Applications of SzRL

L11/12

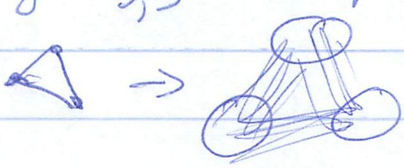
①

1. Weak Blow-up Lemma

Def Given a graph G with an ε-reg. partition $\Pi = (V_0, \dots, V_k)$ and a real number $d \in [0,1]$ (the density threshold), let $R = R(\Pi, d)$ be a graph on $V(R) = \{v_1, \dots, v_k\}$ with $v_i v_j \in E(R)$ iff the pair (V_i, V_j) is ε-reg of density $d_G(V_i, V_j) \geq d$.

The (blow-up) graph R_s of a graph R is obtained from R by replacing each vertex $v_i \in V(R)$ by a set U_i, $i = 1, \dots, |R|$, where the U_i's are indep. ~~sets~~ and disjoint sets, and replacing each edge $v_i v_j$ by $K_{s,s}$ with bipartition (U_i, U_j).

E.g. $(K^r)_s = K_{\underbrace{s, \dots, s}_{r}}$



Lemma 5 (Weak Blow-up Lemma, Komlós-Simonovits, 1996)

For all $d \in (0,1]$, $\Delta \geq 1$ $\exists\, \varepsilon_0 > 0$: $\forall H$, $\Delta(H) \leq \Delta$, $\forall s \geq 1$, $\forall G$ with $\Pi = \{V_0, V_1, \dots, V_k\}$ with $\varepsilon \leq \varepsilon_0$, where $|V_i| \geq \frac{2s}{\cancel{\varepsilon_0}\, d^{\Delta}}$, $i = 1, \dots, k$

$\uparrow$ ε-reg. partition of $V(G)$

$$H \subseteq R_s(\Pi, d) \Rightarrow H \subseteq G$$

Proof (Idea: sequential embedding of vts with a foresight)

Given d & Δ, choose $\varepsilon_0 < d$ and such that

$$(d - \varepsilon_0)^{\Delta} - \Delta \varepsilon_0 \geq \frac{1}{2} d^{\Delta} \qquad (1)$$

[possible, as L-H-S $\to d^{\Delta}$ when $\varepsilon_0 \to 0$]

Let G, H, s, R be given as stated.

Let $\{V_0, V_1, \dots, V_k\}$ be an ε-reg. part. of G, $\varepsilon \le \varepsilon_0$,

$|V_1| = \dots = |V_k| = \ell \geq \frac{2s}{d^\Delta}$; $V(R) = \{v_1, \dots, v_k\}$.

Assume $H \subseteq R_s$, $V(H) = \{u_1, \dots, u_h\}$, $h = |H|$.

$\forall i \; \exists \; j := \sigma(i) : u_i \in U_j$ $\left(V(R_s) = U_1 \cup \dots \cup U_k, \; |U_j| = s \right)$

