Recall that, ~~by Cor. 5~~ (H(w)), $\lim_{n\to\infty} \frac{t_{r-1}(n)}{n^2/2} = \frac{r-2}{r-1}$, i.e.

$\forall \delta\ \exists n_\delta : \forall n \geq n_\delta \quad \frac{t_{r-1}(n)}{n^2/2} > \frac{r-2}{r-1} - \delta$. So

$$\|R\| \geq \frac{1}{2}k^2\left(\frac{t_{r-1}(n)}{n^2/2} + \delta\right) > \frac{1}{2}k^2 \frac{r-2}{r-1} \overset{(H(w))}{\geq} t_{r-1}(k).$$

By Thm 15, $R \supseteq K^r$, so $R_\delta \supseteq K_\delta^r \underset{15}{\Rightarrow} G \supset K_\delta^r$

$[n_0 = \max(n_1, n_\delta)]$. $\square$

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3. Application - Ramsey number for sparse graphs

Ramsey's Thm: $\forall r\ \exists n : \forall G_n : G_n \supset K_r$ or $G_n \supset K_r$.

Clearly, the same is true for any graph H in place of K_r.

$$R(H) = \min\{n : \underset{G_n}{\forall} G_n \supset H \text{ or } \overline{G_n} \supset K_r\}$$

Thm 17 (Chvátal, Rödl, Szemerédi, Trotter, 1983)

$\forall \Delta \geq 1\ \exists c > 0 : \forall H$ with $\Delta(H) \leq \Delta$:

$$R(H) \leq c|H|.$$

Proof Idea: Want to show that if $|G|$ is large enough (but not too large), then $H \subseteq G$ or $H \subseteq \overline{G}$.

Consider an ε-reg. partition of G provided by the regularity lemma. If most pairs have high density, we will try to find H in G; otherwise in $\overline{G}$.

Let $R(\Pi, 0)$ be the reduced graph with no constraints on density. We split $R(\Pi,0) = R' \cup R''$, where R' corresponds to pairs with density $\geq \frac{1}{2}$, R'' – $\leq \frac{1}{2}$.
So, $R' = R(\Pi, \frac{1}{2})$ & $R'' = R(\Pi, \frac{1}{2})$ for $\overline{G}$ (see Problem 8.1)

(*) All we need is a $K^{\Delta+1} \subset R'$ or $K^{\Delta+1} \subset R''$.
We obtain this by finding in $R(\Pi, 0)$ a K^m, where $m = R(\Delta+1)$ (the Ramsey number).
Once we have (*), we blow R' (or R'') by $s \geq \alpha(H)$ to get $H \subset R'_s$ or $H \subset R''_s$ and use the Blow-up Lemma

(Why $\Delta+1$? B/c $\chi(H) \leq \Delta(H)+1 \leq \Delta+1$, so H can be split into $\Delta+1$ independent sets, each $\leq s$.)

Details Fix $\Delta \geq 1$. On inputs $d := \frac{1}{2}$ & Δ,

Lemma 5 (The Blow-up L.) returns ε_0.

Let $m = R(\Delta+1)$ and let $\varepsilon \leq \varepsilon_0$ s.t. $\forall\, k \geq m$

$$2\varepsilon < \frac{1}{m-1} - \frac{1}{k} \qquad (\Rightarrow \varepsilon < 1). \quad (1)$$

On inputs ε, m, Lemma 4 (Reg. L) returns M.

We shall prove Thm 17 with $c = \dfrac{2^{\Delta+1} M}{1-\varepsilon}$

Let H with $\Delta(H) \leq \Delta$ be given, $s := |H|$.
Let G have $|G| = n \geq c|H|$.
We'll show $H \subseteq G$ or $H \subseteq \overline{G}$.

