) = k-1$

$\Rightarrow$ (By ind ass. for $k-1$) $\chi'(G') \le k-1$.
Restoring edges of G of color k & ~~def~~ assigning $c(vu) = k$
$\Rightarrow \chi'(G) \le k$. $\square$