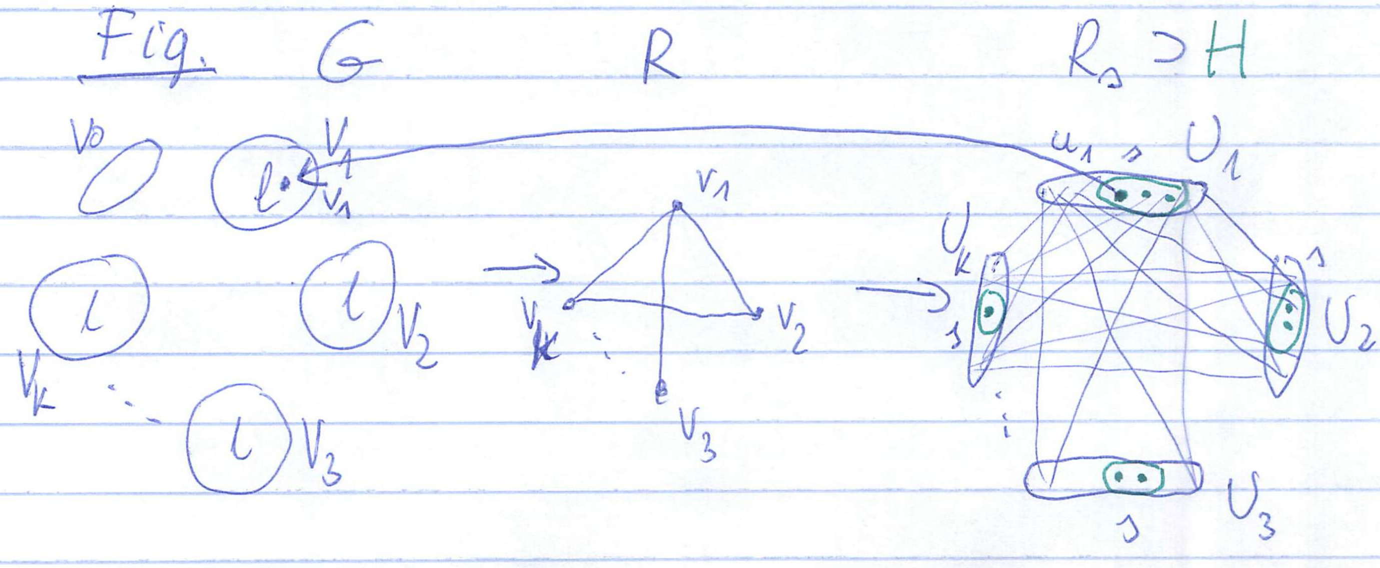
<u>Goal:</u> Find an embedding of H into G, i.e.

$u_i \mapsto v_i \in V_{\sigma(i)}$ s.t. $v_1, \dots, v_h$ are all distinct

& $u_i u_j \in E(H) \Rightarrow v_i v_j \in E(G)$.

We choose images $v_1, \dots, v_h$ inductively, updating "candidate" sets $Y_i^j \subseteq V_{\sigma(i)}$ as we go.
Initially, $Y_i^0 = V_{\sigma(i)}$. After embedding $u_1, \dots, u_j$, $j < i$, if $u_j u_i \in E(H)$, then we delete from Y_i^{j-1} all vts that are <u>not</u> adjacent to v_j.

<u>Fig.</u>



So, $V_{\sigma(i)} = Y_i^0 \supseteq Y_i^1 \supseteq \dots \supseteq Y_i^{i-1} \supseteq Y_i^i = \{v_i\}$

Y_i^j – the candidate set for v_i after $u_1, \dots, u_{i-1}$ embedded

Need to make sure that Y_i^{i-1} is big enough, that is, $\geq s$ (since, maybe, all other $s-1$ vts from U_i has been already embedded)

How to choose the image of u_j so that Y_i^j's don't shrink too much?

Select v_j (= the image of u_j) so that for all $i > j$, $u_i u_j \in H$, the new candidate set $\boxed{Y_i^j := N_G(v_j) \cap Y_i^{j-1}}$ (2)

is not too small.

For this, recall Problem 2 from Set 8: "If (A, B) is ε-reg with density d & $Y \subseteq B$, $|Y| \geq \varepsilon |B|$, then all but at most $\varepsilon |A|$ vts. $v \in A$ have each $\geq (d-\varepsilon)|Y|$ nghbs in Y."