By L4, G has an ε-reg. part. $\{V_0, V_1, \ldots, V_k\}$ with $|V_0| \le \varepsilon n$, $|V_1| = \ldots = |V_k| =: l$, $m \le l \le M$. Then

$$l = \frac{n - |V_0|}{k} \ge n \frac{1-\varepsilon}{M} \ge cs \frac{1-\varepsilon}{M} \ge 2^{\Delta+1} s = \frac{2s}{d^\Delta}. \quad (2)$$

Let $R := R_G(\Pi, 0)$. Then $|R| = k$ &

$$\|R\| \ge \binom{k}{2} - \varepsilon k^2 = \frac{1}{2} k^2 \left(1 - \frac{1}{k} - 2\varepsilon\right)$$

$$\overset{(1)}{\ge} \frac{1}{2} k^2 \left(1 - \frac{1}{k} - \frac{1}{m-1} + \frac{1}{k}\right) = \frac{1}{2} k^2 \frac{m-2}{m-1} \ge t_{m-1}(k).$$

So, by (Turán's) Thm. 15, $R \supset K = K^m$.

We now color the edges of R with 2 colors:

- red if $d_G(V_i, V_j) \ge \frac{1}{2}$
- green if $d_G(V_i, V_j) \le \frac{1}{2}$

$\boxed{\text{was OK.}}$ ~~(so, some edges of R may get both colors)~~

Let $R' \subset R$ consist of red edges, i.e. $R' = R_G(\Pi, \frac{1}{2})$,

and $R'' \subset R$ consist of green edges, i.e. $R'' = R_{\bar{G}}(\Pi, \frac{1}{2})$

[here we use Prob. 8.1 that a pair is ε-reg. in $G \Leftrightarrow$ ε-reg. in $\bar{G}$.]

By def. of m (= Ramsey # for $\Delta+1$), K contains red or green $K^{\Delta+1}$, so $H \subseteq R'_s$ or $H \subseteq R''_s$ (the s-blow-ups of R', R'')

Since $\varepsilon \le \varepsilon_0$, $\ell \ge \frac{2s}{d^s}$ (by (2)), both R' & R'' satisfy the assumptions of Lemma 5, so

$$H \subseteq G \text{ or } H \subseteq \overline{G}. \ \square$$

Something easy at the end (if time permits).

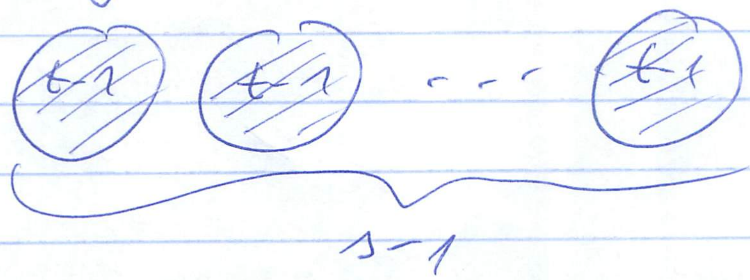
Let $R(H_1, H_2) = \min\{n : \forall G_n : H_1 \subseteq G \text{ or } H_2 \subseteq \overline{G}\}$

Prop. 10 $\forall s, t \ge 1$, T - a tree with $|T| = t$,

$$R(T, K^s) = (s-1)(t-1) + 1.$$

Proof 1) Lower bound by construction:

$G = (s-1) K^{t-1}$



$G \not\supseteq T$, $\overline{G} \not\supseteq K^s$

so $R(T, K^s) > (s-1)(t-1)$.

2) Upper bound: Let G, $|G| = n = (s-1)(t-1)+1$, be given.

Suppose $\overline{G} \not\supseteq K^s$. Then $s > 1$ &

$$\chi(G) \ge \left\lceil \frac{n}{s-1} \right\rceil = t \quad (\text{b/c } \alpha(G) \le s-1).$$

By Lemma 1 (Lecture 3), $G \supseteq H$, $\delta(H) \ge t-1$, so $H \supseteq T$. $\square$
(see Probl. 9.6)