We apply this with $A = V_{\sigma(j)}$, $B = V_{\sigma(i)}$, $Y = Y_i^{j-1}$,

provided $|Y_i^{j-1}| \geq \varepsilon |V_{\sigma(i)}| = \varepsilon \ell$, simultaneously for all $i > j$ with $u_i u_j \in E(H)$ [there are at most Δ such i's]:

all but at most $\Delta \varepsilon \ell$ choices of v_j yield, by (2),

$$|Y_i^j| \geq (d-\varepsilon)|Y_i^{j-1}| \qquad \text{for all } i > j,\ ij \in E(H) \qquad (3)$$

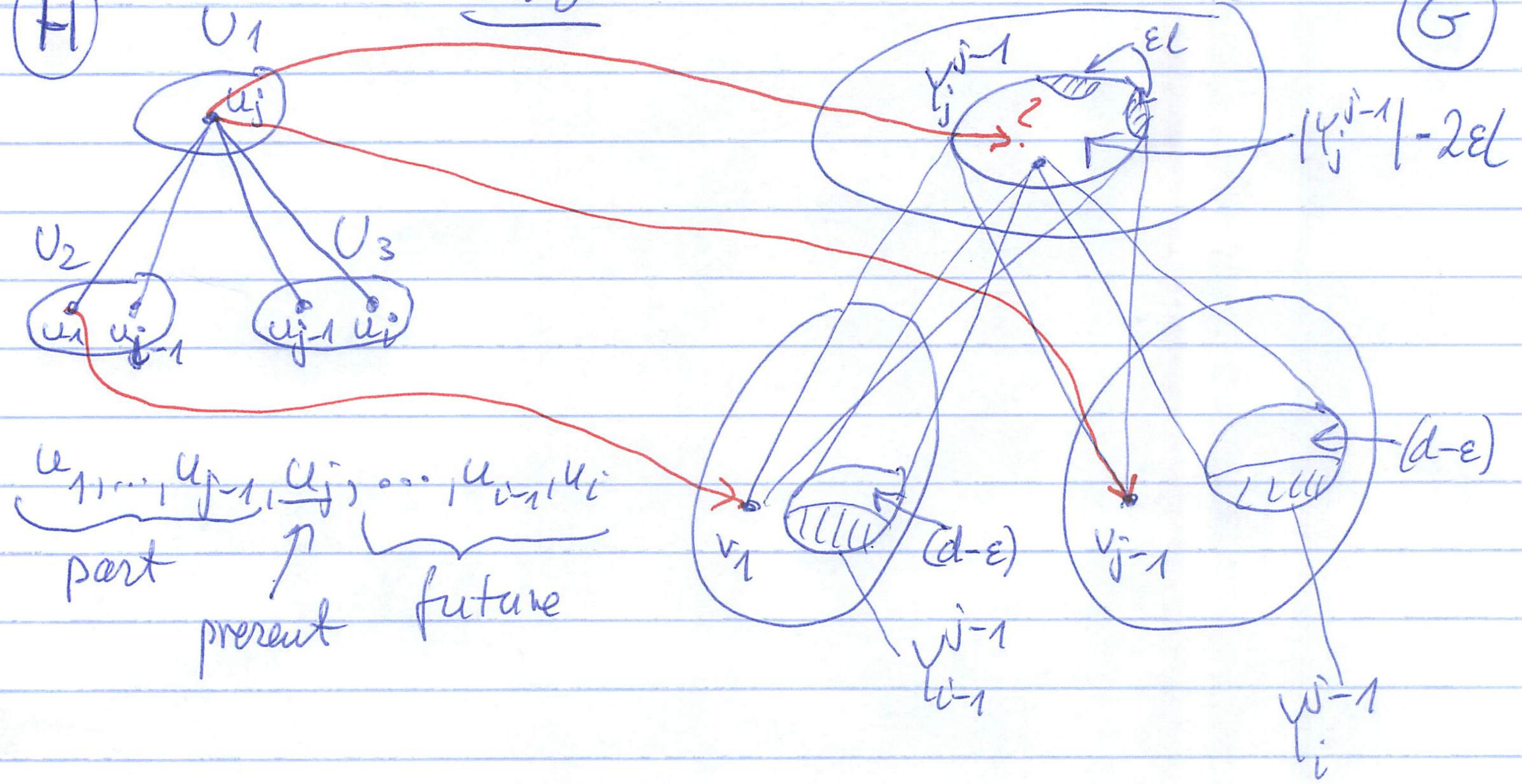
So, we need $|Y_i^{j-1}| \geq \Delta \varepsilon \ell + s$, as well as $|Y_i^{j-1}| \geq \varepsilon \ell$.

Fix i. We have $|Y_i^0| = \ell$ and each time a vx u_j, $j<i$, $u_j u_i \in E(H)$, is embedded, the candidate set "shrinks" by a factor of $(d-\varepsilon)$ [at most], and this happens $\leq \Delta$ times. So, $\forall j \leq i$,

$$|Y_i^{j-1}| - \Delta\varepsilon\ell \overset{(3)}{\geq} (d-\varepsilon)^{\Delta}\ell - \Delta\varepsilon\ell \overset{\varepsilon \leq \varepsilon_0}{\geq} (d-\varepsilon_0)^{\Delta}\ell - \Delta\varepsilon_0\ell \overset{(1)}{\geq} \frac{1}{2}d^{\Delta}\ell \geq s$$

In particular, $|Y_i^{j-1}| \geq \varepsilon\ell$ & $|Y_j^{j-1}| - \Delta\varepsilon\ell \geq s$. □

Fig



2. Application - Erdős-Stone Thm

Def. (of Turán number) Given a graph H & $n \geq |H|$,

$ex(n, H)$ ~~$ex_n(H)$~~ $= \max\{\|G\| : |G| = n \text{ \& } G \not\supseteq H\}$

Problem: Determine $ex(n, H)$ and all graphs G which achieve the extremum.

Historically first: $H = K^r$ – the complete graph.

Clearly, any G with $\chi(G) < r$ is K^r-free, so let us try complete $(r-1)$-partite graphs first. Which of them maximizes $\|G\|$?
Clearly (Exercise!), the one with partition classes as equal as possible, i.e., of sizes $\left\lfloor \frac{n}{r-1} \right\rfloor$ or $\left\lceil \frac{n}{r-1} \right\rceil$.

This (unique) $(r-1)$-partite graph on $n \geq r-1$ vts is called <u>the Turán graph</u> and denoted $T^{r-1}(n)$.

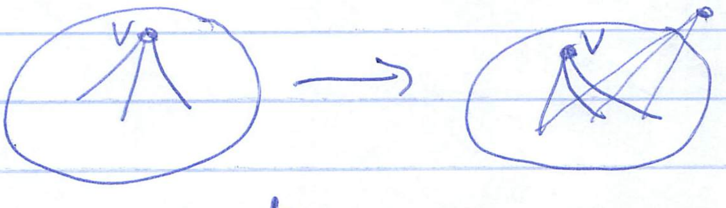
Also $\|T^{r-1}(n)\| = t_{r-1}(n)$. Clearly, $T^{r-1}(n) = K^n$ for $n < r$.

<u>Theorem 15 (Turán, 1941)</u> $\forall\, n \geq r > 1\ \forall\, G \not\supseteq K^r$, $|G| = n$,

$$\|G\| = ex(n, K^r) \Rightarrow G = T^{r-1}(n)$$

In particular, $ex(n, K^r) = t_{r-1}(n)$.

For the proof, we need an operation – <u>vx duplication</u>, by which we mean adding a new vertex v' and joining it to all the nghbs of vx $v \in V(G)$ (but not to v itself)



<u>Notation</u>: G^{+v}

<u>Proof</u> As $T^{k}(n)$ has the most edges among all k-~~$(r-1)$~~-partite n-vx graphs for all $k \leq r-1$, it suffices to show that G is complete multipartite. If not, then $\exists\, y_1, x, y_2 : y_1x, xy_2 \notin E(G)$, while $y_1y_2 \in E(G)$. (Why? b/c: non-adjacency is not an equivalence relation)

If $d(y_1) > d(x)$, then $\|(G-x)^{+y_1}\| > \|G\|$ and still no K^r ↯. So $d(y_1) \leq d(x)$ and similarly $d(y_2) \leq d(x)$. But then, as $2d(x) - d(y_1) - d(y_2) + 1 > 0$,

$\|(G-y_1-y_2)^{+x+x}\| > \|G\|$ and still no K^r ↯ $\square$

We have $t_{r-1}(n) \leq \frac{1}{2} n^2 \frac{r-2}{r-1}$ (with $=$ ~~for~~ when $(r-1)|n$) HW

and $t_{r-1}(n) \sim \frac{1}{2}\frac{r-2}{r-1} n^2$ for large n.

The next result shows that just εn^2 more edges guarantees a copy of K^r_s $\forall$ fixed s.

Theorem 16 (Erdős-Stone, 1946) $\forall r \geq 2, s \geq 1, \varepsilon > 0$

$\exists n_0 : \forall G, |G| = n \geq n_0, \|G\| \geq t_{r-1}(n) + \varepsilon n^2 : G \supset K^r_s$.

Corollary 5 (Erdős-Simonovits, 1966) $\forall H, \|H\| \geq 1$,

$$\lim_{n\to\infty} ex(n,H)\binom{n}{2}^{-1} = \frac{\chi(H)-2}{\chi(H)-1}$$

Proof of Cor. Let $r = \chi(H)$. Then $H \not\subseteq T^{r-1}(n)$ $\forall n$, so $t_{r-1}(n) \leq ex(n,H)$. On the other hand, for sufficiently large s, $H \subseteq K^r_s$, so $ex(n,H) \leq ex(n, K^r_s)$. Now, fix such s. Then, by Thm, $\forall \varepsilon > 0 \ \exists n_0 \ \forall n \geq n_0$

$$ex(n, K^r_s) < t_{r-1}(n) + \varepsilon n^2.$$ Hence,

$$\frac{t_{r-1}(n)}{\binom{n}{2}} \leq \frac{ex(n,H)}{\binom{n}{2}} \leq \frac{ex(n,K_s^r)}{\binom{n}{2}} \leq \frac{t_{r-1}(n)}{\binom{n}{2}} + \frac{2\varepsilon}{1-\frac{1}{n}}$$

$$\leq \frac{t_{r-1}(n)}{\binom{n}{2}} + 4\varepsilon \quad (\text{as } n \geq 2)$$

Since $\frac{t_{r-1}(n)}{\binom{n}{2}} \to \frac{r-2}{r-1}$, so does $\frac{ex(n,H)}{\binom{n}{2}}$ $\square$

<u>Proof of Thm 16</u> Let $r \geq 2$, $s \geq 2$ ($s=1 \Leftarrow$ Thm 15)

Let $\gamma > 0$ (γ will play the role of ε in Thm 16).

If $|G|=n$ & $\|G\| \geq t_{r-1}(n) + \gamma n^2$, then $\gamma < 1$ (why?)

Want to show $K_s^r \subseteq G$ for n large enough (this hides n_0)

Plan: Use reg. lemma and choose d so that $R(\Pi, d) \supseteq K^r$ (by Thm 15). Then $R_s \supseteq K_s^r$, so we may apply Lemma 5.

With $d := \gamma$ & $\Delta := \Delta(K_s^r)$ $[=(r-1)s]$, L5 returns $\varepsilon_0 > 0$.

We apply the reg. lemma with $m > \frac{1}{\gamma}$ & $\varepsilon \leq \varepsilon_0$ and s.t.

$\varepsilon < \frac{\gamma}{2} < 1$ & $\delta := 2\gamma - \varepsilon^2 - 4\varepsilon - d - \frac{1}{m} > 0$;

[possible, as $2\gamma - d - \frac{1}{m} = \gamma - \frac{1}{m} > 0$.]

Reg. lemma returns M. Assume $n \underset{(*)}{\geq} \frac{2Ms}{d^\Delta(1-\varepsilon)}$ $[=: n_1]$

Obviously $n \geq m$.

By reg. lemma $\exists\ \Pi = \{V_0, V_1, \dots, V_k\}$, $m \le k \le M$, an ε-reg partition of G. Let $|V_1| = \dots = |V_k| =: l$.

Then $\boxed{n \geq kl}$ & $l = \dfrac{n - |V_0|}{k} \geq \dfrac{n - \varepsilon n}{M} = n\dfrac{1-\varepsilon}{M} \overset{(*)}{\geq} \dfrac{2s}{d^{\Delta}}$

Let $R = R(\Pi, d)$. We will show $K^r \subseteq R$ by Thm 15.

For this we need to show that $\|R\| > t_{r-1}(k)$.
We gonna use the ass. $\|G\| \geq t_{r-1}(n) + \gamma n^2$.
Let us count the edges of G lying outside the ε-reg. pairs

- inside V_0 – $\leq \binom{|V_0|}{2} \leq \frac{1}{2}(\varepsilon n)^2$
- between V_0 & $V_1 \cup \dots \cup V_k$ – $\leq |V_0|\, kl \leq \varepsilon n k l \leq \varepsilon n^2$
- inside ε-irreg. pairs – $\leq \varepsilon k^2 l^2$
- inside not dense pairs – $\leq \binom{k}{2} d l^2 < \frac{1}{2} k^2 d l^2$
- inside sets $V_1, \dots, V_k$ – $\leq k\binom{l}{2} < \frac{1}{2} k l^2$

Finally, # edges in dense, ε-reg. pairs – $\leq \|R\| l^2$. Thus,

$$\|G\| \leq \frac{1}{2}\varepsilon^2 n^2 + \varepsilon n^2 + \varepsilon k^2 l^2 + \frac{1}{2}k^2 d l^2 + \frac{1}{2} k l^2 + \|R\| l^2$$

$$\Rightarrow \|R\| \geq \frac{1}{2}k^2 \cdot \frac{\|G\| - \frac{1}{2}\varepsilon^2 n^2 - \varepsilon n^2 - \varepsilon k^2 l^2 - \frac{1}{2} d k^2 l^2 - \frac{1}{2} k l^2}{\frac{1}{2}k^2 l^2}$$

$$\geq \frac{1}{2}k^2\left(\frac{t_{r-1}(n) + \gamma n^2 - \frac{1}{2}\varepsilon^2 n^2 - \varepsilon n^2}{n^2/2} - 2\varepsilon - d - \frac{1}{k}\right)$$

$$\geq \frac{1}{2}k^2\left(\frac{t_{r-1}(n)}{n^2/2} \underbrace{+ 2\gamma - \varepsilon^2 - 4\varepsilon - d - \frac{1}{m}}_{\delta}\right) = \frac{1}{2}k^2\left(\frac{t_{r-1}(n)}{n^2/2} + \delta\right)$$

Recall that $\overset{(HW)}{,}$ ~~by Cor. 5~~, $\lim\limits_{n\to\infty} \frac{t_{r-1}(n)}{n^2/2} = \frac{r-2}{r-1}$, i.e.

$\forall \delta\ \exists n_\delta : \forall n \geq n_\delta \quad \frac{t_{r-1}(n)}{n^2/2} > \frac{r-2}{r-1} - \delta$. So

$$\|R\| \geq \frac{1}{2}k^2\left(\frac{t_{r-1}(n)}{n^2/2} + \delta\right) > \frac{1}{2}k^2\frac{r-2}{r-1} \overset{(HW)}{\geq} t_{r-1}(k).$$

By Thm 15, $R \supseteq K^r$, so $R_s \supseteq K_s^r \underset{L5}{\Rightarrow} G \supset K_s^r$

$[n_0 = \max(n_1, n_\delta)]$. $\square$